It is clear that explicit zero-free regions for $\zeta(s)$ can be obtained at least in those cases where the integrals

$$\int_0^1 x^{\varrho-1} G(x_l x)\, dx$$

are of the form $I(\varrho)J(x_l)$, in which case the inequality of Theorem 2 takes the form

$$\sum_{k=1}^{N} \frac{1}{\lambda_k} \sum_{l=1}^{N} |e_{lk} J(x_l)|^2 \leqslant \frac{1}{|I(\varrho)|^2(1-2\beta)}.$$

The integral will be separable when $X \in Q^N$, $0 < x_1 < \ldots < x_N \leqslant 1$, and when $g \in \mathscr{C}$ has the form

$$g(x) = \sum a_\nu h(c_\nu x), \quad 0 < c_\nu \leqslant 1,$$

with $h(x)$ having the form

$$h(x) = \frac{\cos 2\pi x}{x^r}, \quad r \in Z,\ r \geqslant 2.$$

This follows by the same reasoning given in Theorem 2.

We remark that the single greatest impediment to the estimation of the size of the sum appearing in Theorem 1 is the lack of an effective construction of the orthonormal functions F_k, $1 \leqslant k \leqslant N$. A systematic procedure for constructing such functions has been given, but it is ineffective in the sense that the spectrum of the matrix $A_N(X, g)$ is largely unknown, so that good estimates of the sum in Theorem 1 and Theorem 2 are not obtained.

Finally, we observe that the methods of the present paper can be readily adapted to the study of the zeros of Dirichlet L-series, where, in this case, we would begin with the integral

$$\int_0^\infty x^{s-1} \sum_{n=1}^{\infty} \chi(n) f(nx)\, dx.$$

Any zero-free region like that given in Theorem 2 would have important consequences for the distribution of prime numbers.

References

[1] C. Ryavec, *Zero-free regions for* $\zeta(s)$, Mathematika 22 (1975), pp. 92–96.
[2] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford 1951.

UNIVERSITY OF CALIFORNIA
Santa Barbara, California

Received on 10. 3. 1976 (823)

ACTA ARITHMETICA
XXXIII (1977)

On the construction of non-congruence subgroups

by

JULIET BRITTO (Bombay)

The existence of subgroups of finite index in the modular group which are different from its "congruence subgroups" was first pointed out by Klein ([5], § 1, p.63) and several mathematicians have since dealt with the explicit construction of such subgroups (see [7], [8] and [9]). If one looks for "non-congruence subgroups" in analogous situations, one is led naturally in the light of the well-known solution of the "Congruence subgroup problem" and especially [10], to consider the other interesting case namely, that of the group $SL(2, \mathfrak{o})$, where $\mathfrak{o}$ is the ring of algebraic integers in an imaginary quadratic field over the field Q of rational numbers. The problem of constructing "non-congruence subgroups" of $SL(2, \boldsymbol{Z}[\sqrt{-1}])$ has recently been dealt with in a purely algebraic fashion by A. Drillick [2]. We are thankful to Professor S. Raghavan for his guidance during the preparation of this note. We are also thankful to Professor K. G. Ramanathan for his encouragement and to Professor W. Magnus for having given us an opportunity to see the dissertation of Drillick.

In this note, we deal with the construction of non-congruence subgroups of $SL(2, \mathfrak{o})$, i.e. subgroups of finite index but different from "congruence subgroups", when $\mathfrak{o}$ is the ring of integers in an imaginary quadratic field $Q(\sqrt{-d})$ with $d = 1, 2, 3, 7, 11, 5, 6, 15$. In the first five cases, the field is euclidean and B. Fine [3] has given an explicit presentation for $PSL(2, \mathfrak{o})$; in the remaining cases, we use the presentation given by Swan [11].

Our proof is based on the same ideas as of Drillick and perhaps a little simpler.

Let $SL(2, \mathfrak{o})$ denote the group of 2-rowed matrices, with entries in $\mathfrak{o}$ and determinant 1, and $PSL(2, \mathfrak{o})$ the group $SL(2, \mathfrak{o})/\langle \pm I\rangle$, where I is the 2-rowed identity matrix. For any ideal $\mathfrak{q}$ in $\mathfrak{o}$, let $SL(2, \mathfrak{o}/\mathfrak{q})$ denote the group of 2-rowed matrices, with entries in the quotient ring $\mathfrak{o}/\mathfrak{q}$ and determinant 1, and $PSL(2, \mathfrak{o}/\mathfrak{q})$ the quotient group $SL(2, \mathfrak{o}/\mathfrak{q})/\langle \pm I\rangle$.

Let

$$\Gamma(\mathfrak{q}) = \{T \in \mathrm{SL}(2, \mathfrak{o}) \text{ such that } T \equiv I (\mathrm{mod}\, \mathfrak{q})\}$$

and

$$\bar{\Gamma}(\mathfrak{q}) = \{T \in \mathrm{SL}(2, \mathfrak{o}) \text{ such that } T \equiv \pm I (\mathrm{mod}\, \mathfrak{q})\}.$$

Then $\Gamma(\mathfrak{q})$ is called the *principal congruence subgroup modulo* $\mathfrak{q}$ in $\mathrm{SL}(2, \mathfrak{o})$. Any subgroup of $\mathrm{SL}(2, \mathfrak{o})$ containing $\Gamma(\mathfrak{q})$ for some ideal $\mathfrak{q}$ of $\mathfrak{o}$ is called a *congruence subgroup* and is necessarily of finite index. Clearly

$$\mathrm{SL}(2, \mathfrak{o})/\bar{\Gamma}(\mathfrak{q}) \simeq \mathrm{SL}(2, \mathfrak{o}/\mathfrak{q}),$$

$$\bar{\Gamma}(\mathfrak{q})/\langle \pm I\rangle \simeq \Gamma(\mathfrak{q}).$$

LEMMA 1. *Let $\mathfrak{p}$ be any prime ideal of $\mathfrak{o}$ and t any natural number. Then the order of the group $\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}^t)$ is a power of p where p is the rational prime belonging to $\mathfrak{p}$.*

Proof. We know from [4] that $\Gamma(\mathfrak{q})$ has in $\mathrm{SL}(2, \mathfrak{o})$ the index given by

$$[\mathrm{SL}(2, \mathfrak{o}) : \Gamma(\mathfrak{q})] = (N\mathfrak{q})^3 \prod_{\mathfrak{p}|\mathfrak{q}} \left(1 - \frac{1}{(N\mathfrak{p})^2}\right)$$

where $\mathfrak{p}$ runs over all prime ideals dividing $\mathfrak{q}$ and $N\mathfrak{q}$ (respectively $N\mathfrak{p}$) denotes the order of $\mathfrak{o}/\mathfrak{q}$ (resp. $\mathfrak{o}/\mathfrak{p}$). The homomorphism η: $\mathrm{SL}(2, \mathfrak{o}/\mathfrak{p}^t) \to \mathrm{SL}(2, \mathfrak{o}/\mathfrak{p})$ induced by the natural map $\mathfrak{o}/\mathfrak{p}^t \to \mathfrak{o}/\mathfrak{p}$ is surjective and the kernel is $\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}^t)$. Hence $\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}^t)$ has order $(N\mathfrak{p})^{3t-3}$. We know that

$$N\mathfrak{p} = \begin{cases} p^2 & \text{if } p\mathfrak{o} \text{ is prime,} \\ p & \text{otherwise.} \end{cases}$$

Thus $\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}^t)$ is a p-group.

LEMMA 2. *The alternating group A_n, $n \geqslant 7$, is not isomorphic to any composition factor in a composition series for* $\mathrm{SL}(2, \mathfrak{o}/\mathfrak{q})$.

Proof. Let $\mathfrak{q} = \mathfrak{p}_1^{a_1} \ldots \mathfrak{p}_k^{a_k}$, where $\mathfrak{p}_i$, $1 \leqslant i \leqslant k$, are distinct prime ideals and $a_i > 0$ are integers for $1 \leqslant i \leqslant k$. As is well-known, $\mathrm{SL}(2, \mathfrak{o}/\mathfrak{q})$ is the direct product of $\mathrm{SL}(2, \mathfrak{o}/\mathfrak{p}_i^{a_i})$ for $1 \leqslant i \leqslant k$ and any composition factor in a composition series for $\mathrm{SL}(2, \mathfrak{o}/\mathfrak{q})$ is also a composition factor in a composition series for $\mathrm{SL}(2, \mathfrak{o}/\mathfrak{p}_i^{a_i})$ for some i, $1 \leqslant i \leqslant k$. Hence it is enough to prove the lemma, when $\mathfrak{q} = \mathfrak{p}^t$, $\mathfrak{p}$ a prime ideal of $\mathfrak{o}$, $t > 0$ any integer.

Case 1. $2 \in \mathfrak{p}$. Consider the normal series

$$\mathrm{SL}(2, \mathfrak{o})/\Gamma(\mathfrak{p}^t) \supset \Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}^t) \supset \langle I\rangle.$$

In view of Lemma 1, $\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}^t)$ has order a power of 2, hence its composition factors are of order 2. Now

$$\mathrm{SL}(2, \mathfrak{o})/\Gamma(\mathfrak{p}^t)/\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}^t) \simeq \mathrm{SL}(2, \mathfrak{o}/\mathfrak{p})$$

and the order of $\mathrm{SL}(2, \mathfrak{o}/\mathfrak{p})$

$$= \begin{cases} 6 & \text{when 2 splits in } \mathfrak{o}, \\ 60 & \text{when 2 remains prime.} \end{cases}$$

Hence A_n, for $n \geqslant 7$ is not isomorphic to any composition factor.

Case 2. $2 \notin \mathfrak{p}$. Consider the normal series,

$$\mathrm{SL}(2, \mathfrak{o})/\Gamma(\mathfrak{p}^t) \supset \bar{\Gamma}(\mathfrak{p})/\Gamma(\mathfrak{p}^t) \supset \Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}^t) \supset \langle I\rangle.$$

In view of Lemma 1, $\Gamma(\mathfrak{p})/\Gamma(\mathfrak{p}^t)$ is a p-group. Hence its composition factors are all cyclic of order p. Now

$$\mathrm{SL}(2, \mathfrak{o})/\Gamma(\mathfrak{p}^t)/\bar{\Gamma}(\mathfrak{p})/\Gamma(\mathfrak{p}^t) \simeq \mathrm{SL}(2, \mathfrak{o})/\bar{\Gamma}(\mathfrak{p}) \simeq \mathrm{PSL}(2, \mathfrak{o}/\mathfrak{p}).$$

When $N\mathfrak{p} = 3$, the order of $\mathrm{PSL}(2, \mathfrak{o}/\mathfrak{p})$ is 12; hence A_n, $n \geqslant 7$, is not isomorphic to any composition factor. When $N\mathfrak{p} > 3$, $\mathrm{PSL}(2, \mathfrak{o}/\mathfrak{p})$ is simple and A_n, $n \geqslant 7$, is not isomorphic to $\mathrm{PSL}(2, F)$, where F is a field with p or p^2 elements; in fact for a field F_0 with $q = p^t$ elements (and p a prime), it is well-known that the order of $\mathrm{PSL}(2, F_0)$ is $\frac{1}{2}q(q^2-1)$ and $n! = q(q^2-1)$ is impossible for $p > t \geqslant 3$ or $t = 2$, $p \geqslant 5$, or $t = 1$, $p > 5$. This completes the proof of the lemma.

LEMMA 3. *Let the symmetric group S_n or the alternating group A_n for some $n \geqslant 7$, be the quotient of* $\mathrm{SL}(2, \mathfrak{o})$ *by a subgroup K. Then K is a non-congruence subgroup.*

Proof. (i) Let, if possible, $A_n \simeq \mathrm{SL}(2, \mathfrak{o})/K$ for $n \geqslant 7$, with $K \supset \Gamma(\mathfrak{q})$ for some ideal $\mathfrak{q}$. Then any composition series for $\mathrm{SL}(2, \mathfrak{o}/\mathfrak{q})$ contains a factor isomorphic to A_n for some $n \geqslant 7$, which is a contradiction to Lemma 2.

(ii) Let, if possible, $S_n \simeq \mathrm{SL}(2, \mathfrak{o})/K$ with $K \supset \Gamma(\mathfrak{q})$ for some ideal $\mathfrak{q}$ and for some $n \geqslant 7$. Then $A_n \simeq H/K$, for a normal subgroup H of $\mathrm{SL}(2, \mathfrak{o})$. The normal series

$$\mathrm{SL}(2, \mathfrak{o})/\Gamma(\mathfrak{q}) \supset H/\Gamma(\mathfrak{q}) \supset K/\Gamma(\mathfrak{q}) \supset \langle I\rangle$$

has a simple factor isomorphic to A_n, for some $n \geqslant 7$, a contradiction to Lemma 2. Hence the kernel K is a non-congruence subgroup.

Remark. Even if we have a homomorphism η of a normal subgroup H of finite index in $\mathrm{SL}(2, \mathfrak{o})$ onto A_n or S_n, for $n \geqslant 7$, the kernel K of η is a non-congruence subgroup of $\mathrm{SL}(2, \mathfrak{o})$.

LEMMA 4. *If* $\mathrm{SL}(2, \mathfrak{o})$ *contains one non-congruence subgroup then it contains infinitely many such subgroups.*

Proof. Let G be a non-congruence subgroup of $\mathrm{SL}(2, \mathfrak{o})$. Then $G \cap \Gamma(\mathfrak{q})$ for any ideal $\mathfrak{q}$ of $\mathfrak{b}$ is of finite index in $\mathrm{SL}(2, \mathfrak{o})$ and is evidently a non-congruence subgroup. Let S be an infinite subset of the set of rational primes. For each p in S, let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$ containing p. We assert that $G \cap \Gamma(\mathfrak{p})$ are distinct for infinitely many p's in S. Let, if possible,

$G \cap \Gamma(\mathfrak{p}) = H$ for infinitely many p's in S. Then $N\mathfrak{p}(N\mathfrak{p}^2-1)$, the index of $\Gamma(\mathfrak{p})$ in $SL(2, \mathfrak{o})$ divides h, the index of H in $SL(2, \mathfrak{o})$ for infinitely many primes p, this is clearly impossible.

THEOREM. *Let $\mathfrak{o}$ be the ring of integers of $Q(\sqrt{-d})$, where $d = 1, 2, 3, 7, 11, 5, 6, 15$. Then* $SL(2, \mathfrak{o})$ *contains infinitely many non-congruence subgroups.*

Proof. In view of Lemma 4, it is enough to exhibit one non-congruence subgroup of $SL(2, \mathfrak{o})$. For the construction of one such subgroup, we use an explicit presentation for $PSL(2, \mathfrak{o})$ given in [3] and [11] for the d's mentioned.

In the sequel, we mean by a product $\sigma_1\sigma_2$ of two permutations σ_1, σ_2 on n symbols, the permutation obtained by applying σ_1 first and then σ_2.

We deal with the various d's one by one, as follows.

$d = 1$.

$$PSL(2, \mathfrak{o}) = \{a, l, t, u;\ a^2 = l^2 = (al)^2 = (tl)^2 = (ul)^2 = (at)^3 = (ual)^3 = \text{identity},\ tu = ut\}.$$

The map which sends a, l, t, u, to $(14)(25)(67)$, $(15)(24)(67)$, $(153)(2674)$, $(27)(46)$ respectively clearly extends to a homomorphism of $PSL(2, \mathfrak{o})$ into S_7. We assert that it is onto. In fact the image contains (52), (5271346) (for, $(alata)^5 \to (25)$, $(ualat^4a) \to (5271346)$) and hence S_7, by ([6], p. 49). Hence S_7 is a quotient group of $SL(2, \mathfrak{o})$ and the kernel is a non-congruence subgroup, by Lemma 3.

$d = 2$.

$$PSL(2, \mathfrak{o}) = \{a, t, u;\ tu = ut,\ a^2 = (at)^3 = (u^{-1}aua)^2 = \text{identity}\}.$$

Since for every $n \geqslant 9$, S_n is generated by two elements α, β of order 2 and 3 respectively [1], it is a quotient of $PSL(2, \mathfrak{o})$; a map φ with $\varphi(a) = \alpha$, $\varphi(t) = \alpha\beta$ and $\varphi(u) =$ identity will do. Thus S_n, for every $n \geqslant 9$, is a quotient of $SL(2, \mathfrak{o})$ and the kernel is a non-congruence subgroup.

$d = 3$.

Let G^* be the commutator subgroup of $PSL(2, \mathfrak{o})$. Then

$$G^* = \{j, m, t, u;\ j^2 = m^2 = t^2 = u^2 = (tj)^3 = (mtu)^3 = (umj)^3 = (mt)^2 = \text{identity}\}.$$

Let φ be the map which sends m, t, u, j to $(13)(24)$, $(12)(34)(56)$, $(17)(23)(56)$, $(16)(24)(35)$ respectively. Since $\varphi(t)\varphi(j) = (145)(263)$, $\varphi(m)\varphi(t)\varphi(u) = (147)$, $\varphi(u)\varphi(m)\varphi(j) = (175)(263)$, $\varphi(m)\varphi(t) = (14)(23)(56)$, φ extends to a homomorphism of G^* into S_7. Since $\varphi(um) = (17342)(56)$, $\varphi(um)^5 = (56)$. Also $\varphi(mtumjtjmj) = (1364752)$. Hence $(5624173) \in \varphi(G^*)$. Thus S_7 is a quotient group of G^* hence also of the commutator subgroup G of $SL(2, \mathfrak{o})$. Since G is of finite index in $SL(2, \mathfrak{o})$, we get a non-congruence subgroup of $SL(2, \mathfrak{o})$, by the Remark following Lemma 3.

$d = 7$.

$$PSL(2, \mathfrak{o}) = \{a, t, u;\ tu = ut,\ a^2 = (at)^3 = (u^{-1}auat)^2 = \text{identity}\}.$$

If we set $\varphi(a) = (12)(34)(56)$, $\varphi(t) = (15947)(2368)$, $\varphi(u) = (2368)$ then clearly $\varphi(t)\varphi(u) = \varphi(u)\varphi(t)$, $\varphi(a)\varphi(t) = (137)(258)(469)$ and $\varphi(u)^{-1}\varphi(a) \times \varphi(u)\varphi(a)\varphi(t) = (17)(25)(49)$; hence φ extends to a homomorphism of $PSL(2, \mathfrak{o})$ into S_9. We show that φ is surjective as follows. In fact,

$$(12)(34)(56)(17)(25)(49) = (15627)(394),$$

$$\big((15627)(394)\big)^6 = (15627) \in \varphi\big(PSL(2, \mathfrak{o})\big),$$

$$(15627)(2368) = (15827)(36);$$

hence

$$\big((15827)(36)\big)^5 = (36) \in \varphi\big(PSL(2, \mathfrak{o})\big).$$

Also

$$(15947)(15827) = (1827594), \quad (1827594)(15947)(15947) = (1825749),$$

$$(15947)(12)(34)(56)(74951)(2368) = (1827396),$$

$$(1825749)(1827396) = (125398746), \quad \text{and} \quad (125398746)^5 = (369182754).$$

Since (36) and (369182754) generate S_9, $\varphi\big(PSL(2, \mathfrak{o})\big) = S_9$. Hence it follows that S_9 is a quotient group of $SL(2, \mathfrak{o})$ and the kernel is a non-congruence subgroup by Lemma 3.

$d = 11$.

$$PSL(2, \mathfrak{o}) = \{a, t, u;\ tu = ut,\ a^2 = (at)^3 = (u^{-1}auat)^3 = \text{identity}\}.$$

The map φ which sends a, t, u to $(12)(34)(56)(78)$, $(1245)(3678)$, (1245) respectively extends to a homomorphism of $PSL(2, \mathfrak{o})$ into S_8. Further,

$$(1245)(12)(34)(56)(78)(3678) = (265)(347), \quad (265)(347)(1245) = (126)(3547)$$

and hence

$$\big((126)(3547)\big)^4 = (126) \in \varphi\big(PSL(2, \mathfrak{o})\big).$$

From $(1245)(126) = (16)(245)$, we have

$$\big((16)(245)\big)^3 = (16) \in \varphi\big(PSL(2, \mathfrak{o})\big).$$

Moreover,

$$(126)(3678)(1245)(16) = (14562783)$$

and so

$$(14562783)^3 = (16842357) \in \varphi\big(PSL(2, \mathfrak{o})\big).$$

Since (16) and (16842357) generate S_8, $\varphi(\mathrm{PSL}(2,\mathfrak{o})) = S_8$, hence S_8 is also a quotient group of $\mathrm{SL}(2,\mathfrak{o})$ and the kernel is a non-congruence subgroup.

$d = 5$.

$$\mathrm{PSL}(2,\mathfrak{o}) = \{T, U, A, B, C; TU = UT,\ A^2 = B^2 = (TA)^3 = (AB)^2 = (AUBU^{-1})^2 = \text{identity}; ACA = TCT^{-1}, UBU^{-1}CB = TCT^{-1}\}.$$

Since S_n, for any $n \geqslant 9$, is generated by two elements α, β of order 2 and 3 respectively, the map φ with $\varphi(B) = \varphi(U) = \varphi(C) = \text{identity}$, $\varphi(A) = \alpha$, $\varphi(T) = \beta\alpha$, extends to a homomorphism of $\mathrm{PSL}(2,\mathfrak{o})$ onto S_n, $n \geqslant 9$. Hence S_n, for any $n \geqslant 9$, is a quotient of $\mathrm{SL}(2,\mathfrak{o})$ and the kernel is a non-congruence subgroup.

$d = 6$.

$$\mathrm{PSL}(2,\mathfrak{o}) = \{T, U, A, B, C; TU = UT,\ A^2 = B^2 = (TA)^3 = (ATB)^3 = (ATUBU^{-1})^3 = \text{identity};\ AC = CA, T^{-1}CTUBU^{-1} = BC\}.$$

The map φ defined by $\varphi(U) = \varphi(B) = \varphi(C) = \text{identity}$, $\varphi(A) = (12)(34)(56)$, $\varphi(T) = (1458)(26937)$ gives a homomorphism of $\mathrm{PSL}(2,\mathfrak{o})$ into S_9. We have

$$\varphi(ATA) = (15947)(2368).$$

Hence

$$\varphi(ATA)^5 = (2368) \in \varphi(\mathrm{PSL}(2,\mathfrak{o})).$$

Since

$$(2368)(1458)(26937) = (14586)(27)(39),$$

we have

$$(27)(39) \in \varphi(\mathrm{PSL}(2,\mathfrak{o})).$$

Further $(26937)(27)(39) = (263)$ and so

$$(2368)(263) = (68) \in \varphi(\mathrm{PSL}(2,\mathfrak{o})).$$

Now

$$\varphi(TA) = (137)(258)(469),\quad (137)(258)(469)(263) = (125869437),$$

and hence

$$(869437125) \in \varphi(\mathrm{PSL}(2,\mathfrak{o})).$$

Since (86) and (869437125) generate S_9, S_9 is a quotient group of $\mathrm{PSL}(2,\mathfrak{o})$ and hence of $\mathrm{SL}(2,\mathfrak{o})$ and the kernel is a non-congruence subgroup.

$d = 15$.

$$\mathrm{PSL}(2,\mathfrak{o}) = \{T, U, A, C; TU = UT,\ A^2 = (TA)^3 = \text{identity};\ AC = CA,\ UCUAT = TAUCU\}.$$

Since, for every $n \geqslant 9$, S_n is generated by two elements α, β of order 2 and 3 respectively, the map φ with $\varphi(U) = \text{identity}$, $\varphi(A) = \alpha$, $\varphi(C) = \alpha$, $\varphi(T) = \beta\alpha$ gives a homomorphism of $\mathrm{PSL}(2,\mathfrak{o})$ onto S_n, for $n \geqslant 9$. Hence S_n, for $n \geqslant 9$, are quotients of $\mathrm{SL}(2,\mathfrak{o})$ and the kernels are non-congruence subgroups. Thus for all the rings $\mathfrak{o}$, mentioned in the theorem, $\mathrm{SL}(2,\mathfrak{o})$ contains infinitely many non-congruence subgroups (in view of Lemma 4) and the proof of the theorem is complete.

References

[1] I. M. S. Dey and J. Wiegold, *Generators for alternating and symmetric groups*, J. Austral. Math. Soc. 12 (1971), pp. 63–68.

[2] A. Drillick, *The Picard group*, Ph. D. Thesis, New York University, New York 1971.

[3] B. Fine, *The structure of* $\mathrm{PSL}_2(R)$; *R, the ring of integers in a euclidean quadratic imaginary number field*, Ann. of Math.-Studies N° 79, *Discontinuous groups and Riemann surfaces*, 1973, pp. 145–170.

[4] A. Hurwitz, *Mathematische Werke 2*, pp. 244–268, Basel 1933.

[5] F. Klein, *Zur Theorie der elliptischen Modulfunktionen*, Math. Ann. 17 (1880), pp. 62–70.

[6] A. G. Kurosh, *The theory of groups*, vol. I, Chelsea, New York 1960.

[7] H. Maass, *Lectures on modular functions of one complex variable*, Tata Institute of Fundamental Research, Bombay 1964.

[8] W. Magnus, *Non-euclidean tesselations and their groups*, Academic Press, New York and London 1974.

[9] R. A. Rankin, *The modular group and its subgroup*, The Ramanujan Institute, University of Madras, 1969.

[10] J. P. Serre, *Le problème des groupes de congruence pour* SL_2, Ann. of Math. 92 (1970), pp. 489–527.

[11] R. G. Swan, *Generators and relations for certain special linear groups*, Advances in Math. 6 (1971), pp. 1–77.

CENTRE OF ADVANCED STUDY IN MATHEMATICS
UNIVERSITY OF BOMBAY

Received on 29. 3. 1976 (832)