ACTA ARITHMETICA
XXXIII (1977)

Homogeneous additive equations and Waring's problem

by

R. C. VAUGHAN (London)

1. Introduction. Let $k \geqslant 3$ be a natural number. Davenport and Lewis [7] define $\Gamma^*(k)$ and $G^*(k)$ as follows. If $c_1, \ldots, c_s$ are integers such that for every prime power p^m the congruence

$$c_1 x_1^k + c_2 x_2^k + \ldots + c_s x_s^k \equiv 0 \pmod{p^m} \tag{1.1}$$

has a solution with $x_1, \ldots, x_s$ not all divisible by p, then $c_1, \ldots, c_s$ are said to satisfy the *congruence condition*. The number $\Gamma^*(k)$ is the least number s such that every set of s integers $c_1, \ldots, c_s$ satisfies the congruence condition, and $G^*(k)$ is the least number such that if $s \geqslant G^*(k)$ and $c_1, \ldots, c_s$ are any s integers, not all the same sign when k is even, which satisfy the congruence condition, then the equation

$$c_1 x_1^k + c_2 x_2^k + \ldots + c_s x_s^k = 0 \tag{1.2}$$

has a solution in integers $x_1, \ldots, x_s$, not all of which are zero.

The major part of their paper is devoted to showing that

$$\Gamma^*(k) \leqslant k^2+1, \tag{1.3}$$

with equality whenever $k+1$ is a prime. However, their Theorem 2 implies that when $k \geqslant 18$

$$G^*(k) \leqslant k^2+1. \tag{1.4}$$

They also indicate that the methods of Davenport ([3], [5]) will give this when $k \leqslant 6$, and observe that it seems doubtful whether the solubility of (1.2) for $s \geqslant k^2+1$ can be proved for all the intermediate values $k = 7, \ldots, 17$ by existing methods. Our purpose is to reduce the gap.

THEOREM. *We have* $G^*(9) \leqslant 91$, $G^*(10) \leqslant 107$, $G^*(11) \leqslant 122$, $G^*(12) \leqslant 137$, $G^*(13) \leqslant 153$, $G^*(14) \leqslant 168$, $G^*(15) \leqslant 184$, $G^*(16) \leqslant 200$, $G^*(17) \leqslant 216$.

COROLLARY. *When* $11 \leqslant k \leqslant 17$ *we have* (1.4).

As far as $k = 7, 8$ are concerned, the method of Davenport [5] when adapted to this problem is still the most effective and gives $G^*(7) \leqslant 53$ and $G^*(8) \leqslant 73$.

The argument used here is an adaptation of one of Vinogradov [10], Chapter IV, related to the estimation of $G(k)$ in Waring's problem (see also Chen [1]). For large k it gives

$$\limsup_{k\to\infty}\frac{G^*(k)}{k\log k}\leqslant 3 \tag{1.5}$$

and by comparison the method of Davenport and Lewis gives this with the 3 replaced by 4. By adapting another method of Vinogradov [11] it is possible to show that

$$\limsup_{k\to\infty}\frac{G^*(k)}{k\log k}\leqslant 2. \tag{1.6}$$

Although there is quite a wide range of choice of the parameters involved in the proof of (1.6), it appears that the argument used here is always more effective when k is less than about 50000.

We observe that when applied to Waring's problem our method gives the above theorem with G^* replaced by G. In particular this improves on the known bounds for $G(9)$ and $G(10)$ due to Cook [2].

We further note that to prove the theorem it suffices to assume that $c_1, \ldots, c_s$ are all non-zero.

Throughout, δ is a fixed but sufficiently small positive real number in terms of $c, c_1, \ldots, c_s$ and k, where $c, c_1, \ldots, c_s$ are non-zero integers and k is a natural number with $k\geqslant 9$. Formulae containing ε hold for every sufficiently small positive ε and the implied constants in the $\asymp, O, \ll$ and $\gg$ symbols depend at most on $\varepsilon, c, c_1, \ldots, c_s, k$, and δ.

2. Preliminary lemmas

LEMMA 1. *Let a_n $(n = M+1, \ldots, M+N)$ and b_r $(r = 1, \ldots, R)$ be complex numbers, suppose that the x_r $(r = 1, \ldots, R)$ are real numbers which are distinct modulo one, and define*

$$\delta = \min\|x_r - x_t\|$$

where the minimum is taken over all pairs r, t with $r\neq t$, and where $\|u\|$ denotes the distance of u from the nearest integer. Then

$$\Big|\sum_{n=M+1}^{M+N}\sum_{r=1}^{R}a_nb_re(nx_r)\Big|\leqslant\Big(\sum_{n=M+1}^{M+N}|a_n|^2\sum_{r=1}^{R}|b_r|^2(N+\delta^{-1})\Big)^{1/2}.$$

Proof. At once from Cauchy's inequality and Theorem 1 of Montgomery and Vaughan [9].

LEMMA 2. *Suppose that $c\neq 0$, X is a real number with $X>1$, $|\alpha - a/q|\leqslant q^{-1}X^{-2k}$, $(a, q) = 1$, $q\leqslant X^{2k}$, a_n $(|n|<X^{2k})$ are complex numbers, $|\alpha - a/q|\geqslant q^{-1}X^{2-4k}$ when $q\leqslant X$, and*

$$S(\alpha) = \sum_{n,p,p_1,p_2,r}a_ne(\alpha cp_1^kp_2^kr^k+\alpha np^k)$$

where the summations are over $|n|<X^{2k}$, $\frac14X^2<p<\frac12X^2$, $r<X^{\delta/k}$ and $p_1, p_2<X^{1-\delta}$. Then

$$S(\alpha)\ll X^{3+k}\Big(\sum_{|n|<X^{2k}}|a_n|^2\Big)^{1/2}.$$

This is essentially Lemma 2 of Vinogradov [10], Chapter IV, but the use of the factor

$$\sum_{p_1,p_2,r}e(\alpha cp_1^kp_2^kr^k) \tag{2.1}$$

is new. The purpose of the apparently superfluous variable r is to ensure that when the variable x_{s_0}, where s_0 is to be defined, of (1.2) appears in the singular series it is summed over a complete set of residues, rather than a reduced set. This is of paramount importance, for otherwise the congruence condition cannot be met.

Proof. We first of all treat (2.1). Since $|\alpha - a/q|\leqslant q^{-1}X^{-2k}$ and the number of different prime divisors of q is $O(q^\varepsilon)$ we have

$$\sum_{p_1,p_2,r}e(\alpha cp_1^kp_2^kr^k) = \sum_r\sum_{\substack{p_1,p_2\\(p_1p_2,q)=1}}e\Big(\frac{a}{q}cp_1^kp_2^kr^k\Big)+O(X^{2-\delta}q^{-1}+q^\varepsilon X^{1-\delta/2}). \tag{2.2}$$

For a given r, let $a' = acr^k/(q, cr^k)$, $q' = q/(q, cr^k)$ so that $(a', q') = 1$ and $q\geqslant q'\gg qX^{-\delta}$. If $(b, q') = 1$, then the number of solutions of $n^k\equiv b$ $(\bmod q')$ with $1\leqslant n\leqslant q'$ is $O(q^\varepsilon)$. Hence, by Cauchy's inequality,

$$\Big|\sum_{\substack{p_1,p_2\\(p_1p_2,q)=1}}e\Big(\frac{a'}{q'}p_1^kp_2^k\Big)\Big|^2\ll X^{1-\delta}q^\varepsilon(1+X^{1-\delta}/q')\sum_{m=1}^{q'}\Big|\sum_{p_2\nmid q}e\Big(\frac{m}{q'}p_2^k\Big)\Big|^2$$
$$\ll X^{2-2\delta}q^{2\varepsilon}(1+X^{1-\delta}/q')^2q'.$$

Thus, by (2.2),

$$\sum_{p_1,p_2,r}e(\alpha cp_1^kp_2^kr^k)\ll X^{2-\delta}q^{-1/2}+X^{1-\delta/2}q^{1/2}. \tag{2.3}$$

The proof now divides into two parts according as $q>X$ or $q\leqslant X$. This largely follows Vinogradov. Suppose first that $q>X$. We have

$$\sum_{n,p}a_ne(\alpha np^k) = \sum_{r=1}^{q}\sum_n\sum_{\substack{p\\p^k\equiv r(\bmod q)}}e\Big(n\Big(\frac{a}{q}r+\Big(\alpha-\frac{a}{q}\Big)p^k\Big)\Big). \tag{2.4}$$

Let $\varrho(r)$ be the number of primes p which satisfy $p^k\equiv r(\bmod q)$ and enumerate them as $p_1(r), \ldots, p_{\varrho(r)}(r)$. Then

$$\varrho(r)\ll(X^2q^{-1}+1)q^\varepsilon. \tag{2.5}$$

Let $\varrho = \max \varrho(r)$, define $b_r^{(j)}$ to be 1 if $j \leqslant \varrho(r)$ and 0 otherwise, and for convenience define $p_j(r)$ to be 0 if $j > \varrho(r)$. Then, by (2.4),

$$(2.6)\qquad \sum_{n,p} a_n e(\alpha n p^k) = \sum_{j=1}^{\varrho} \sum_{n} \sum_{r=1}^{q} a_n b_r^{(j)} e\left(n\left(\frac{a}{q}r + \left(\alpha - \frac{a}{q}\right) p_j(r)^k\right)\right).$$

For a fixed j, consider the numbers

$$x_r = \frac{a}{q}r + \left(\alpha - \frac{a}{q}\right) p_j(r)^k \qquad (r = 1, \ldots, q).$$

Modulo one, the numbers ar/q are distinct and spaced $1/q$ apart. Moreover

$$|\alpha - a/q| p_j(r)^k < q^{-1} x^{-2k} (\tfrac{1}{2}X^2)^k \leqslant \tfrac{1}{4} q^{-1}.$$

Thus the x_r are spaced at least $\frac{1}{2}q^{-1}$ apart modulo one. Hence, by Lemma 1,

$$\sum_{n} \sum_{r=1}^{q} a_n b_r^{(j)} e\left(n\left(\frac{a}{q}r + \left(\alpha - \frac{a}{q}\right) p_j(r)^k\right)\right) \ll \left(\sum_{n} |a_n|^2 \sum_{r=1}^{q} |b_r^{(j)}|^2 (X^{2k} + q)\right)^{1/2}$$
$$\ll \Big(\sum_{n} |a_n|^2\Big)^{1/2} (X^k + q^{1/2}) \min(X, q^{1/2}).$$

Hence, by (2.5) and (2.6),

$$\sum_{n,p} a_n e(\alpha n p^k) \ll (X^2 q^{-1} + 1) q^{\varepsilon} \Big(\sum_{n} |a_n|^2\Big)^{1/2} (X^k + q^{1/2}) \min(X, q^{1/2}).$$

If $q > X^2$, then this gives the lemma at once, and if $X < q \leqslant X^2$, then it follows easily from this and (2.3).

Now suppose that $q \leqslant X$. Define $b_r^{(s)}$ to be 0 unless $\frac{1}{4}X^2 < qr + s < \frac{1}{2}X^2$ and $qr + s$ is prime in which case define it to be 1. Then

$$(2.7)\qquad \sum_{n,p} a_n e(\alpha n p^k) = \sum_{s=0}^{q-1} \sum_{n} \sum_{r} a_n b_r^{(s)} e\left(n\left(\frac{a}{q}s^k + \left(\alpha - \frac{a}{q}\right)(qr+s)^k\right)\right).$$

For a fixed s, take

$$x_r = \frac{a}{q}s^k + \left(\alpha - \frac{a}{q}\right)(qr+s)^k.$$

The x_r are all contained in an interval of length at most $\frac{1}{4}q^{-1}$. Thus

$$\|x_r - x_t\| = \left|\alpha - \frac{a}{q}\right| |(qr+s)^k - (qt+s)^k|$$

so that

$$\min_{r \neq t} \|x_r - x_t\| \gg X^{-2k}.$$

Thus, by (2.7) and Lemma 1,

$$\sum_{n,p} a_n e(\alpha n p^k) \ll q \Big(\sum_{n} |a_n|^2\Big)^{1/2} (X^{2+2k} q^{-1})^{1/2}.$$

This with (2.3) gives the desired conclusion.

The next lemma is an extension of Theorem 2 of Davenport and Erdös [6]. When $s > 3$ it is apparently new.

LEMMA 3. *Suppose that $s \geqslant 3$, $c_1, \ldots, c_s$ are non-zero integers,*

$$(2.8)\qquad \theta = 1 - \frac{1}{k}, \qquad \tau_2 = \frac{k^2 - \theta^{s-3}}{k^2 + k - k\theta^{s-3}}, \qquad \tau_3 = \frac{k^2 - k - 1}{k^2 + k - k\theta^{s-3}}$$

and S denotes the number of solutions of

$$\sum_{j=1}^{s} c_j (r_j^k - t_j^k) = 0$$

with

$$X < r_1, t_1 < 2X, \quad X^{\tau_2} < r_2, t_2 < 2X^{\tau_2}, \quad X^{\tau_3\theta^{j-3}} < r_j, t_j < 2X^{\tau_3\theta^{j-3}} \ (3 \leqslant j \leqslant s).$$

Then

$$S \ll X^{1+\tau_2+\tau_3+\tau_3\theta+\ldots+\tau_3\theta^{s-3}+\varepsilon}.$$

Proof. Let S_m denote the number of solutions of

$$\sum_{j=1}^{m} c_j (r_j^k - t_j^k) = 0$$

with $r_m \neq t_m$. Since $S_1 = 0$ we have

$$(2.9)\qquad S = \sum_{m=2}^{s} S_m X^{\tau_3\theta^{m-2}+\ldots+\tau_3\theta^{s-3}} + O(X^{1+\tau_2+\ldots+\tau_3\theta^{s-3}}).$$

Moreover,

$$(2.10)\qquad S_2 \ll X^{2\tau_2+\varepsilon} \ll X^{1+\tau_2}.$$

We write

$$(2.11)\qquad S_m = S_m' + 2S_m'' \quad (m \geqslant 3)$$

where S_m' is the number of solutions with $r_1 = t_1$ and S_m'' the number with $r_1 > t_1$. Then

$$(2.12)\qquad S_m' \ll X T_m$$

where T_m is the number of solutions of

$$(2.13)\qquad \sum_{j=2}^{m} c_j (r_j^k - t_j^k) = 0.$$

If $m = 3$, then at once

$$T_3 \ll X^{\tau_2+\tau_3}. \tag{2.14}$$

If $m > 3$, then since $\tau_3 < \tau_2\theta$, given any set of $t_2, t_3, \ldots, t_m$, the number of choices for $r_2, r_3, \ldots, r_m$ for which (2.13) holds is $\ll 1$. Hence

$$T_m \ll X^{\tau_2+\tau_3+\ldots+\tau_3\theta^{m-3}} \quad (m > 3). \tag{2.15}$$

We now turn to the treatment of S_m''. The number of choices for r_2, t_2 is $\ll X^{2\tau_2}$. For any such choice we have

$$A + c_1(r_1^k - t_1^k) + \sum_{j=3}^{m} c_j(r_j^k - t_j^k) = 0 \tag{2.16}$$

where A is fixed. Let $h = r_1 - t_1$. Then $r_1^k - t_1^k > hX^{k-1}$. Also

$$A + \sum_{j=3}^{m} c_j(r_j^k - t_j^k) \ll X^{k\tau_2}.$$

Hence $0 < h \ll X^{k\tau_2-k+1}$, and (2.16) can be rewritten in the form

$$A + c_1\big((t_1+h)^k - t_1^k\big) \ll X^{k\tau_3}. \tag{2.17}$$

For a given h, let t and $t+j$ be two possible values of t_1 for which (2.17) holds. Then

$$(t+j+h)^k - (t+j)^k - (t+h)^k + t^k \ll X^{k\tau_3},$$

whence $hjX^{k-2} \ll X^{k\tau_3}$. Thus the number of possible choices for t_1 is

$$\ll 1 + X^{k\tau_3-k+2}h^{-1}.$$

For given r_1, t_1, (2.16) becomes

$$A_1 + \sum_{j=3}^{m} c_j(r_j^k - t_j^k) = 0 \tag{2.18}$$

where A_1 is fixed. The number of choices for $t_3, \ldots, t_{m-1}$ is $\ll X^{\tau_3+\ldots+\tau_3\theta^{m-4}}$ and for any such choice the number of choices for $r_3, \ldots, r_{m-1}$ is $\ll 1$.

Given $t_3, \ldots, t_{m-1}, r_3, \ldots, r_{m-1}$, (2.18) becomes

$$A_2 + c_m(r_m^k - t_m^k) = 0$$

and since $r_m \neq t_m$ the number of choices for r_m, t_m is $\ll X^{\varepsilon}$. Thus

$$S_m'' \ll X^{2\tau_2} \sum_{0<h\ll X^{k\tau_2-k+1}} (1 + X^{k\tau_3-k+2}h^{-1}) X^{\tau_3+\ldots+\tau_3\theta^{m-4}+\varepsilon}. \tag{2.19}$$

The lemma now follows from this, (2.9), …, (2.12), (2.14) and (2.15).

For future reference we note that by (2.8),

$$1 + \tau_2 + \tau_3 + \ldots + \tau_3\theta^{s-3} = k\left(1 - \frac{k^3 - 3k^2 + k + 2}{k^3 + k^2 - k^2\theta^{s-3}}\,\theta^{s-3}\right). \tag{2.20}$$

LEMMA 4. *Suppose that* $1 \leqslant r \leqslant k-2$, $0 < \nu < 1$, $\mathcal{U}$ *and* $\mathcal{V}$ *are finite subsets of* $\mathbf{Z}^n$, $f\colon \mathcal{U}\times\mathcal{V} \to [-X^{\nu+k-1}\delta^{-1}, X^{\nu+k-1}\delta^{-1}] \cap \mathbf{Z}$ *and write*

$$r(m, \boldsymbol{v}) = |\{(x, \boldsymbol{u})\colon X < x < 2X, \boldsymbol{u} \in \mathcal{U}, cx^k + f(\boldsymbol{u}, \boldsymbol{v}) = m\}|,$$

$$R(m, \boldsymbol{v}) = |\{\boldsymbol{u}\colon \boldsymbol{u} \in \mathcal{U}, f(\boldsymbol{u}, \boldsymbol{v}) = m\}|,$$

$$S = \sum_m \sum_{\boldsymbol{v}\in\mathcal{V}} r(m, \boldsymbol{v})^2 \quad \textit{and} \quad T = \sum_m \sum_{\boldsymbol{v}\in\mathcal{V}} R(m, \boldsymbol{v})^2.$$

Then

$$S \ll XT + X^{\nu+1-2^{-r}}T + X^{(\nu+1)(1-2^{-r})-r2^{-r}+\varepsilon}T^{1-2^{-r}}|\mathcal{U}|^{2^{1-r}}|\mathcal{V}|^{2^{-r}}$$

where $|\mathcal{U}|, |\mathcal{V}|$ *denote the cardinalities of* $\mathcal{U}$ *and* $\mathcal{V}$ *respectively.*

Proof. This follows that of Theorem 1 of Davenport [4] with one important modification, due to the fact that f may not be one-to-one.

It suffices to prove the result when $|\mathcal{V}| = 1$. For then the more general result follows by summing over all possible $\boldsymbol{v}$ and applying Hölder's inequality to the last expression on the right. We henceforth suppress the $\boldsymbol{v}$. Let

$$\mathcal{H}_j = \{\boldsymbol{h}\colon \boldsymbol{h} = (h_1, \ldots, h_j);\ h_i > 0;\ h_1 \ll X^{\nu};\ h_2, \ldots, h_j \ll X\}$$

and

$$\varrho_j(\boldsymbol{h}, m) = |\{(x, \boldsymbol{u})\colon X < x < 2X;\ \boldsymbol{u} \in \mathcal{U};\ c\Delta_j(x^k, h_1, \ldots, h_j) + f(\boldsymbol{u}) = m\}| \tag{2.21}$$

where Δ_j is the usual jth iterate of the forward difference operator. Now let

$$N_j = \sum_{\boldsymbol{h}\in\mathcal{H}_j} \sum_m R(m)\varrho_j(\boldsymbol{h}, m). \tag{2.22}$$

Then

$$S \ll XT + N_1 \tag{2.23}$$

and, by Cauchy's inequality,

$$N_j^2 \ll X^{\nu+j-1}T \sum_{\boldsymbol{h}} \sum_m \varrho_j(\boldsymbol{h}, m)^2 \ll X^{\nu+j-1}T(X^{\nu+j}T + N_{j+1}).$$

Therefore

$$N_j \ll X^{\nu+j-1/2}T + (X^{\nu+j-1}TN_{j+1})^{1/2}.$$

Hence, by induction on r,

$$N_1 \ll X^{\nu+1-2^{-r}}T + X^{(\nu+1)(1-2^{-r})-r2^{-r}}T^{1-2^{-r}}N_{r+1}^{2^{-r}}. \tag{2.24}$$

By (2.21) and (2.22),

$$N_{r+1} \ll X^{\varepsilon}\Big(\sum_{m} R(m)\Big)^2 = X^{\varepsilon}|\mathscr{U}|^2.$$

This with (2.23) and (2.24) gives the desired conclusion.

As an immediate corollary we have

LEMMA 5. *In addition to the premises of Lemma 4 suppose that*

$$T \ll X^{\varepsilon}|\mathscr{U}||\mathscr{V}|, \quad |\mathscr{U}| \ll X^{\alpha(\nu+k-1)}, \quad \nu \leqslant 2^{-r}$$

and

$$\nu \leqslant \big(r+1-\alpha(k-1)\big)/(2^r-1+\alpha)$$

where $0<\alpha<1$. *Then*

$$S \ll X^{1+\varepsilon}|\mathscr{U}||\mathscr{V}|.$$

The next lemma follows by adapting the proof of Theorem 4 of Davenport [4] in the same way that we adapted the proof of his Theorem 1 to give our Lemma 4.

LEMMA 6. *We assume the hypothesis of Lemma 4 with* $k=9$ *and suppose further that* $r=6$ *or* $r\leqslant 5$ *and* r *is odd, that*

$$r(m,p,\boldsymbol{v}) = |\{(x,\boldsymbol{u})\colon X<x<2X,\ p\nmid x,\ \boldsymbol{u}\in\mathscr{U},\ cx^9+p^9f(\boldsymbol{u},\boldsymbol{v})=m\}|$$

and that

$$S' = \sum_{\substack{p\ll X^{(1-\nu)/9}\\ 9|p+1}}\sum_{m}\sum_{\boldsymbol{v}\in\mathscr{V}} r(m,p,\boldsymbol{v})^2.$$

Then

$$S' \ll X^{\tau}T + X^{\tau+\nu-2^{-r}}T + X^{(\tau+\nu)(1-2^{-r})-r2^{-r}+\varepsilon}T^{1-2^{-r}}|\mathscr{U}|^{2^{1-r}}|\mathscr{V}|^{2^{-r}}$$

where $\tau=(10-\nu)/9$.

LEMMA 7. *In addition to the premises of Lemma 6 we assume that*

$$T \ll X^{\varepsilon}|\mathscr{U}||\mathscr{V}|, \quad |\mathscr{U}| \ll X^{(\nu+8)\alpha}, \quad \nu \leqslant 2^{-r}$$

and

$$\nu \leqslant \frac{9r+10-72\alpha}{9\cdot 2^r+9\alpha-8}$$

where $0<\alpha<1$. *Then*

$$S' \ll X^{(10-\nu)/9+\varepsilon}|\mathscr{U}||\mathscr{V}|.$$

Proof. Immediate by Lemma 6.

3. Definitions. The case $k>9$. Let

$$\theta = 1-\frac{1}{k} \tag{3.1}$$

and

$$s_1 = 3+\left[\frac{\log\left(6k-24+\dfrac{443}{14k}\right)}{-\log\theta}\right]. \tag{3.2}$$

We shall form the variables $x_1,\ldots,x_s$ into four groups, the first two containing s_1 variables each, the third s_2+1 where s_2 is yet to be defined, and the fourth the remainder.

Let

$$\begin{gathered} t=20\ (k=10), \quad t=24\ (k=11), \quad t=27\ (k=12),\\ t=4+\left[\left(\log\left(\frac{k2^{k-2}+k^3-2k^2+2}{k^3+k^2}\right)\right)\Big/(-\log\theta)\right]\ (k\geqslant 13), \end{gathered} \tag{3.3}$$

$$\alpha(m) = 1-\frac{k^3-3k^2+k+2}{k^3+k^2-k^2\theta^{m-3}}\theta^{m-3}, \tag{3.4}$$

$$t_1 = \min(s_1, t), \tag{3.5}$$

$$\alpha_1 = 1-(2^{k-2}+k-1)\left(k+\left(\frac{2^{k-2}\theta^{-1}}{2^{k-2}-1}\right)^{s_1-t_1}\left(\frac{2^{k-2}+k-1}{1-\alpha(t_1)}-k\right)\right)^{-1}, \tag{3.6}$$

$$\alpha_2 = 1-\frac{1}{k}+4(1-\alpha_1), \tag{3.7}$$

$$s_2 = \begin{cases} 1+t+\left[\left(\log\left(\dfrac{k\alpha_2+2^{k-2}-1}{1-\alpha_2}\cdot\dfrac{1-\alpha(t)}{k\alpha(t)+2^{k-2}-1}\right)\right)\Big/\log\left(\dfrac{2^{k-2}\theta^{-1}}{2^{k-2}-1}\right)\right] \\ \qquad\qquad (\alpha_2>\alpha(t)), \\ 4+\left[\left(\log\left(\dfrac{k^3-2k^2-\alpha_2k^2+k+2}{(k^3+k^2)(1-\alpha_2)}\right)\right)\Big/(-\log\theta)\right] \quad (\alpha_2\leqslant\alpha(t)) \end{cases} \tag{3.8}$$

and

$$s_0 = s_0(k) = 2s_1+s_2+1. \tag{3.9}$$

In particular this gives $s_0(10)=107$, $s_0(11)=122$, $s_0(12)=137$, $s_0(13)=153$, $s_0(14)=168$, $s_0(15)=184$, $s_0(16)=200$, $s_0(17)=216$, and we shall show that for $s\geqslant s_0$, (1.2) has a non-trivial solution providing that the congruence condition is satisfied, and this establishes the theorem when $k>9$.

Let

$$\theta_m = \theta \quad (s_1-t_1+4 \leqslant m \leqslant s_1), \tag{3.10}$$

$$\theta_{s_1-t_1+3} = \frac{k^2-k-1}{k^2-\theta^{t_1-3}}, \tag{3.11}$$

$$\theta_{s_1-t_1+2} = \frac{k^2-\theta^{t_1-3}}{k^2+k-\theta^{t_1-3}} \tag{3.12}$$

and define inductively on s_1-t_1+2-i for $s_1-t_1+1 \geqslant i \geqslant 1$

$$\mu_i = \frac{1}{k}\sum_{j=i}^{s_1}\prod_{r=i+1}^{j}\theta_r \tag{3.13}$$

and

$$\theta_i = \frac{\theta 2^{k-2}}{2^{k-2}-1+\mu_i}. \tag{3.14}$$

Now define

$$\lambda_{2i-1} = \lambda_{2i} = \prod_{j=2}^{i}\theta_j \quad (i = 1, \ldots, s_1) \tag{3.15}$$

and

$$\lambda_i = \frac{1}{2}\prod_{j=2}^{i-2s_1}\theta_{j+s_1-s_2} \quad (i = 2s_1+1, \ldots, 2s_1+s_2). \tag{3.16}$$

When k is even we are given that not all the c_j are of the same sign. We can also assume this when k is odd, since we can always replace c_j by $-c_j$ and $-x_j^k$ by $(-x_j)^k$. Then by relabeling we can further suppose that

$$c_1 > 0, \quad c_2 < 0. \tag{3.17}$$

Let P be a large real number and write

$$P_1 = |c_2|^{1/k}P, \quad P_2 = c_1^{1/k}P, \quad P_3 = P^{1-\delta}, \tag{3.18}$$

$$f_{2i-2+j}(\alpha) = \sum_{P_j^{\lambda_{2i}} < x < 2P_j^{\lambda_{2i}}} e(\alpha c_{2i-2+j}x^k) \quad (i = 1, \ldots, s_1;\ j = 1, 2), \tag{3.19}$$

$$F(\alpha) = \prod_{i=1}^{2s_1} f_i(\alpha), \tag{3.20}$$

$$g_i(\alpha) = \sum_{P_3^{\lambda_i} < x < 2P_3^{\lambda_i}} e(\alpha c_i x^k) \quad (i = 2s_1+1, \ldots, 2s_1+s_2), \tag{3.21}$$

$$H(\alpha) = \sum_{\frac{1}{4}P^{1/2} < p < \frac{1}{2}P^{1/2}} \prod_{i=1}^{s_2} g_{2s_1+i}(\alpha p^k), \tag{3.22}$$

$$h(\alpha) = \sum_{p_1, p_2 < P^{\frac{1}{2}(1-\delta)}} \sum_{r < P^{\delta/4k}} e(\alpha c_{s_0} p_1^k p_2^k r^k), \tag{3.23}$$

$$h_i(\alpha) = \sum_{x < P^{1/2}} e(\alpha c_i x^k) \quad (i = s_0+1, \ldots, s), \tag{3.24}$$

$$E(\alpha) = \prod_{i=s_0+1}^{s} h_i(\alpha) \tag{3.25}$$

and

$$\mathcal{N}(P) = \int_0^1 F(\alpha)H(\alpha)h(\alpha)E(\alpha)\,d\alpha. \tag{3.26}$$

Clearly $\mathcal{N}(P)$ is the number of solutions of (1.2) with the variables restricted in various ways. We shall show that the congruence condition implies that $\mathcal{N}(P) \to \infty$ as $P \to \infty$. This will establish the theorem when $k > 9$.

4. Definitions. The case $k = 9$. Let

$$\theta = \tfrac{8}{9}, \quad t = 8, \quad s_1 = 32, \quad s_2 = 26, \quad s_0 = 91, \tag{4.1}$$

$$\theta_m = \theta \quad (28 \leqslant m \leqslant 32), \tag{4.2}$$

$$\theta_{27} = \frac{71}{81-\theta^5}, \tag{4.3}$$

$$\theta_{26} = \frac{81-\theta^5}{90-9\theta^5}, \tag{4.4}$$

and inductively on $26-i$,

$$\mu_i = \frac{1}{9}\sum_{j=i}^{32}\prod_{r=i+1}^{j}\theta_r \quad (1 \leqslant i \leqslant 25) \tag{4.5}$$

and

$$\theta_i = \begin{cases} \dfrac{8}{9} + \dfrac{1}{9}\min\left(\dfrac{1}{32}, \dfrac{55-72\mu_i}{280+9\mu_i}\right) & (i = 25, 24), \\[2ex] \dfrac{8}{9} + \dfrac{1}{9}\min\left(\dfrac{1}{64}, \dfrac{64-72\mu_i}{568+9\mu_i}\right) & (i = 23, \ldots, 20), \\[2ex] \dfrac{8}{9} + \dfrac{1}{9}\min\left(\dfrac{1}{128}, \dfrac{8(1-\mu_i)}{127+\mu_i}\right) & (i = 19, \ldots, 2). \end{cases} \tag{4.6}$$

Let

$$\lambda_{2i-1} = \lambda_{2i} = \prod_{j=2}^{i} \theta_j \ (i \leqslant 32), \qquad \lambda_i = \frac{1}{2} \prod_{j=2}^{i-2s_1} \theta_{j+s_1-s_2} \quad (65 \leqslant i \leqslant 90). \tag{4.7}$$

As in the case $k > 9$ we can assume that (3.17) holds and further let

$$P_1 = |c_2|^{1/9} P, \quad P_2 = c_1^{1/9} P, \quad P_3 = P^{1-\delta} \tag{4.8}$$

and

$$Q_i = P^{\lambda_i - \frac{1}{9}\lambda_{i+1}} \ (i \leqslant 64), \quad Q_i = P_3^{\lambda_i - \frac{1}{9}\lambda_{i+1}} \ (65 \leqslant i \leqslant 90) \tag{4.9}$$

where P is large.

Let

$$f_{2i-2+j}(\alpha) = \sum_{P_j^{\lambda_{2i}} < x < 2P_j^{\lambda_{2i}}} e(\alpha c_{2i-2+j} x^9) \quad (i \leqslant 32, j = 1, 2) \tag{4.10}$$

and

$$F(\alpha) = \Big(\prod_{i=1}^{36} f_i(\alpha)\Big) \sum_{x_{37}} \cdots \sum_{x_{48}} \sum_{p_{19}} \cdots \sum_{p_{24}} e\big(\alpha A(\boldsymbol{x}, \boldsymbol{p})\big) \prod_{i=49}^{64} f_i(\alpha p_{19}^9 \ldots p_{24}^9) \tag{4.11}$$

where

$$A(\boldsymbol{x}, \boldsymbol{p}) = c_{37} x_{37}^9 + c_{38} x_{38}^9 + p_{19}^9 \big(c_{39} x_{39}^9 + c_{40} x_{40}^9 + p_{20}^9 \big(c_{41} x_{41}^9 + \ldots + p_{23}^9 (c_{47}^9 x_{47}^9 + c_{48} x_{48}^9) \ldots\big)\big) \tag{4.12}$$

and the summations are over

$$P_1^{\lambda_{2i-1}} < x_{2i-1} < 2P_1^{\lambda_{2i-1}}, \quad P_2^{\lambda_{2i}} < x_{2i} < 2P_2^{\lambda_{2i}}, \quad p_i < Q_{2i}, \quad p_i \nmid x_{2i} x_{2i-1}, \quad 9 \,|\, p_i + 1. \tag{4.13}$$

Let

$$g_i(\alpha) = \sum_{P_3^{\lambda_i} < x < 2P_3^{\lambda_i}} e(\alpha c_i x^9) \quad (i = 65, \ldots, 90) \tag{4.14}$$

and

$$H(\alpha) = \sum_p \Big(\prod_{i=65}^{76} g_i(\alpha p^9)\Big) \sum_{x_{77}} \cdots \sum_{x_{82}} \sum_{p_{77}} \cdots \sum_{p_{82}} e\big(\alpha p^9 B(\boldsymbol{x}, \boldsymbol{p})\big) \prod_{i=83}^{90} g_i(\alpha p^9 p_{77}^9 \ldots p_{82}^9) \tag{4.15}$$

where

$$B(\boldsymbol{x}, \boldsymbol{p}) = c_{77} x_{77}^9 + p_{77}^9 \big(c_{78} x_{78}^9 + p_{78}^9 (c_{79} x_{79}^9 + \ldots + p_{81}^9 c_{82} x_{82}^9) \ldots\big) \tag{4.16}$$

and the summations are over

$$P_3^{\lambda_i} < x_i < 2P_3^{\lambda_i}, \quad p_i < Q_i, \quad p_i \nmid x_i, \quad 9 \,|\, p_i + 1, \quad \tfrac{1}{4} P^{1/2} < p < \tfrac{1}{2} P^{1/2}. \tag{4.17}$$

Further, let

$$h(\alpha) = \sum_{p_1, p_2 < P^{(1-\delta)/4}} \sum_{r < P^{\delta/36}} e(\alpha c_{91} p_1^9 p_2^9 r^9), \tag{4.18}$$

$$h_i(\alpha) = \sum_{x < P^{1/2}} e(\alpha c_i x^9) \quad (92 \leqslant i \leqslant s), \tag{4.19}$$

$$E(\alpha) = \prod_{i=92}^{s} h_i(\alpha) \tag{4.20}$$

and

$$\mathcal{N}(P) = \int_0^1 F(\alpha) H(\alpha) h(\alpha) E(\alpha) \, d\alpha. \tag{4.21}$$

5. The Farey dissection. Let

$$\mathfrak{M}(q, a) = \Big\{\alpha: \Big|\alpha - \frac{a}{q}\Big| < q^{-1} P^{1/2-k}\Big\} \tag{5.1}$$

denote a typical major arc and write

$$\mathfrak{M} = \bigcup_{q \leqslant P^{1/4}} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}(q, a) \tag{5.2}$$

for their union and

$$\mathfrak{m} = (P^{1/2-k}, 1 + P^{1/2-k}) \setminus \mathfrak{M} \tag{5.3}$$

for their complement, the minor arcs.

Let

$$\eta = \delta^2, \tag{5.4}$$

define

$$\mathfrak{N}(q, a) = \Big\{\alpha: \Big|\alpha - \frac{a}{q}\Big| < q^{-1} P^{k\eta - k}\Big\} \tag{5.5}$$

to be a truncated major arc and

$$\mathfrak{N} = \bigcup_{q \leqslant P^{k\eta}} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{N}(q, a) \tag{5.6}$$

to be their union.

The point of our definitions in §§ 3 and 4 is to make the most effective use of the lemmas of § 2 on the minor arcs, and to keep the variables $x_1, \ldots, x_s$ of (1.2) explicit.

The estimation of the major arcs are, as usual, nothing more than a matter of technique.

We proceed now to examine the minor arcs.

LEMMA 8. *We have*

$$\int_0^1 |F(a)|\,da \ll P^{s-k\mu_1}F(0). \tag{5.7}$$

Proof. We first of all consider the case $k>9$. By Schwarz's inequality, (3.19), (3.20) and Parseval's identity the square of the integral on the left of (5.7) is majorized by the product of the two expressions

$$\sum_m r_j(m)^2 \quad (j=1,2) \tag{5.8}$$

where

$$r_j(m) = \Big|\Big\{(x_1, \ldots, x_{s_1}):\ \sum_{r=1}^{s_1} c_{2r-2+j}x_r^k = m,\ P_j^{\lambda_{2i}} < x_i < 2P_j^{\lambda_{2i}}\Big\}\Big|.$$

By first of all invoking Lemma 3 with $s=t_1$ and then successively applying Lemma 5 with $r=k-2$ we find, providing that

$$k-1 < k\theta_i \leqslant k-1+2^{2-k} \quad (2\leqslant i\leqslant s_1-t_1), \tag{5.9}$$

that

$$\sum_m r_j(m)^2 \ll P^{\lambda_2+\lambda_4+\ldots+\lambda_{2s_1}+s} \ll P^{s-k\mu_1}F(0).$$

This gives the desired conclusion when $k>9$ on establishing (5.9). For $k=10$, 11 and 12 it can be checked by direct calculation. For $k>12$ we observe that

$$\min\Big(k2^{k-2}+k^3-2k^2+2,\ (k^3+k^2)\Big(\frac{k-1}{k}\Big)\Big(6k-24+\frac{443}{14k}\Big)\Big)$$
$$> k^4-4k^3+5k^2+k-2+2^{2-k}(k^3-3k^2+k+2).$$

Thus, by (3.2), (3.3) and (3.5),

$$(k^3+k^2)\theta^{3-t_1} > k^4-4k^3+5k^2+k-2+2^{2-k}(k^3-3k^2+k+2).$$

Therefore

$$(k-1+2^{2-k})(k^3-3k^2+k+2)\theta^{t_1-3} < k^3+k^2-k^2\theta^{t_1-3}.$$

Hence, by (2.8), (2.20), (3.10), …, (3.13),

$$0 < 1-\mu_{s_1-t_1+1} = \frac{k^3-3k^2+k+2}{k^3+k^2-k^2\theta^{t_1-3}}\theta^{t_1-3} < \frac{1}{k-1+2^{2-k}}$$

and this with (3.14) gives (5.9) for $i=s_1-t_1+1$. Since μ_i is a decreasing function of i it follows at once that (5.9) also holds when $i\leqslant s_1-t_1$.

The proof in the case $k=9$ is similar. We first of all observe that by (4.11) and (4.12) the integral in question is bounded by

$$\int \sum_{p_{19}}\cdots\sum_{p_{24}} |F_1F_2|\Big(\prod_{i\,\mathrm{odd}}|f_i|\Big)\Big(\prod_{i\,\mathrm{even}}|f_i|\Big)$$

where F_1 contains the x_i with i odd and F_2 the ones with i even.

By the Cauchy–Schwarz inequality this is bounded by the square root of the product of the two expressions

$$\int \sum_{p_{19}}\cdots\sum_{p_{24}} |F_j|^2 \prod_i |f_{2i-2+j}|^2 \quad (j=1,2)$$

and by Parseval's identity this is

$$\sum_{p_{19}}\cdots\sum_{p_{24}}\sum_m r_1(m, p\ \ , \ldots, p_{24})^2$$

with

$$r_1(m, p_{19}, \ldots, p_{24})$$
$$= \big|\{\boldsymbol{x}:\ c_1x_1^9+c_3x_3^9+\ldots+c_{37}x_{37}^9+p_{19}^9(c_{39}x_{39}^9+\ldots+p_{24}^9(c_{49}x_{49}^9+\ldots+c_{63}x_{63}^9)\ldots)$$
$$= m,\ P_1^{\lambda_{2i-1}} < x_{2i-1} < 2P_1^{\lambda_{2i-1}},\ p_j \nmid x_{2j-1}\}\big|$$

with a similar expression for r_2. We first of all use Lemma 3 with $s=8$, then apply Lemma 7 twice with $r=5$ and four times with $r=6$. Finally we apply Lemma 5 successively eighteen times with $r=7$. The choice of the parameters in (4.2,) …, (4.6) ensures that the hypotheses of the lemmas are satisfied.

Thus we have

$$\sum_{p_{19}}\cdots\sum_{p_{24}}\sum_m r_j(m, p_{19}, \ldots, p_{24})^2 \ll Q_{38}Q_{40}\ldots Q_{48}P^{\lambda_1+\lambda_3+\ldots+\lambda_{63}+s}$$
$$\ll P^{2s-9\mu_1}F(0).$$

This completes the proof of the lemma.

LEMMA 9. *We have* $\mu_1 = a_1$ $(k>9)$, $\mu_7 > 0.96149$ $(k=9)$ *and* $\mu_1 > 0.98185$ $(k=9)$.

Proof. Consider first $k>9$. By (3.1), (3.10), …, (3.13),

$$k\mu_{s_1-t_1+1} = 1+\frac{k^2-\theta^{t_1-3}}{k^2+k-k\theta^{t_1-3}}+\frac{k^2-k-1}{k^2+k-k\theta^{t_1-3}}k(1-\theta^{t_1-2})$$

and thus, by (3.4),

$$\mu_{s_1-t_1+1} = \alpha(t_1).$$

If $s_1 \leqslant t$, then by (3.5) and (3.6) we have $\mu_1 = \alpha_1$. Thus we can suppose that $s_1 > t = t_1$. By (3.13) and (3.14), for $i \leqslant s_1 - t_1$,

$$\mu_i = \frac{1}{k} + \theta_{i+1}\mu_{i+1} = \frac{1}{k} + \left(1 - \frac{1}{k}\right)\frac{2^{k-2}\mu_{i+1}}{2^{k-2}-1+\mu_{i+1}}.$$

On rearrangement this becomes

$$\frac{2^{k-2}-1+k\mu_i}{1-\mu_i} = \frac{2^{k-2}\theta^{-1}}{2^{k-2}-1}\cdot\frac{2^{k-2}-1+k\mu_{i+1}}{1-\mu_{i+1}}.$$

Hence, by (3.5) and (5.10),

$$\frac{2^{k-2}-1+k\mu_1}{1-\mu_1} = \left(\frac{2^{k-2}\theta^{-1}}{2^{k-2}-1}\right)^{s_1-t}\frac{2^{k-2}-1+k\alpha(t)}{1-\alpha(t)}.$$

On rearrangement and comparison with (3.6) we obtain $\mu_1 = \alpha_1$ once more.

Now consider $k = 9$. By (4.1), …, (4.5)

$$\mu_{25} = \frac{14444893}{22587741}.$$

By (4.5),

$$\mu_i = 1/9 + \theta_{i+1}\mu_{i+1}.$$

We now use this with (4.6) to successively calculate the value of μ_i. We find that

$$\mu_{16} > 0.88136.$$

Then, since $1024(1-\mu_i) < 127 + \mu_i$ for $i \leqslant 16$ we can use the formula

$$\frac{1-\mu_i}{127+9\mu_i} = \left(\frac{127}{144}\right)^{16-i}\frac{1-\mu_{16}}{127+9\mu_{16}}$$

to give the desired lower bounds for μ_7 and μ_1.

LEMMA 10. *Let* $\alpha \in \mathfrak{m}$. *Then*

$$H(\alpha)h(\alpha) \ll P^{-2\delta-k+k\mu_1}H(0)h(0).$$

Proof. Suppose first of all that $k > 9$. Choose a, q so that $(a, q) = 1$, $|\alpha - a/q| \leqslant q^{-} P^{-k/2}$ and $q \leqslant P^{k/2}$. Then, by (5.1), (5.2) and (5.3), whenever $q \leqslant P^{1/4}$ we have

$$|\alpha - a/q| \geqslant q^{-1}P^{-k+1/2}.$$

Hence, by (3.16), (3.18), (3.21), (3.22) and (3.23), the hypothesis of Lemma 2 is satisfied with $X = P^{1/4}$ and $S(\alpha) = H(\alpha)h(\alpha)$. Thus

$$H(\alpha)h(\alpha) \ll P^{3/4+k/4}\Big(\sum_m r(m)^2\Big)^{1/2} \tag{5.11}$$

where

$$r(m) = \Big|\Big\{\boldsymbol{x}\colon \sum_{i=2s_1+1}^{2s_1+s_2} c_i x_i^k = m,\ P_3^{\lambda_i} < x_i < 2P_3^{\lambda_i}\Big\}\Big|.$$

The sum $\sum r(m)^2$ is estimated in the same way as the analogous sums arising in the proof of Lemma 8. Thus

$$\sum_m r(m)^2 \ll P_3^{\lambda_{2s_1+1}+\ldots+\lambda_{2s_1+s_2}+\varepsilon},$$

whence, by (3.8), (3.18) and the same argument as in the first part of the proof of Lemma 9,

$$\sum_m r(m)^2 \ll P^{-6\delta-\frac{1}{2}k\alpha_2}P_3^{2\lambda_{2s_1+1}+\ldots+2\lambda_{2s_1+s_2}}.$$

Hence, by (3.21), (3.22), (3.23) and (5.11),

$$H(\alpha)h(\alpha) \ll P^{-2\delta-\frac{1}{4}+\frac{1}{4}k(1-\alpha_2)}H(0)h(0).$$

The desired result then follows from (3.7) and Lemma 9.

In the case $k = 9$ we follow the same argument to begin with. This gives (5.11) with

$$r(m) = \sum_{p_{77}<Q_{77}} \cdots \sum_{p_{82}<Q_{82}} r(m, p_{77}, \ldots, p_{82})$$

where

$$\begin{aligned} &r(m, p_{77}, \ldots, p_{82}) \\ &= \Big|\Big\{\boldsymbol{x}\colon\ c_{65}x_{65}^9 + \ldots + c_{77}x_{77}^9 + p_{77}^9\big(c_{78}x_{78}^9 + \ldots + p_{82}^9(c_{83}x_{83}^9 + \ldots + c_{90}x_{90}^9)\ldots\big), \\ &\qquad P_3^{\lambda_i} < x_i < 2P_3^{\lambda_i},\ p_j \nmid x_j,\ 9 \mid p_j+1\Big\}\Big|. \end{aligned}$$

Hence, by Cauchy's inequality,

$$\sum_m r(m)^2 \ll Q_{77}\ldots Q_{82}\sum_{p_{77}}\cdots\sum_{p_{82}}\sum_m r(m, p_{77}, \ldots, p_{82})^2.$$

We now follow the argument of the case $k = 9$ of Lemma 8. This gives

$$\sum_{p_{77}}\cdots\sum_{p_{82}}\sum_m r(m, p_{77}, \ldots, p_{82})^2 \ll Q_{77}\ldots Q_{82}P^{\lambda_{65}+\ldots+\lambda_{90}+\varepsilon}.$$

Thus, by (4.5) and (4.7),

$$\sum_m r(m)^2 \ll P_3^{\varepsilon-\frac{9}{2}\mu_7}Q_{77}^2\ldots Q_{82}^2P_3^{2\lambda_{65}+\ldots+2\lambda_{90}}.$$

Hence, by (5.11), (4.14), (4.15) and (4.18),

$$H(a)h(a) \ll P^{2+\delta}P_3^{-\frac{9}{4}\mu_7}H(0)h(0).$$

The proof is now completed by appealing to Lemma 9.

Combining Lemmas 8 and 10 establishes

$$\int_{\mathfrak{m}} F(a)H(a)h(a)E(a)\,da \ll P^{-k-\delta}F(0)H(0)h(0)E(0), \tag{5.12}$$

and concludes the discussion of the minor arcs.

6. The truncation of the major arcs. Let

$$S_i(q, a) = \sum_{r=1}^{q} e(ac_i r^k/q), \tag{6.1}$$

$$W_i(\beta) = \sum_{P_j^{k\lambda_i}<x<P_j^{k\lambda_i}} \frac{1}{k}x^{1/k-1}e(\beta c_i x) \quad (2\,|\,i-j,\ i\leqslant 2s_1,\ j=1,2), \tag{6.2}$$

$$f_i^*(a, q, a) = q^{-1}S_i(q, a)W_i\left(a-\frac{a}{q}\right) \tag{6.3}$$

and

$$f_i^*(a) = \begin{cases} f_i^*(a, q, a) & (a \in \mathfrak{M}(q, a)),\\ 0 & (a \notin \mathfrak{M}).\end{cases} \tag{6.4}$$

We note that by (3.2) and (4.1), $s_1 > k$.

LEMMA 11. *We have*

$$\int_{\mathfrak{M}} \left|\left(\prod_{i=1}^{2k+2} f_i(a)\right) - \left(\prod_{i=1}^{2k+2} f_i^*(a)\right)\right| da \ll P^{-k-\delta}\prod_{i=1}^{2k+2} f_i(0).$$

Proof. Let $a \in \mathfrak{M}(q, a)$. Since $\lambda_i \geqslant \theta^k\left(1-\frac{1}{k(k-1)}\right) > \frac{1}{4}$ $(i \leqslant 2k+2)$ it follows from Lemma 8 of Davenport [3] that

$$f_i(a) - f_i^*(a) \ll q^{3/4+\varepsilon}. \tag{6.5}$$

By Lemma 3 of Hardy and Littlewood [8],

$$S_i(q, a) \ll q^{1-1/k}, \quad (a, q) = 1. \tag{6.6}$$

Thus, by (6.2), ..., (6.5), for $q \leqslant P^{1/4}$,

$$f_i(a), f_i^*(a) \ll q^{-1/k}P^{\lambda_i} \quad (i \leqslant 2k+2). \tag{6.7}$$

By Lemma 9 of Davenport [3],

$$f_i(a), f_i^*(a) \ll q^{-1/k}P\left(1+P^k\left|a-\frac{a}{q}\right|\right)^{-1} \quad (i = 1, 2). \tag{6.8}$$

Thus, by (6.5) and (6.7),

$$\left(\prod_{i=1}^{2k+2} f_i(a)\right) - \left(\prod_{i=1}^{2k+2} f_i^*(a)\right) \ll q^{\varepsilon-\frac{5}{4}-\frac{1}{k}}\left(1+P^k\left|a-\frac{a}{q}\right|\right)^{-1}\prod_{i=1}^{2k+1} P^{\lambda_i}.$$

The lemma follows easily from this.

LEMMA 12. *We have*

$$\int_{\mathfrak{M}\setminus\mathfrak{N}} \prod_{i=1}^{2k+2} |f_i^*(a)|\,da \ll P^{-k-\eta}\prod_{i=1}^{2k+2} f_i(0).$$

Proof. Let $a \in \mathfrak{M}\setminus\mathfrak{N}$. Then, by (5.2) and (5.6), there exist a, q such that $|a-a/q| < q^{-1}P^{-k+1/2}$, $(a, q) = 1$ and $1 \leqslant a \leqslant q \leqslant P^{1/4}$, and moreover such that if $q \leqslant P^{k\eta}$, then $|a-a/q| \geqslant q^{-1}P^{k\eta-k}$. By (6.7) and (6.8),

$$\prod_{i=1}^{2k+2} |f_i^*(a)| \ll q^{-2-2/k}\left(1+P^k\left|a-\frac{a}{q}\right|\right)^{-2}\prod_{i=1}^{2k+2} f_i(0).$$

Thus the integral in question is

$$\ll \Big(\sum_{q\leqslant P^{k\eta}} q^{-1-2/k}P^{-2k}\int_{q^{-1}P^{k\eta-k}}^{\infty} \beta^{-2}\,d\beta + \sum_{q>P^{k\eta}} q^{-1-2/k}P^{-k}\Big)\prod_{i=1}^{2k+2} f_i(0)$$

and this gives the lemma.

By Lemmas 11 and 12 we see that

$$\int_{\mathfrak{M}\setminus\mathfrak{N}} F(a)H(a)h(a)E(a)\,da \ll P^{-k-\eta}F(0)H(0)h(0)E(0). \tag{6.9}$$

7. The truncated major arcs. Let

$$\mathfrak{S}_1(n, X) = \sum_{q\leqslant X} q^{-2k-2}\sum_{\substack{a=1\\(a,q)=1}}^{q} e(an/q)\prod_{i=1}^{2k+2} S_i(q, a). \tag{7.1}$$

LEMMA 13. *Suppose that* $|n| < P^{k-\delta}$. *Then*

$$\int_{\mathfrak{N}} e(an)\prod_{i=1}^{2k+2} f_i(a)\,da = \left(J(P)\mathfrak{S}_1(n, P^{k\eta}) + O(P^{-\eta})\right)P^{-k}\prod_{i=1}^{2k+2} f_i(0)$$

where $J(P) \asymp 1$.

Proof. By Lemma 11 it suffices to prove the result with each f_i in the integrand replaced by f_i^*. Let $\alpha \in \mathfrak{N}(q, a)$. By (5.5), (6.7) and (6.8),

$$\int_{\mathfrak{N}(q,a)} e(\alpha n) \prod_{i=1}^{2k+2} f_i^*(\alpha)\,d\alpha = \int_{\mathfrak{N}(q,a)} e\left(\frac{a}{q}n\right) \prod_{i=1}^{2k+2} f_i^*(\alpha)\,d\alpha + O\left(q^{-3-2/k} P^{k\eta-\delta-k} \prod_{i=1}^{2k+2} f_i(0)\right)$$

$$= \int_{\frac{a}{q}-\frac{1}{2}}^{\frac{a}{q}+\frac{1}{2}} e\left(\frac{a}{q}n\right) \prod_{i=1}^{2k+2} f_i^*(\alpha, q, a)\,d\alpha + O\left(q^{-1-2/k} P^{-k-k\eta} \prod_{i=1}^{2k+2} f_i(0)\right).$$

Thus, by (6.3) and on summing over all the $\mathfrak{N}(q, a)$, we have

$$\int_{\mathfrak{N}} e(\alpha n) \prod_{i=1}^{2k+2} f_i(\alpha)\,d\alpha = J_1(P)\mathfrak{S}_1(n, P^{k\eta}) + O\left(P^{-k-\eta} \prod_{i=1}^{2k+2} f_i(0)\right). \tag{7.2}$$

By (6.2),

$$J_1(P) = \sum_{x_1} \dots \sum_{x_{2k+2}} k^{-2k-2} (x_1 \dots x_{2k+2})^{1/k-1}$$

where the summations are over

$$P_j^{k\lambda_{2i}} < x_{2i-2+j} < 2^k P_j^{k\lambda_{2i}}$$

subject to $c_1 x_1 + \dots + c_{2k+2} x_{2k+2} = 0$. It now follows from (3.17), (3.18) and (4.8) that

$$J_1(P) \asymp P^{-k} \prod_{i=1}^{2k+2} f_i(0).$$

This with (7.2) gives the lemma.

LEMMA 14. *We have*

$$\int_{\mathfrak{N}} F(\alpha)H(\alpha)h(\alpha)E(\alpha)\,d\alpha = \big(J(P)\mathfrak{S} + O(P^{-\eta})\big) P^{-k} F(0)H(0)h(0)E(0) \tag{7.3}$$

where

$$\mathfrak{S} = \sum_{q=1}^{\infty} q^{-2s} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \prod_{i=1}^{s} S_i(q, a). \tag{7.4}$$

Proof. We consider the case $k = 9$. The case $k > 9$ is similar but simpler. By Lemma 13 the integral on the left of (7.3) is

$$J(P)P^{-9}\left(\sum_{i=1}^{20} f_i(0)\right) \sum_{\boldsymbol{x}} \sum_{\boldsymbol{p}} \mathfrak{S}_1\big(D(\boldsymbol{x}, \boldsymbol{p}), P^{9\eta}\big) + O\big(P^{-9-\eta} F(0)H(0)h(0)E(0)\big) \tag{7.5}$$

where

$$D(\boldsymbol{x}, \boldsymbol{p}) = c_{21}x_{21}^9 + \dots + c_{36}x_{36}^9 + A(\boldsymbol{x}', \boldsymbol{p}') + p_{19}^9 \dots p_{24}^9 (c_{49}x_{49}^9 + \dots + c_{64}x_{64}^9) + $$
$$+ p^9\big(c_{65}x_{65}^9 + \dots + c_{76}x_{76}^9 + B(\boldsymbol{x}'', \boldsymbol{p}'') + p_{77}^9 \dots p_{82}^9 (c_{83}x_{83}^9 + \dots + c_{90}x_{90}^9)\big) +$$
$$+ c_{91}p_1^9 p_2^9 x_{91}^9 + c_{92}x_{92}^9 + \dots + c_s x_s^9,$$

A and B are given by (4.12) and (4.16), and the summations are over

$$P_1^{\lambda_{2i-1}} < x_{2i-1} < 2P_1^{\lambda_{2i-1}}, \quad P_2^{\lambda_{2i}} < x_{2i} < 2P_2^{\lambda_{2i}} \quad (i \leqslant 32),$$
$$p_i < Q_i, \quad p_i \nmid x_{2i}x_{2i-1}, \quad 9 \mid p_i + 1 \quad (19 \leqslant i \leqslant 24),$$
$$P_3^{\lambda_i} < x_i < 2P_3^{\lambda_i} \quad (65 \leqslant i \leqslant 90),$$
$$p_i < Q_i, \quad p_i \nmid x_i, \quad 9 \mid p_i + 1 \quad (77 \leqslant i \leqslant 82),$$
$$\tfrac{1}{4}P^{1/2} < p < \tfrac{1}{2}P^{1/2},$$
$$p_1, p_2 < P^{(1-\delta)/4},$$
$$x_{91} < P^{\delta/36}$$

and

$$x_i < P^{1/2} \quad (i \geqslant 92).$$

By (7.1), the multiple sum in (7.5) can be written in the form

$$\sum_{q \leqslant P^{9\eta}} q^{-20} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{\boldsymbol{p}} \left(\prod_{i=1}^{20} S_i(q, a)\right) \sum_{\boldsymbol{x}} e\big(aD(\boldsymbol{x}, \boldsymbol{p})/q\big).$$

For each variable x_i we can replace the sum over the x_i by an expression of the form

$$P_j^{\lambda_i} q^{-1} S_i(q, am_i^9) + O(q)$$

(λ_i is $1/2$ when $i \geqslant 92$ and $\delta/36$ when $i = 91$, and P_j is P when $i \geqslant 91$) unless $37 \leqslant i \leqslant 48$ or $77 \leqslant i \leqslant 82$ in which case we obtain an expression of the form

$$P_j^{\lambda_i} q^{-1}\left(S_i(q, am_i^9) - \frac{1}{p_i^*} S_i\big(q, a(m_i p_i^*)^9\big)\right) + O(q).$$

The number m_i is either 1 or a product of prime numbers. In view of (6.6) and (5.4) the contribution from the O terms can be accomodated in the error term in (7.5). Since the number of different prime divisors of q is $O(q^\varepsilon)$ we can, for each q in the range, neglect those $\boldsymbol{p}$ for which at least one element divides q. But then for the $\boldsymbol{p}$ that remain we can replace all the m_i and $m_i p_i^*$ by one. We can then add back those $\boldsymbol{p}$ we neglected, the total error in doing so being easily accomodated in the error term of (7.5).

Now we observe that

$$P_j^{\lambda_i} = \sum_{P_j^{\lambda_i} < x_i < 2P_j^{\lambda_i}} 1 + O(1) = \sum_{x_i < P_j^{\lambda_i}} 1 + O(1)$$

and

$$P_j^{\lambda_i} \frac{p_i^* - 1}{p_i^*} = \sum_{\substack{P_j^{\lambda_i} < x_i < 2P_j^{\lambda_i} \\ p_i^* \nmid x_i}} 1 + O(1).$$

The O terms are again easily accomodated in the error term in (7.5) and the resulting main term is

$$J(P)P^{-9}F(0)H(0)h(0)E(0) \sum_{q \leqslant P^{9\eta}} q^{-s} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \prod_{i=1}^{s} S_i(q, a).$$

By (6.6) we can complete the summation to infinity with a negligible error term, and this gives the lemma.

8. Completion of the proof. By (3.26), (4.21), (5.3) and (5.12),

$$\mathcal{N}(P) = \int_{\mathfrak{M}} F(a)H(a)h(a)E(a)\,da + O\big(P^{-k-\delta}F(0)H(0)h(0)E(0)\big)$$

and by (6.9) and Lemma 14

$$\int_{\mathfrak{M}} F(a)H(a)h(a)E(a)\,da = \big(J(P)\mathfrak{S} + O(P^{-\eta})\big)P^{-k}F(0)H(0)h(0)E(0).$$

The hypothesis that the congruence condition is satisfied ensures that by standard arguments $\mathfrak{S} \gg 1$. Thus $\mathcal{N}(P) \to \infty$ as $P \to \infty$ which proves the theorem.

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Received on 2. 7. 1975 (738)