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Inequalities for zeros of $\zeta(s)$

by

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It was shown in [1] that the specific integral representation of the Riemann zeta-function

$$-\frac{\zeta(s)}{s} = \int_0^\infty \{u\} u^{-s-1}\,du \tag{1}$$

could be used to construct zero-free regions for $\zeta(s)$ in the critical strip $\mathscr{S} = \{s = \sigma + it\colon 0 < \sigma < 1\}$. The basis of the method used in [1] is to (A) decompose $\zeta(s)$ into the sum of two functions, $\zeta(s) = \zeta_1(s) + \zeta_2(s)$, so that whenever $\zeta(\varrho) = 0$, the equality $|\zeta_1(\varrho)| = |\zeta_2(\varrho)|$ holds; and to then (B) use the assumption $\varrho = \beta + i\gamma$, $0 < \beta < \frac{1}{2}$, with Bessel's inequality to deduce zero-free regions for $\zeta(s)$.

In the present paper, we separate this method from the special representation (1) and present it within a general framework which clearly displays the relationship between the natural numbers and certain simple, but effective, aspects of L^2-theory. The essence of this relationship is the construction of functions $f_k(x)$, $1 \leqslant k \leqslant N$ $(x \geqslant 1)$ for which

$$\int_1^\infty \sum_{n=1}^\infty f_j(nx) \sum_{n=1}^\infty f_k(nx)\,dx = \delta_{jk}, \quad 1 \leqslant j,\ k \leqslant N.$$

A systematic procedure is given for constructing such functions, and they are used to obtain inequalities for the zeros of $\zeta(s)$ in $\mathscr{S}$. This is the result of Theorem 2. In Theorem 3 we deduce explicit zero-free regions for $\zeta(s)$.

Throughout the paper we shall exclusively use functions in class $\mathscr{C}$.

DEFINITION. We let $\mathscr{C}$ denote the class of real-valued, continuous functions, $f(x)$, $x \geqslant 0$, for which there exists a $\sigma_0 > 1$ such that

$$x^{\sigma-1} f(x) \in L^1(0, \infty), \quad 0 < \sigma < \sigma_0, \tag{2}$$

$$\int_0^\infty f(x)\,dx = 0, \tag{3}$$

$$(4)\qquad |f(x)| \leqslant x^{-\sigma_0}, \quad x\to\infty,$$

$$(5)\qquad F(x) = \sum_{n=1}^{\infty} f(nx) \text{ converges for } x \geqslant 0, \text{ and}$$

$$x^{\sigma-1}F(x) \in L^1(0,\infty), \quad 0<\sigma<\sigma_0.$$

LEMMA 1. *Let f be a function in $\mathscr{C}$. Then for some $\sigma_0 > 1$,*

$$(6)\qquad \zeta(s)\int_0^{\infty} x^{s-1}f(x)\,dx = \int_0^{\infty} x^{s-1}F(x)\,dx,$$

holds for all $s = \sigma+it$, $0<\sigma<\sigma_0$.

Proof. The conditions (4) and (5) guarantee that the steps

$$\int_0^{\infty} x^{\sigma-1}F(x)\,dx = \sum_{n=1}^{\infty}\int_0^{\infty} x^{\sigma-1}f(nx)\,dx$$
$$= \sum_{n=1}^{\infty} n^{-\sigma}\int_0^{\infty} x^{\sigma-1}f(x)\,dx = \zeta(\sigma)\int_0^{\infty} x^{\sigma-1}f(x)\,dx$$

are valid for $1<\sigma<\sigma_0$. The conditions (4) and (5), together with (2) and (3), show that (6) holds for all $s=\sigma+it$, $0<\sigma<\sigma_0$, by analytic continuation. This completes the proof of Lemma 1.

A variation of Lemma 1 can be found on p. 28 of [2]. Now suppose that functions $f_k(x)$, $1\leqslant k\leqslant N$, from $\mathscr{C}$ have been found so that

$$(7)\qquad \int_1^{\infty} F_j(x)F_k(x)\,dx = \delta_{jk}, \quad 1\leqslant j,k\leqslant N.$$

For each f_k, we use Lemma 1 to write

$$\zeta(s)\int_0^{\infty} x^{s-1}f_k(x)\,dx = \int_0^{1} x^{s-1}F_k(x)\,dx + \int_1^{\infty} x^{s-1}F_k(x)\,dx,$$

$0<\sigma\leqslant 1$. If $\zeta(\varrho)=0$, $\varrho=\beta+i\gamma$, $0<\beta<\frac{1}{2}$, then by Bessel's inequality, since $x^{\varrho-1}\in L^2(1,\infty)$,

$$(8)\qquad \sum_{k=1}^{N}\Big|\int_0^{1} x^{\varrho-1}F_k(x)\,dx\Big|^2 = \sum_{k=1}^{N}\Big|\int_1^{\infty} x^{\varrho-1}F_k(x)\,dx\Big|^2 \leqslant \frac{1}{1-2\beta}.$$

But $x^{\varrho-1}\in L^2(1,\infty)$ if and only if $x^{\varrho-1}\notin L^2(0,1)$; and thus we are led to explore the possibility that, by properly choosing the functions f_k (satisfying (7)), the sum

$$(9)\qquad \sum_{k=1}^{N}\Big|\int_0^{1} x^{\varrho-1}F_k(x)\,dx\Big|^2$$

can be made sufficiently large so as to contradict (8). This would establish the Riemann hypothesis.

The sums (9) are not known to be unbounded. We shall, however, be able to obtain zero-free regions for $\zeta(s)$ using the method just described. This will be accomplished in Theorem 3. First we give a systematic procedure for constructing functions f_k, $1\leqslant k\leqslant N$, which satisfy (7).

Let $X=(x_1,x_2,\ldots,x_N)$ be a point in E^N, Euclidean N-space; and let Q^N denote the subspace of E^N defined by the inequalities $0<x_1<x_2<\ldots<x_N$. For any $g\in\mathscr{C}$, define a function $G(u,v)$, $u,v>0$,

$$G(u,v) = \int_1^{\infty} G(ux)G(vx)\,dx,$$

where $G(x) = \sum_{n=1}^{\infty} g(nx)$. For any $X\in Q^N$, consider the $N\times N$ real, symmetric matrix $A = A_N(X,g)$ defined by

$$A = \|G(x_j,x_k)\|, \quad 1\leqslant j,k\leqslant N.$$

It is known that A has N non-negative eigenvalues $\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_N\geqslant 0$, $\lambda_k=\lambda_k(X,g)$ (since A is a Gram matrix), with corresponding orthonormal eigenvectors $e_1,e_2,\ldots,e_N$, $e_k=e_k(X,g)=(e_{1k},e_{2k},\ldots,e_{Nk})$. We choose X and g so that, in fact, A is positive definite, in which case the smallest eigenvalue λ_N is positive. When this is the case, the functions

$$(10)\qquad f_k(x) = \frac{1}{\lambda_k^{1/2}}\sum_{l=1}^{N} e_{lk}\,g(x_l x)$$

are functions in $\mathscr{C}$ and satisfy (7), since

$$\int_1^{\infty} F_j(x)F_k(x)\,dx = \int_1^{\infty}\frac{1}{\lambda_j^{1/2}}\sum_{m=1}^{N} e_{mj}G(x_m x)\frac{1}{\lambda_k^{1/2}}\sum_{l=1}^{N} e_{lk}G(x_l x)\,dx$$
$$= \frac{1}{(\lambda_j\lambda_k)^{1/2}}\sum_{m=1}^{N}\sum_{l=1}^{N} e_{mj}e_{lk}G(x_l,x_m)$$
$$= \frac{1}{(\lambda_j\lambda_k)^{1/2}}(Ae_k,e_j) = \frac{1}{(\lambda_j\lambda_k)^{1/2}}(\lambda_k e_k,e_j) = \delta_{jk}.$$

It follows that we have

THEOREM 1. *Let $g\in\mathscr{C}$ and $X\in Q^N$ be chosen so that $A=A_N(X,g)$ is positive definite. With this choice of X and g, define functions f_k, $1\leqslant k\leqslant N$, by (10). Then if $\zeta(\varrho)=0$, $\varrho=\beta+i\gamma$, $0<\beta<\frac{1}{2}$, the inequality*

$$\sum_{k=1}^{N}\Big|\int_0^{1} x^{\varrho-1}F_k(x)\,dx\Big|^2 \leqslant \frac{1}{1-2\beta}$$

holds.

Proof. The inequality of Theorem 1 is simply the inequality given in (8) with the orthonormal functions f_k, $1 \leqslant k \leqslant N$, given by (10).

No method to effectively estimate the magnitude of the sum in Theorem 1 is known when g and X are unrestricted. When g and X are suitably restricted to a subclass of $\mathscr{C}$ and Q^N, however, then it is possible to disentangle ϱ from the other parameters to obtain explicit zero-free regions for $\zeta(s)$. We illustrate this procedure in Theorem 2 with a special choice of g; and afterwards, we discuss how other choices of g also lead to explicit zero-free domains.

Put

$$h(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2.$$

We consider a linear combination of h in order to have a function in $\mathscr{C}$. Thus, put

$$g(x) = \sum_{\nu=1}^{4} a_\nu h(c_\nu x), \tag{11}$$

with $a_\nu \in \boldsymbol{R}$ and $0 < c_\nu \leqslant 1$, $1 \leqslant \nu \leqslant 4$, chosen so as to satisfy

(a) $$\sum a_\nu = 0,$$

(b) $$\sum a_\nu c_\nu = 0,$$

(c) $$\sum a_\nu c_\nu^{-1} = 0,$$

(d) $$\sum a_\nu c_\nu^2 = D \neq 0.$$

Since $h(x)$ satisfies (2) and (4) with $\sigma_0 = 2$, so does $g(x)$; and it is easy to see that condition (c) guarantees that (3) holds. We now show that (a) and (b) imply (5). (The condition (d) will be needed for another reason in Theorem 2.) When $x > 0$, we have

$$\sum_{n=1}^{\infty} h(nx) = \frac{\frac{1}{12} - B_2(x)}{x^2},$$

where $B_2(x)$ is the second Bernoulli function; i.e., the periodic extension of $\frac{1}{6} - x + x^2$, $0 \leqslant x \leqslant 1$, with period 1. Hence,

$$G(x) = \sum_{n=1}^{\infty} g(nx) = -x^{-2} \sum_{\nu=1}^{4} a_\nu B_2(c_\nu x).$$

Since $0 < c_\nu \leqslant 1$, $B_2(c_\nu x) = \frac{1}{6} - c_\nu x + (c_\nu x)^2$, $0 \leqslant x \leqslant 1$. It follows from (a) and (b) that

$$-x^{-2} \sum a_\nu B_2(c_\nu x) = -\sum a_\nu c_\nu^2, \quad 0 \leqslant x \leqslant 1,$$

in which case condition (5) holds.

THEOREM 2. *Let $g(x)$ be defined by (11), and choose $X \in Q^N$ subject to the restriction $0 < x_1 < x_2 < \ldots < x_N \leqslant 1$. If $\varrho = \beta + i\gamma$, $0 < \beta < \frac{1}{2}$, is a zero of $\zeta(s)$, we have*

$$D^2 \sum_{k=1}^{N} \frac{(e_k, X^2)^2}{\lambda_k} \leqslant \frac{|\varrho|^2}{1-2\beta},$$

where, as usual, $\lambda_k(X)$ and $e_k = e_k(X)$, $1 \leqslant k \leqslant N$, are the eigenvalues and eigenvectors of $A_N(X, g)$, and where we have put $X^2 = (x_1^2, x_2^2, \ldots, x_N^2)$.

Proof. Let X and g be chosen subject to the conditions stated in Theorem 2. We consider the $N \times N$ matrix $A_N(X, g)$ defined as usual. Let $\lambda_1, \ldots, \lambda_N$ and $e_1, \ldots, e_N$, $e_k = (e_{1k}, e_{2k}, \ldots, e_{Nk})$ denote the eigenvalues and corresponding eigenvectors of A. We shall show at the end of the proof that $\lambda_k > 0$. Let us assume this is true for now. Put

$$f_k(x) = \frac{1}{\lambda_k^{1/2}} \sum_{l=1}^{N} e_{lk} g(x_l x), \quad 1 \leqslant k \leqslant N.$$

Then we know that the N functions

$$F_k(x) = -\frac{1}{\lambda_k^{1/2}} \sum_{l=1}^{N} e_{lk} \sum_{\nu=1}^{4} a_\nu B_2(x_l c_\nu x)$$

are orthonormal, and that the inequality of Theorem 1 holds:

$$\sum_{k=1}^{N} \left| \int_0^1 x^{\varrho-1} F_k(x)\, dx \right|^2 \leqslant \frac{1}{1-2\beta}.$$

But it is now a simple matter to evaluate the integrals; *viz.*

$$\int_0^1 x^{\varrho-1} F_k(x)\, dx = -\frac{1}{\lambda_k^{1/2}} \sum_{l=1}^{N} e_{lk} \sum_{\nu=1}^{4} a_\nu c_\nu^2 x_l^2 \varrho^{-1} = -\frac{D(e_k, X^2)}{\lambda_k^{1/2} \varrho}.$$

Substituting this value of the integral into the inequality of Theorem 1 yields the inequality of Theorem 2.

To finish the theorem we need to show that $A_N(X, g)$ is positive definite. This will be accomplished if we can show that for each (fixed) choice of the real number $t_1, \ldots, t_N$, and for each $X \in Q^N$, the function

$$T(x) = \sum_{l=1}^{N} t_l g(x_l x)$$

is the zero function only when $t_1 = t_2 = \ldots = t_N = 0$. But if l_0 is the largest index for which $t_l \neq 0$, and if $c = \max_{1 \leqslant \nu \leqslant 4} c_\nu$, then the derivative of $T(x)$ has a jump discontinuity at $x = (c x_{l_0})^{-1}$, in which case T does not vanish identically. This completes the proof of Theorem 2.

It is clear that explicit zero-free regions for $\zeta(s)$ can be obtained at least in those cases where the integrals

$$\int_0^1 x^{\varrho-1}G(x_l x)\,dx$$

are of the form $I(\varrho)J(x_l)$, in which case the inequality of Theorem 2 takes the form

$$\sum_{k=1}^{N}\frac{1}{\lambda_k}\sum_{l=1}^{N}|e_{lk}J(x_l)|^2 \leqslant \frac{1}{|I(\varrho)|^2(1-2\beta)}.$$

The integral will be separable when $X\in Q^N$, $0<x_1<\ldots<x_N\leqslant 1$, and when $g\in\mathscr{C}$ has the form

$$g(x)=\sum a_\nu h(c_\nu x),\quad 0<c_\nu\leqslant 1,$$

with $h(x)$ having the form

$$h(x)=\frac{\cos 2\pi x}{x^r},\quad r\in Z,\ r\geqslant 2.$$

This follows by the same reasoning given in Theorem 2.

We remark that the single greatest impediment to the estimation of the size of the sum appearing in Theorem 1 is the lack of an effective construction of the orthonormal functions F_k, $1\leqslant k\leqslant N$. A systematic procedure for constructing such functions has been given, but it is ineffective in the sense that the spectrum of the matrix $A_N(X,g)$ is largely unknown, so that good estimates of the sum in Theorem 1 and Theorem 2 are not obtained.

Finally, we observe that the methods of the present paper can be readily adapted to the study of the zeros of Dirichlet L-series, where, in this case, we would begin with the integral

$$\int_0^\infty x^{s-1}\sum_{n=1}^{\infty}\chi(n)f(nx)\,dx.$$

Any zero-free region like that given in Theorem 2 would have important consequences for the distribution of prime numbers.

References

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On the construction of non-congruence subgroups

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The existence of subgroups of finite index in the modular group which are different from its "congruence subgroups" was first pointed out by Klein ([5], § 1, p.63) and several mathematicians have since dealt with the explicit construction of such subgroups (see [7], [8] and [9]). If one looks for "non-congruence subgroups" in analogous situations, one is led naturally in the light of the well-known solution of the "Congruence subgroup problem" and especially [10], to consider the other interesting case namely, that of the group $SL(2,\mathfrak{o})$, where $\mathfrak{o}$ is the ring of algebraic integers in an imaginary quadratic field over the field Q of rational numbers. The problem of constructing "non-congruence subgroups" of $SL(2,\boldsymbol{Z}[\sqrt{-1}])$ has recently been dealt with in a purely algebraic fashion by A. Drillick [2]. We are thankful to Professor S. Raghavan for his guidance during the preparation of this note. We are also thankful to Professor K. G. Ramanathan for his encouragement and to Professor W. Magnus for having given us an opportunity to see the dissertation of Drillick.

In this note, we deal with the construction of non-congruence subgroups of $SL(2,\mathfrak{o})$, i.e. subgroups of finite index but different from "congruence subgroups", when $\mathfrak{o}$ is the ring of integers in an imaginary quadratic field $Q(\sqrt{-d})$ with $d=1,2,3,7,11,5,6,15$. In the first five cases, the field is euclidean and B. Fine [3] has given an explicit presentation for $PSL(2,\mathfrak{o})$; in the remaining cases, we use the presentation given by Swan [11].

Our proof is based on the same ideas as of Drillick and perhaps a little simpler.

Let $SL(2,\mathfrak{o})$ denote the group of 2-rowed matrices, with entries in $\mathfrak{o}$ and determinant 1, and $PSL(2,\mathfrak{o})$ the group $SL(2,\mathfrak{o})/\langle\pm I\rangle$, where I is the 2-rowed identity matrix. For any ideal $\mathfrak{q}$ in $\mathfrak{o}$, let $SL(2,\mathfrak{o}/\mathfrak{q})$ denote the group of 2-rowed matrices, with entries in the quotient ring $\mathfrak{o}/\mathfrak{q}$ and determinant 1, and $PSL(2,\mathfrak{o}/\mathfrak{q})$ the quotient group $SL(2,\mathfrak{o}/\mathfrak{q})/\langle\pm I\rangle$.