Homogeneous additive equations and Waring's problem

by

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1. Introduction. Let $k \geq 3$ be a natural number. Davenport and Lewis [7] define $I^s(k)$ and $G^s(k)$ as follows. If $c_1, \ldots, c_s$ are integers such that for every prime power $p^m$ the congruence

$$c_1x_1^k + c_2x_2^k + \ldots + c_sx_s^k \equiv 0 \pmod{p^m} \quad (1.1)$$

has a solution with $x_1, \ldots, x_s$ not all divisible by $p$, then $c_1, \ldots, c_s$ are said to satisfy the congruence condition. The number $I^s(k)$ is the least number $s$ such that every set of $s$ integers $c_1, \ldots, c_s$ satisfies the congruence condition, and $G^s(k)$ is the least number such that if $s \geq G^s(k)$ and $c_1, \ldots, c_s$ are any $s$ integers, not all the same sign when $k$ is even, which satisfy the congruence condition, then the equation

$$c_1x_1^k + c_2x_2^k + \ldots + c_sx_s^k = 0 \quad (1.2)$$

has a solution in integers $x_1, \ldots, x_s$, not all of which are zero.

The major part of their paper is devoted to showing that

$$I^s(k) \leq k^2 + 1, \quad (1.3)$$

with equality whenever $k+1$ is a prime. However, their Theorem 2 implies that when $k \geq 18$

$$G^s(k) \leq k^2 + 1. \quad (1.4)$$

They also indicate that the methods of Davenport ([3], [5]) will give this when $k \leq 6$, and observe that it seems doubtful whether the solubility of (1.2) for $s \geq k^2 + 1$ can be proved for all the intermediate values $k = 7, \ldots, 17$ by existing methods. Our purpose is to reduce the gap.

**Theorem.** We have $G^s(9) \leq 91$, $G^s(10) \leq 107$, $G^s(11) \leq 122$, $G^s(12) \leq 137$, $G^s(13) \leq 153$, $G^s(14) \leq 168$, $G^s(15) \leq 184$, $G^s(16) \leq 200$, $G^s(17) \leq 216$.

**Corollary.** When $11 \leq k \leq 17$ we have (1.4).

As far as $k = 7, 8$ are concerned, the method of Davenport [5] when adapted to this problem is still the most effective and gives $G^s(7) \leq 53$ and $G^s(8) \leq 73$. 
The argument used here is an adaptation of one of Vinogradov [10], Chapter IV, related to the estimation of $G(k)$ in Waring's problem (see also Chen [1]). For large $k$ it gives

$$\limsup_{k \to \infty} \frac{G^*(k)}{\log k} \leq 3$$

and by comparison the method of Davenport and Lewis gives this with the $3$ replaced by $4$. By adapting another method of Vinogradov [11] it is possible to show that

$$\limsup_{k \to \infty} \frac{G^*(k)}{\log k} \leq 2.$$  

Although there is quite a wide range of choice of the parameters involved in the proof of (1.6), it appears that the argument used here is always more effective when $k$ is less than about 50,000.

We observe that when applied to Waring's problem our method gives the above theorem with $G^*$ replaced by $G$. In particular this improves on the known bounds for $G(9)$ and $G(10)$ due to Cook [2].

We further note that to prove the theorem it suffices to assume that $a_1, \ldots, a_r$ are all non-zero.

Throughout, $\delta$ is a fixed but sufficiently small positive real number in terms of $c, c_1, \ldots, c_r$ and $k$, where $c, c_1, \ldots, c_r$ are non-zero integers and $k$ is a natural number with $k \geq 0$. Formulae containing $\epsilon$ hold for every sufficiently small positive $\epsilon$ and the implied constants in the $\asymp$, $O$, $\ll$ and $\gg$ symbols depend at most on $c, c_1, \ldots, c_r$, $\epsilon$ and $\delta$.

2. Preliminary lemmas

**Lemma 1.** Let $a_n (n = M + 1, \ldots, M + N)$ and $b_r (r = 1, \ldots, R)$ be complex numbers, suppose that the $\pi_r$ $(r = 1, \ldots, R)$ are real numbers which are distinct modulo one, and define

$$\delta = \min|a_n - x_r|$$

where the minimum is taken over all pairs $r, t$ with $r \neq t$, and where $\|u\|$ denotes the distance of $u$ from the nearest integer. Then

$$\sum_{n = M + 1}^{M + N} a_n b_r e(n \pi_r) \ll \sum_{n = M + 1}^{M + N} |a_n|^2 \sum_{r = 1}^{R} |b_r|^2 (N + \delta^{-1})^{1/2}.$$  

**Proof.** At once from Cauchy's inequality and Theorem 1 of Montgomery and Vaughan [9].

**Lemma 2.** Suppose that $c \neq 0$, $X$ is a real number with $X > 1$, $|a - q| \leq q^{-1}X^{-1/2}$, $a_n ([n] < X^{2k})$ are complex numbers, $|a - q| \geq q^{-1}X^{1/4}$ when $q < X$, and

$$S(a) = \sum_{n \equiv 0 \mod m} a_n e(\alpha p_n f + \beta p_n^m)$$

where the summations are over $|n| < X^{2k}$, $\frac{1}{2} < \frac{1}{r} < X$, $r < X^{0k}$ and $p_1, p_2 < X^{-1/2}$. Then

$$S(a) \ll Y^{2k} \left( \sum_{|n| < X^{2k}} |a_n|^2 \right)^{1/2}.$$  

This is essentially Lemma 2 of Vinogradov [10], Chapter IV, but the use of the factor

$$\sum_{p_1, p_2 \leq r} e(\alpha p_1 f + \beta p_2 g)$$

is new. The purpose of the apparently superfluous variable $r$ is to ensure that when the variable $x$, where $x$ is to be defined, of (1.2) appears in the singular series it is summed over a complete set of residues, rather than a reduced set. This is of paramount importance, for otherwise the congruence condition cannot be met.

**Proof.** We first of all treat (2.1). Since $|a - q| \leq q^{-1}X^{-1/2}$ and the number of different prime divisors of $q$ is $O(q^{1/3})$ we have

$$\sum_{p_1, p_2 \leq r} e(\alpha p_1 f + \beta p_2 g) = \sum_{p_1, p_2 \leq r} e\left( \frac{q}{\varphi}\left( \frac{1}{p_1 f} + \frac{1}{p_2 g} \right) \right) = O(X^{-\frac{1}{2}}X^{-\frac{1}{2}} + q^{1/2}X^{-1/2}).$$

For a given $r$, let $a' = a \varphi(r)/q$, $q' = q/(\varphi(r))$ so that $(a', q') = 1$ and $q' \geq q', X^{-1/2}$. If $(b, q') = 1$, then the number of solutions of $n^b = b$ (mod $q'$) with $1 \leq n \leq q'$ is $O(q')$. Hence, by Cauchy's inequality,

$$\sum_{p_1, p_2 \leq r} e\left( \frac{a'}{q} p_1 f + \frac{\beta}{q} p_2 g \right) \ll X^{1/4} q' \sum_{n = 1}^{q'} \sum_{p_1, p_2 \leq r} \left( \frac{\beta}{q} p_2 g \right)^2 \ll X^{1/4} q' (1 + X^{-1/2}/q') \ll X^{1/2} q' (1 + X^{-1/4}) q'^{1/2}.$$  

Thus, by (2.2),

$$\sum_{p_1, p_2 \leq r} e(\alpha p_1 f + \beta p_2 g) \ll X^{-1/2} q'^{1/2} + X^{-1/4} q'^{1/2}.$$  

The proof now divides into two parts according as $q > X$ or $q < X$. This largely follows Vinogradov. Suppose first that $q > X$. We have

$$\sum_{n, p} a_n e(\alpha n p^k) = \sum_{n = 1}^{q} \sum_{p \equiv 0 \mod q} e\left( \frac{\alpha n p^k}{q} \right) = \sum_{p \equiv 0 \mod q} e\left( \frac{\alpha p^k}{q} \right).$$

Let $\varphi(r)$ be the number of primes $p$ which satisfy $p^k = r (\text{mod } q)$ and enumerate them as $p_1(r), \ldots, p_{\varphi(r)}(r)$. Then

$$\varphi(r) \ll (X^2 q^{-1} + 1) q'^{1/2}.$$
Let \( q = \max \rho(r) \), define \( b^{(j)} \) to be 1 if \( j \leq \rho(r) \) and 0 otherwise, and for convenience define \( p_j(r) \) to be 0 if \( j > \rho(r) \). Then, by (2.4),

\[
(2.6) \quad \sum_{n,p} a_n e(ans) = \sum_{r=1}^q \sum_{n=1}^q \sum_{r=1}^q a_n b^{(j)} e \left( n \left( \frac{a}{q} r + \left( a - \frac{a}{q} \right) p_j(r) s \right) \right).
\]

For a fixed \( j \), consider the numbers

\[
\alpha_r = \frac{a}{q} r + \left( a - \frac{a}{q} \right) p_j(r) s \quad (r = 1, \ldots, q).
\]

Modulo one, the numbers \( \alpha_r \) are distinct and spaced \( 1/q \) apart. Moreover

\[
|\alpha - \alpha| p_j(r) s < q^{-1} \theta^{-2k} \left( \frac{q}{X} \right)^2 q^{-2} \leq \frac{1}{q} q^{-1}.
\]

Thus the \( \alpha_r \) are spaced at least \( \frac{1}{q} q^{-1} \) apart modulo one. Hence, by Lemma 1,

\[
(2.5) \quad \sum_{n,p} a_n e(ans) = \left( \sum_{n=1}^q |a_n|^2 \sum_{r=1}^q b^{(j)} e \left( \left( \frac{q}{X} \right)^2 q^{-2} \left( \frac{q}{X} \right)^2 q^{-2} \min(X, q^{1/2}) \right) \right)^{1/2} \leq \left( \sum_{n=1}^q |a_n|^2 \left( X^2 q^{-1} + q^{-2} \right) \min(X, q^{1/2}) \right)^{1/2}.
\]

Hence, by (2.5) and (2.6),

\[
\sum_{n,p} a_n e(ans) = \left( X^3 q^{-1} + 1 \right) q^2 \left( \sum_{n=1}^q |a_n|^2 \left( X^2 q^{-1} + q^{-1} \right) \min(X, q^{1/2}) \right)^{1/2}.
\]

If \( q > X^2 \), then this gives the lemma at once, and if \( X < q < X^2 \), then it follows easily from this and (2.3).

Now suppose that \( q \leq X \). Define \( b^{(j)} \) to be 0 unless \( \frac{1}{2} X^2 < q r + s < \frac{1}{2} X^2 \) and \( q r + s \) is prime in which case define it to be 1. Then

\[
(2.7) \quad \sum_{n,p} a_n e(ans) = \sum_{r=1}^q \sum_{n=1}^q \sum_{r=1}^q a_n b^{(j)} e \left( n \left( \frac{a}{q} r + \left( a - \frac{a}{q} \right) (q r + s) \right) \right).
\]

For a fixed \( s \), take

\[
\alpha_r = \frac{a}{q} r + \left( a - \frac{a}{q} \right) (q r + s) s.
\]

The \( \alpha_r \) are all contained in an interval of length at most \( \frac{1}{q} q^{-1} \). Thus

\[
\| x_r - \alpha_i \| = \left| a - \frac{a}{q} \right| \left| (q r + s) s - (q i + s) s \right|
\]

so that

\[
\min_{r < i} \| x_r - \alpha_i \| \geq X^{-2k},
\]

Thus, by (2.7) and Lemma 1,

\[
\sum_{n,p} a_n e(ans) \ll q^2 \left( \sum_{n=1}^q |a_n|^2 \right)^{1/2} \left( \frac{X^2 q^{-1} + q^{-2}}{}\right)^{1/2} X^{-2k}.
\]

This with (2.3) gives the desired conclusion.

The next lemma is an extension of Theorem 3 of Davenport and Erdös [6]. When \( s > 3 \) it is apparently new.

**Lemma 3.** Suppose that \( s \geq 3 \), \( a_1, \ldots, a_s \) are non-zero integers,

\[
(2.8) \quad \theta = \frac{1}{k^2} - \frac{1}{k}, \quad \tau_s = \frac{k^2 - \theta^{-3}}{k^2 + k - k \theta^{-3}}, \quad \tau_s = \frac{k^2 - k - 1}{k^2 + k - k \theta^{-3}}
\]

and \( S \) denotes the number of solutions of

\[
\sum_{j=1}^s \gamma_j (r_j - t_j) = 0
\]

with \( X < r_1, t_1 < 2X, X^2 < r_2, t_2 < 2X^2, X^{1/2} < r_j, t_j < 2X^{1/2} \) for \( 3 \leq j \leq s \).

Then

\[
S \ll X^{1 + \theta + \tau_1 + \tau_2 + \cdots + \tau_s + \theta^2 - 1 + \epsilon}.
\]

**Proof.** Let \( S_m \) denote the number of solutions of

\[
\sum_{j=1}^m \gamma_j (r_j - t_j) = 0
\]

with \( r_m \neq t_m \). Since \( S_1 = 0 \) we have

\[
(2.9) \quad S = \sum_{m=2}^s S_m X^{\tau_1 + \tau_2 + \cdots + \tau_s + \theta^2 - 1} + O(X^{1 + \theta + \tau_1 + \tau_2 + \cdots + \tau_s + \theta^2 - 1}).
\]

Moreover,

\[
(2.10) \quad S_1 \ll X^{1 + \theta + \tau_1} \ll X^{1 + \theta + \tau_2}.
\]

We write

\[
(2.11) \quad S_m = S'_m + 2S''_m \quad (m \geq 3)
\]

where \( S'_m \) is the number of solutions with \( r_1 = t_1 \) and \( S''_m \) the number with \( r_1 > t_1 \). Then

\[
(2.12) \quad S'_m \ll XT_m
\]

where \( T_m \) is the number of solutions of

\[
(2.13) \quad \sum_{j=1}^m \gamma_j (r_j - t_j) = 0.
\]
If \( m = 3 \), then at once
\[
T_3 \ll X^{3 \nu_3 + 1}.
\]

If \( m > 3 \), then since \( r_3 \leq r_2 \theta^2 \), given any set of \( t_1, t_2, \ldots, t_m \), the number of choices for \( r_1, r_2, \ldots, r_m \) for which (2.13) holds is \( \ll 1 \). Hence
\[
T_m \ll X^{r_3 + r_2 + \cdots + r_m \theta^{m-2}} \quad (m > 3).
\]

We now turn to the treatment of \( S_m' \). The number of choices for \( r_3, t_3 \) is \( \ll X^{r_3^2} \). For any such choice we have
\[
A + c_1 (r_3^2 - t_3^2) + \sum_{j=3}^m c_j (r_j^2 - t_j^2) = 0
\]
where \( A \) is fixed. Let \( h = r_3 - t_3 \). Then \( r_3^2 - t_3^2 > h X^{r_3-1} \). Also
\[
A + \sum_{j=3}^m c_j (r_j^2 - t_j^2) \ll X^{r_3^2/2}.
\]

Hence \( 0 < h \ll X^{r_3^2-k+1} \), and (2.16) can be rewritten in the form
\[
A + c_1 (t_3 + h)^2 - t_3^2 \ll X^{r_3^2}.
\]

For a given \( h \), let \( t \) and \( t+j \) be two possible values of \( t_3 \) for which (2.17) holds. Then
\[
(t+j+h)^2 - (t+j)^2 - (t+h)^2 + t^2 \ll X^{r_3^2},
\]
whence \( h X^{r_3^2-1} \ll X^{r_3^2} \). Thus the number of possible choices for \( t_3 \) is
\[
\ll 1 + X^{r_3^2-k+1}.
\]

For given \( r_3, t_3 \), (2.16) becomes
\[
A_3 + \sum_{j=3}^m c_j (r_j^2 - t_j^2) = 0
\]
where \( A_3 \) is fixed. The number of choices for \( t_3, \ldots, t_{m-1} \) is \( \ll X^{r_3^2 + \cdots + r_m \theta^{m-4}} \) and for any such choice the number of choices for \( r_3, \ldots, r_{m-1} \) is \( \ll 1 \).

Given \( t_3, \ldots, t_{m-1}, r_3, \ldots, r_{m-1} \), (2.18) becomes
\[
A + c_m (r_m^2 - t_m^2) = 0
\]
and since \( r_m \neq t_m \) the number of choices for \( r_m, t_m \) is \( \ll X^r \). Thus
\[
S_m' \ll X^{r_3^2} \sum_{0 < h < X^{r_3^2-k+1}} (1 + X^{r_3^2-k+2} h^{-1}) X^{r_3^2 + \cdots + r_m \theta^{m-4} + h}.
\]

The lemma now follows from this, (2.9), ..., (2.12), (2.14) and (2.15).

For future reference we note that by (2.8),
\[
(2.20) \quad 1 + r_3 + r_3 + \cdots + r_3 \theta^{m-2} = \kappa \left( \frac{k^2 - 2k} {k^2 + 2} \theta^{m-2} \theta^2 \right).
\]

**Lemma 4.** Suppose that \( 1 \leq r \leq k - 2 \), \( 0 < r < 1 \), \( \mathcal{X} \) and \( \mathcal{Y} \) are finite subsets of \( X^n \), \( f : \mathcal{X} \times \mathcal{Y} \rightarrow \{-X^r, X^{r-k} \} \times \{X^{r-k} - 1 \} \times \mathbb{Z} \) and write
\[
r(m, v) = \left| \{(x, y) : x < x < 2X, u \in \mathcal{X}, v \in \mathcal{Y}, f(x, y) = m \} \right|,
\]
\[
R(m, v) = \left| \{(u, v) : u \in \mathcal{X}, f(u, v) = m \} \right|,
\]
\[
S = \sum_{m} r(m, v) \quad \text{and} \quad T = \sum_{m} R(m, v).
\]

Then
\[
S \ll XT + X^{r+1} + \sum_{m} r(m, v) \quad \text{and} \quad T \ll \sum_{m} R(m, v).
\]

where \( |\mathcal{X}|, |\mathcal{Y}| \) denote the cardinalities of \( \mathcal{X} \) and \( \mathcal{Y} \) respectively.

**Proof.** This follows that of Theorem 1 of Davenport [4] with one important modification, due to the fact that \( f \) may not be one-to-one.

It suffices to prove the result when \( |\mathcal{Y}| = 1 \). For then the more general result follows by summing over all possible \( v \) and applying Hölder's inequality to the last expression on the right. We henceforth suppress the \( v \). Let
\[
\mathcal{X}_j = \{ h : h = (h_1, \ldots, h_j) ; h_j > 0 ; h_j \leq X^r ; h_2, \ldots, h_j \leq X \}
\]
and
\[
\mathcal{X}_f(h, m) = \left| \{(x, y) : x \leq x < 2X, u \in \mathcal{X}, v \in \mathcal{Y}, f(x, y) = m \} \right|
\]
where \( \mathcal{X}_f \) is the usual \( j \)th iterate of the forward difference operator. Now let
\[
N_j = \sum_{h \in \mathcal{X}_j} R(h \cdot m).
\]

Then
\[
S \ll XT + N_1
\]
and, by Cauchy's inequality,
\[
N_j \ll X^{r+1} T \sum_{h \in \mathcal{X}_j} \mathcal{X}_f(h, m)^{1/2} \ll X^{r+1} T (X^{r+1} + N_{j+1}).
\]

Therefore
\[
N_j \ll X^{r+1} T \sum_{h \in \mathcal{X}_j} \mathcal{X}_f(h, m)^{1/2} \ll X^{r+1} T (X^{r+1} + N_{j+1}).
\]

Hence, by induction on \( r \),
\[
N_1 \ll X^{r+1} T \sum_{h \in \mathcal{X}_j} \mathcal{X}_f(h, m)^{1/2} \ll X^{r+1} T (X^{r+1} + N_{j+1}).
\]
By (2.21) and (2.22),
\[ N_{r+1} \ll X^r \left( \sum_m R(m)^2 \right)^{\frac{1}{2}} = X^r |\mathcal{A}|^2. \]

This with (2.23) and (2.24) gives the desired conclusion.

As an immediate corollary we have

**Lemma 5.** In addition to the premises of Lemma 4 suppose that
\[ T \ll X^r |\mathcal{A}| |\mathcal{A}'|, \quad |\mathcal{A}| \ll X^{(r+\varepsilon)\cdot}, \quad r \ll 2^{-r} \]
and
\[ r \ll (r+1-a(k-1))/(2^{-r}+a) \]
where \(0 < a < 1\). Then
\[ S \ll X^r |\mathcal{A}| |\mathcal{A}'|. \]

The next lemma follows by adapting the proof of Theorem 4 of Davenport [4] in the same way that we adapted the proof of his Theorem 1 to give our Lemma 4.

**Lemma 6.** We assume the hypothesis of Lemma 4 with \(k = 9\) and suppose further that \(r = 6\) or \(r \leq 6\) and \(r \) is odd, that
\[ r(m, p, v) = |\{(x, u), x < x < 2X, p|x, u \in \mathcal{U}, c^3 + p^2f(u, v) = m\}| \]
and that
\[ S' = \sum_{m \in \mathcal{U}} \sum_{v} \sum_{r} r(m, p, v)^2. \]

Then
\[ S' \ll X^r T_1 + X^{r+\varepsilon} T_1 + X^{r+\varepsilon} T_1 + X^{r+\varepsilon} T_1 + X^{r+\varepsilon} T_1 |\mathcal{A}| |\mathcal{A}'| \]
where \(r = (10-r)/9\).

**Lemma 7.** In addition to the premises of Lemma 6 we assume that
\[ T \ll X^r |\mathcal{A}| |\mathcal{A}'|, \quad |\mathcal{A}| \ll X^{(r+\varepsilon)\cdot}, \quad r \ll 2^{-r} \]
and
\[ r \ll 9r + 10 - 72 \alpha \]
where \(0 < \alpha < 1\). Then
\[ S' \ll X^{(r-\varepsilon)\cdot} |\mathcal{A}| |\mathcal{A}'|. \]

**Proof.** Immediate by Lemma 6.

3. Definitions. The case \(k > 9\). Let
\[ \theta = 1 - \frac{1}{k} \]
and
\[ s_1 = s + \left[ \frac{\log \left( \frac{6k-24 + 443}{14k} \right)}{-\log \theta} \right]. \]

We shall form the variables \(x_1, \ldots, x_s\) into four groups, the first two containing \(s_1\) variables each, the third \(s_2+1\) where \(s_2\) is yet to be defined, and the fourth the remainder.

Let
\[ t = 20 (k = 10), \quad t = 24 (k = 11), \quad t = 27 (k = 12), \]
\[ t = 4 + \left( \frac{\log \left( \frac{k^2-2 - k^2 + 2k^2 + 9}{k^2 + k^2} \right)}{-\log \theta} \right) (k \geq 13), \]
\[ \alpha(m) = 1 - \frac{k^2 - 3k^2 + k^2 + 2k^2}{k^2 + k^2 - k^2} \theta^{m-3}, \]
\[ t = \min(t_1, t), \]
\[ s_1 = 1 - (2k^2 - k^2) + \left( \frac{k^2 - 2k^2 + 1 + k^2 - 1}{1 - \alpha(t_1)} - k^2 \right)^{-1}, \]
\[ s_2 = 1 - \frac{1}{k} + 4(1 - \alpha_2), \]
\[ s_3 = 1 - \frac{1}{k} + 4(1 - \alpha_2), \]
\[ s_4 = \left[ \frac{\log \left( \frac{k^2 - 2k^2 + 1}{k^2 - 2k^2 + 1} \right)}{-\log \theta} \right] (\alpha_2 > \alpha(t)), \]
\[ 4 + \left( \frac{\log \left( \frac{k^2 - 2k^2 + 1}{k^2 + k^2} \right)}{-\log \theta} \right) (\alpha_2 \leq \alpha(t)). \]

In particular this gives \(s_1(10) = 107\), \(s_1(11) = 122\), \(s_1(12) = 137\), \(s_1(13) = 133\), \(s_1(14) = 168\), \(s_1(15) = 184\), \(s_1(16) = 200\), \(s_1(17) = 216\), and we shall show that for \(a > s_1, (1.2)\) has a non-trivial solution providing that the congruence condition is satisfied, and this establishes the theorem when \(k > 9\).
\[ \theta_m = \Theta \quad (s_1 - t_1 + 4 \leq m \leq s_1), \]
\[ \theta_{t_1 - t_1 + \delta} = \frac{k^2 - \delta}{k^2 - \delta^{i+3}}, \]
\[ \theta_{\eta - t_1 + \delta} = \frac{k^2 - \delta^{i+3}}{k^2 + k - \delta^{i+3}}. \]

and define inductively on \( s_1 - t_1 + 2 - i \) for \( s_1 - t_1 + 1 \geq i \geq 1 \)

\[ \mu_i = \frac{1}{k} \sum_{j=1}^{s_1} \prod_{r=i+1}^{j} \theta_r, \]
and
\[ \theta_i = \frac{\delta^{2k-1}}{2^{k-1} - 1 + \mu_i}. \]

Now define
\[ \lambda_{2s_1 - 1} = \lambda_{2s_1} = \prod_{j=2}^{s_1} \theta_j \quad (i = 1, \ldots, s_1) \]
and
\[ \lambda_i = \frac{1}{2} \prod_{j=1}^{i-2s_1} \theta_{j+2s_1}, \quad (i = 2s_1 + 1, \ldots, 2s_1 + s_2). \]

When \( k \) is even we are given that not all the \( c_j \) are of the same sign.
We can also assume this when \( k \) is odd, since we can always replace \( c_j \) by \(-c_j\) and \(-s_j^k\) by \((-s_j^k)\). Then by relabeling we can further suppose that
\[ c_1 > 0, \quad c_2 < 0. \]

Let \( P \) be a large real number and write
\[ P_1 = |c_1|^{1/2}P, \quad P_2 = c_0^{1/2}P, \quad P_3 = P^{3/4}. \]

\[ f_{1-4}(a) = \sum_{a \leq 2s_1 + j < 2s_1 + 2} e(a c_{2s_1 - j} x^k), \quad (i = 1, \ldots, s_1; j = 1, 2), \]
and
\[ F(a) = \prod_{i=1}^{2s_1} f_i(a), \]
\[ g_i(a) = \sum_{a \leq 2s_1 + 1 < 2s_1 + s_2} e(a c_i x^k), \quad (i = 2s_1 + 1, \ldots, 2s_1 + s_2). \]

Clearly \( \mathcal{N}(P) \) is the number of solutions of (1.2) with the variables restricted in various ways. We shall show that the congruence condition implies that \( \mathcal{N}(P) \to P \) as \( P \to \infty \). This will establish the theorem when \( k > 9 \).

4. Definitions. The case \( k = 9 \). Let
\[ \theta = \Theta, \quad t = 8, \quad s_1 = 32, \quad s_2 = 26, \quad s_3 = 91, \]
\[ \theta_m = \Theta \quad (28 \leq m \leq 32), \]
\[ \theta_{2t} = \frac{71}{81 - \Theta^9}, \]
\[ \theta_{2t} = \frac{81 - \Theta^9}{90 - \Theta^9}, \]
and inductively on \( 26 - i \),
\[ \mu_i = \frac{1}{9} \sum_{j=i}^{32} \prod_{r=i+1}^{j} \theta_r \quad (1 \leq i \leq 25). \]
and
\[ \theta_i = \begin{cases} \frac{8}{9} + \frac{1}{9} \min \left( \frac{1}{32}, \frac{55}{280 + 9\mu_i} \right), & (i = 25, 24), \\ \frac{8}{9} + \frac{1}{9} \min \left( \frac{1}{64}, \frac{64 - 72\mu_i}{568 + 9\mu_i} \right), & (i = 23, \ldots, 20), \\ \frac{8}{9} + \frac{1}{9} \min \left( \frac{1}{128}, \frac{8(1-\mu_i)}{127 + \mu_i} \right), & (i = 19, \ldots, 2). \end{cases} \]
and the summations are over

\[ P_{1}^2 < q_i < 2P_{1}^2, \quad p_i < Q_i, \quad p_i + q_i, \quad 9 | p_i + 1, \quad \frac{1}{2}P_{1/2} < p < \frac{1}{2}P_{1/2}. \]

Further, let

\[ h_i(a) = \sum_{e < P_{1/2}} e(aq, e^2) \quad (92 \leq i \leq a), \]

and

\[ \mathcal{N}(P) = \int E(a) \psi(a) h(a) E(a) da. \]

5. The Farey dissection. Let

\[ \mathfrak{M}(q, a) = \{ a; \ | a - a/q | < q^{-1}P_{1/2-k} \} \]

denote a typical major arc and write

\[ \mathfrak{M} = \bigcup_{q \in P_{1/2}} \mathfrak{M}(q, a) \]

to denote their union and

\[ m = (P_{1/2-k}, 1 + P_{1/2-k}) \setminus \mathfrak{M} \]

to denote their complement, the minor arcs.

Let

\[ \eta = \delta^k \]

define

\[ R(q, a) = \{ a; \ | a - a/q | < q^{-1}P_{1/2-k} \} \]

to be a truncated major arc and

\[ R = \bigcup_{q \in P_{1/2} \setminus \mathfrak{M}(q, a)} R(q, a) \]

to denote their union.

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The point of our definitions in §§ 3 and 4 is to make the most effective use of the lemmas of § 2 on the minor arcs, and to keep the variables $a_1, \ldots, a_n$ of (1.2) explicit.

The estimation of the major arcs are, as usual, nothing more than a matter of technique.

We proceed now to examine the minor arcs.

**Lemma 8.** We have

$$\int_0^1 |F(a)| \, da \ll P^{-k\alpha_0} F(0).$$

**Proof.** We first of all consider the case $k > 9$. By Schwarz's inequality (3.19), (3.20) and Parseval's identity the square of the integral on the left of (5.7) is majorized by the product of the two expressions

$$\sum_{m} r_j(m)^2 \quad (j = 1, 2),$$

where

$$r_j(m) = \left| \left( x_1, \ldots, x_n \right); \sum_{r \geq s} c_{i_0-i+r} a_i^k = m, \ P_j^{2k} < x_i < 2P_j^{2k} \right|.$$  

By first of all invoking Lemma 3 with $s = t_1$ and then successively applying Lemma 5 with $r = k - 2$ we find, providing that

$$k - 1 < k \theta_3 \leq k - 1 + 2^{2-k} \quad (2 \leq i \leq s_1 - t_1),$$

that

$$\sum_{m} r_j(m)^2 \ll P_j^{2k+4} \cdots + 2s_1 + 4 \ll P^{-k\alpha_0} F(0).$$

This gives the desired conclusion when $k > 9$ on establishing (5.9). For $k = 10, 11$ and 12 it can be checked by direct calculation. For $k > 12$ we observe that

$$\min \left( k^{2-k} + k - 2k^2 + 2, (k^2 + k) \left( \frac{k-1}{k} \right) \left( 6k - 24 + \frac{44k}{14k} \right) \right) > k^2 - 4k^2 + 5k^2 + k - 2 + 2^{2-k}(k^2 - 3k^2 + k + 2).$$

Thus, by (3.2), (3.3) and (3.3),

$$(k^2 + k) \theta^{3-k} > k^2 - 4k^2 + 5k^2 + k - 2 + 2^{2-k}(k^2 - 3k^2 + k + 2).$$

Therefore

$$(k - 1 + 2^{2-k})(k^2 - 3k^2 + k + 2) \theta^{3-k} < k^2 + k^2 - k^2 \theta^{3-k}.$$ 

Hence, by (2.8), (2.20), (3.10), \ldots, (3.13),

$$0 < 1 - \mu_{s_1 - t_1 + 1} = \frac{k^2 - 3k^2 + k + 2}{k^2 + k^2 - k^2 \theta^{3-k} + k - 1} \frac{1}{k - 1 + 2^{2-k}}$$

and this with (3.14) gives (5.9) for $i = s_1 - t_1$. Since $\mu_i$ is a decreasing function of $i$ it follows at once that (3.9) also holds when $i \leq s_1 - t_1$.

The proof in the case $k = 9$ is similar. We first of all observe that by (4.11) and (4.12) the integral in question is bounded by

$$\int \sum_{p_{19}} \cdots \sum_{p_{24}} |F_i| \, p_{19} \cdots p_{24} \left( \prod_{i} \left| f_i \right| \right) \left( \prod_{l \leq \alpha_0} \left| f_l \right| \right)$$

where $F_i$ contains the $x_i$ with $i$ odd and $F_{2j}$ the ones with $i$ even.

By the Cauchy–Schwarz inequality this is bounded by the square root of the product of the two expressions

$$\int \sum_{p_{19}} \cdots \sum_{p_{24}} |F_i|^2 \, p_{19} \cdots p_{24} \left( \prod_{i} \left| f_i \right|^2 \right) \left( \prod_{l \leq \alpha_0} \left| f_l \right|^2 \right) \quad (j = 1, 2)$$

and by Parseval's identity this is

$$\sum_{p_{19}} \cdots \sum_{p_{24}} \sum_{m} r_j(m, p, \ldots, p_{24})^2$$

with

$$r_j(m, p_{19}, \ldots, p_{24}) = \left| \left( x; c \right); c_1 a_1^{k_1} + \cdots + c_{t_j} a_{t_j}^{k_{t_j}} + p_{19} c_{t_j + 1} a_{t_j + 1}^{k_{t_j + 1}} + \cdots + p_{24} c_{t_j + s} a_{t_j + s}^{k_{t_j + s}} \right| = m, \ P_{2j+1}^{2k} < x_{2j} < 2P_{2j+1}^{2k}, \ P_j \, c \right|$$

with a similar expression for $r_{2j}$. We first of all use Lemma 3 with $s = 8$, then apply Lemma 7 twice with $r = 5$ and four times with $r = 6$. Finally we apply Lemma 5 successively eighteen times with $r = 7$. The choice of the parameters in (4.2), \ldots, (4.6) ensures that the hypotheses of the lemmas are satisfied.

Thus we have

$$\sum_{p_{19}} \cdots \sum_{p_{24}} \sum_{m} r_j(m, p_{19}, \ldots, p_{24})^2 \ll Q_{24} Q_{48} Q_{96} P_{48}^{1/2} \cdots P_{50}^{1/2} \cred \ll P^{-k\alpha_0} F(0).$$

This completes the proof of the lemma.

**Lemma 9.** We have $\mu_1 = a_1$ (k > 9), $\mu_2 > 0.96149$ (k = 9) and $\mu_2 > 0.95185$ (k = 9).

**Proof.** Consider first $k > 9$. By (3.1), (3.10), \ldots, (3.13),

$$k \mu_{s_1 - t_1 + 1} = 1 + \frac{k^2 - \theta^{3-k}}{k^2 + k - k \theta^{3-k} + k^2}$$

and this with (3.14) gives (5.9) for $i = s_1 - t_1$. Since $\mu_i$ is a decreasing function of $i$ it follows at once that (3.9) also holds when $i \leq s_1 - t_1$.
and thus, by (3.4),
\[ \mu_{t_1-t} = c(t). \]

If \( s_1 \leq t \), then by (3.5) and (3.6) we have \( \mu_1 = a_1 \). Thus we can suppose that \( s_1 > t = t_1 \). By (3.13) and (3.14), for \( s_1 \leq s_1 \equiv t_1 \),

\[ \mu_s = \frac{1}{k} + \theta_{t_1} \mu_{t_1} = \frac{1}{k} + \left( 1 - \frac{1}{k} \right) \frac{2^{k-1} \mu_{t_1}}{2^{k-2} - 1 + \mu_{t_1}}. \]

On rearrangement this becomes

\[ \frac{2^{k-2} - 1 + k\mu_s}{1 - \mu_s} = \frac{2^{k-2} - 1 + k\mu_{t_1}}{2^{k-2} - 1}. \]

Hence, by (3.5) and (5.10),

\[ \frac{2^{k-2} - 1 + k\mu_s}{1 - \mu_s} = \left( \frac{2^{k-2} - 1 + k\mu_{t_1}}{2^{k-2} - 1} \right)^{s_1-t}. \]

On rearrangement and comparison with (3.6) we obtain \( \mu_2 = a_1 \) once more.

Now consider \( k = 9 \). By (4.1), ..., (4.5)

\[ \mu_{95} = \frac{14444593}{22857741}. \]

By (4.5),

\[ \mu_s = 1/9 + \theta_{t_1} \mu_{t_1}. \]

We now use this with (4.6) to successively calculate the value of \( \mu_9 \). We find that

\[ \mu_{95} < 0.88136. \]

Then, since \( 1034(1 - \mu_9) < 127 + \mu_9 \) for \( i \leq 16 \) we can use the formula

\[ \frac{1 - \mu_9}{127 + 9\mu_9} = \frac{127}{144} \]

to give the desired lower bounds for \( \mu_9 \) and \( \mu_{95} \).

**LEMMA 10.** Let \( k \in \mathbb{N} \). Then

\[ H(a) \cdot h(a) \ll D^{11-k-\alpha} H(0) h(0). \]

**Proof.** Suppose first of all that \( k > 0 \). Choose \( a, q \) so that \( (a, q) = 1, |a - a/q| \ll P^{-k/2} \) and \( q \ll P^{\alpha/2} \). Then, by (5.1), (5.2) and (5.3), whenever \( q \ll P^{\alpha/2} \) we have

\[ |a - a/q| \gg q^{-1} P^{-k+3/2}. \]

Hence, by (3.16), (3.18), (3.21), (3.22) and (3.23), the hypothesis of Lemma 2 is satisfied with \( X = P^{14} \) and \( S(a) = H(a) \cdot h(a) \).

Thus

\[ H(a) \cdot h(a) \ll P^{11-k+\alpha} \sum_m r(m)^2 \]

where

\[ r(m) = \sum_{\xi I} c_i \alpha_i \xi = m, P_{\xi I} < x_1 < 2P_{\xi I}. \]

The sum \( \sum_m r(m)^2 \) is estimated in the same way as the analogous sums arising in the proof of Lemma 8. Thus

\[ \sum_m r(m)^2 \ll P^{11-k+\alpha} \sum_{\xi I} P_{\xi I}^2 \sum_{\xi I} P_{\xi I}^2 \]

whence, by (3.8), (3.18) and the same argument as in the first part of the proof of Lemma 9,

\[ \sum_m r(m)^2 \ll P^{11-k+\alpha} \sum_{\xi I} P_{\xi I}^2 \sum_{\xi I} P_{\xi I}^2. \]

Hence, by (3.21), (3.22), (3.23) and (5.11),

\[ H(a) \cdot h(a) \ll P^{11-k+\alpha} \sum_{\xi I} P_{\xi I}^2 \sum_{\xi I} P_{\xi I}^2. \]

The desired result then follows from (3.7) and Lemma 9.

In the case \( k = 9 \) we follow the same argument to begin with. This gives (5.11) with

\[ r(m) = \sum_{\xi I} r(m, \xi, \ldots, \xi). \]

where

\[ r(m, \xi, \ldots, \xi) = \left( \sum_{\xi I} c_i \alpha_i \xi = m, P_{\xi I} < x_1 < 2P_{\xi I} \right)^2. \]

Hence, by Cauchy's inequality,

\[ \sum_m r(m)^2 \ll \sum_{\xi I} Q_{\xi I} \sum_{\xi I} r(m, \xi, \ldots, \xi)^2. \]

We now follow the argument of the case \( k = 9 \) of Lemma 8. This gives

\[ \sum_m r(m)^2 \ll \sum_{\xi I} P_{\xi I}^2 \sum_{\xi I} P_{\xi I}^2. \]

Thus, by (4.5) and (4.7),

\[ \sum_m r(m)^2 \ll P_{\xi I}^2 \sum_{\xi I} Q_{\xi I} \sum_{\xi I} P_{\xi I}^2. \]
Hence, by (5.11), (4.14), (4.15) and (4.18),
\[ H(a) h(a) \leq P^{\delta - 2} P_{3}^{\delta - 8} H(0) h(0). \]
The proof is now completed by appealing to Lemma 9.
Combining Lemmas 8 and 10 establishes
\[ \int_{a} F(a) H(a) h(a) E(a) da \leq P^{-\delta - 4} F(0) H(0) h(0) E(0), \]
and concludes the discussion of the minor arcs.

6. The truncation of the major arcs. Let
\[ S_{i}(g, a) = \sum_{r_{i} = 1}^{g} e(a r_{i}^{k} / q), \]
\[ W_{i}(\beta) = \sum_{\delta_{i} k_{i} x_{i} + a \leq \beta} \frac{1}{k_{i}!} e(\beta x_{i}) \quad (2i - j, i \leq 2s_{i}, j = 1, 2), \]
\[ f_{i}^{*}(a, q, a) = q^{-1} S_{i}(g, a) W_{i}(a - a / q) \]
and
\[ f_{i}^{*}(a) = f_{i}^{*}(a, q, a) \quad (a \in \mathfrak{M}(g, a)), \]
\[ 0 \quad (a \notin \mathfrak{M}). \]

We note that by (3.2) and (4.1), \( s_{i} > k \).

Lemma 11. We have
\[ \int_{\mathfrak{M}} \left| \prod_{i=1}^{2k+2} f_{i}(a) - \prod_{i=1}^{2k+2} f_{i}^{*}(a) \right| da \leq P^{-\delta - 2k} \prod_{i=1}^{2k+2} f_{i}(0). \]

Proof. Let \( a \in \mathfrak{M}(g, a) \). Since \( \delta_{i} \geq g^{(1 - \frac{1}{k(k - 1)})} > \frac{1}{k} \) (\( i \leq 2k + 2 \)) it follows from Lemma 8 of Davenport [3] that
\[ f_{i}(a) - f_{i}^{*}(a) \leq q^{2k+2}. \]
By Lemma 3 of Hardy and Littlewood [8],
\[ S_{i}(g, a) \leq q^{1 - \delta k}, \quad (a, q) = 1. \]
Thus, by (6.3), ..., (6.5), for \( q \leq P^{\delta k} \),
\[ f_{i}(a), f_{i}^{*}(a) \leq q^{-1} P^{\delta k} \quad (i \leq 2k + 2). \]

By Lemma 9 of Davenport [3],
\[ f_{i}(a), f_{i}^{*}(a) \leq q^{-1} P^{1 + P^{\delta k} a - a / q} \quad (i = 1, 2). \]
Thus, by (6.5) and (6.7),
\[ \left( \prod_{i=1}^{2k+2} f_{i}(a) \right) - \left( \prod_{i=1}^{2k+2} f_{i}^{*}(a) \right) \leq q^{-1 - \delta k} \left( 1 + P^{\delta k} a - a / q \right)^{-1} \left( \prod_{i=1}^{2k+2} f_{i}(0) \right). \]
The lemma follows easily from this.

Lemma 12. We have
\[ \int_{\mathfrak{M} \setminus \mathfrak{N}} \prod_{i=1}^{2k+2} |f_{i}(a)| da \leq P^{-\delta - 2k} \prod_{i=1}^{2k+2} f_{i}(0). \]

Proof. Let \( a \in \mathfrak{M} \setminus \mathfrak{N} \). Then, by (5.2) and (5.6), there exist \( \alpha, \eta \) such that \( |a - a / q| < q^{-1} P^{-k/12} \), \( (a, q) = 1 \) and \( 1 \leq a \leq q \leq P^{\delta k} \) and moreover such that if \( q \leq P^{\delta k} \), then \( |a - a / q| \geq q^{-1} P^{-k/2} \). By (6.7) and (6.8),
\[ \int_{\mathfrak{M} \setminus \mathfrak{N}} \prod_{i=1}^{2k+2} |f_{i}(a)| \leq q^{-1 - \delta k} \left( 1 + P^{\delta k} a - a / q \right)^{-1} \left( \prod_{i=1}^{2k+2} f_{i}(0) \right). \]

Thus the integral in question is
\[ \ll \left( \sum_{a \in \mathfrak{N}} q^{-1 - \delta k} P^{-s} \right) \int_{-\infty}^{\infty} \left| \sum_{a \in \mathfrak{N}} q^{1 - \delta k} P^{-k} \prod_{i=1}^{2k+2} f_{i}(0) \right| \]
and this gives the lemma.

By Lemmas 11 and 12 we see that
\[ \int_{\mathfrak{N} \setminus \mathfrak{R}} F(a) H(a) h(a) E(a) da \leq P^{-\delta - 4} F(0) H(0) h(0) E(0). \]

7. The truncated major arcs. Let
\[ \mathcal{S}_{i}(n, X) = \sum_{a \in \mathfrak{N}} q^{-1 - \delta k} \sum_{a \in \mathfrak{N}} q^{1 - \delta k} P^{\delta k} \prod_{i=1}^{2k+2} S_{i}(g, a). \]

Lemma 13. Suppose that \( |n| \leq P^{\delta k} \). Then
\[ \int_{\mathfrak{N} \setminus \mathfrak{R}} \left( \sum_{a \in \mathfrak{N}} f_{i}(a) \right) da = (J(P) \mathcal{S}_{i}(n, P^{\delta k}) + O(P^{-\delta})) \prod_{i=1}^{2k+2} f_{i}(0) \]
where \( J(P) \approx 1 \).
Proof. By Lemma 11 it suffices to prove the result with each \( f_i \) in the integrand replaced by \( f_i^* \). Let \( a \in \mathbb{R}(q, a) \). By (5.5), (6.7) and (6.8),

\[
\sum_{a \in \mathbb{R}(q, a)} \left( \sum_{i=1}^{\frac{2k+3}{2}} f_i(a) \right) da = \int \left( \sum_{a \in \mathbb{R}(q, a)} \right) \left( \sum_{i=1}^{\frac{2k+3}{2}} f_i(a) \right) da + O \left( \left( q^{-1/2} \sum_{i=1}^{\frac{2k+3}{2}} f_i(0) \right) \right).
\]

Thus, by (6.3) and on summing over all the \( \mathbb{R}(q, a) \), we have

\[
(7.2) \quad \sum_{a \in \mathbb{R}(q, a)} \left( \sum_{i=1}^{\frac{2k+3}{2}} f_i(a) \right) da = J_1(P) \Xi_1(n, P^{\alpha_2}) + O \left( \left( P^{-3/2} \sum_{i=1}^{\frac{2k+3}{2}} f_i(0) \right) \right).
\]

By (6.2),

\[
J_1(P) = \sum_{a_1} \ldots \sum_{a_{2k+3}} \left( x_{a_1} \ldots x_{a_{2k+3}} \right)^{1/k-1}
\]

where the summations are over

\[
P_{2k+4} < a_{2k+2} < a_{2k+3}
\]

subject to \( c_1 a_1 + \ldots + c_{2k+3} a_{2k+3} = 0 \). It now follows from (3.17), (3.18) and (4.8) that

\[
J_1(P) \approx P^{-k} \sum_{i=1}^{\frac{2k+3}{2}} f_i(0).
\]

This with (7.2) gives the lemma.

**Lemma 14.** We have

\[
(7.3) \quad \int \left( F(a) H(a) h(a) E(a) \right) da = \left( J_1(P) \right) \Xi_1 \left( P^{-1} \right) F(0) H(0) E(0)
\]

where

\[
\Xi_1 = \sum_{i=1}^{\lambda_i} q^{-\alpha_2} \sum_{a \in \mathbb{R}(q, a)} \prod_{i=1}^{n} S_i(q, a).
\]

**Proof.** We consider the case \( k = 9 \). The case \( k > 9 \) is similar but simpler. By Lemma 13 the integral on the left of (7.3) is

\[
(7.5) \quad J_1(P) P^{-1} \left( \sum_{i=1}^{\lambda_i} f_i(0) \right) = \sum_{a \in \mathbb{R}(q, a)} \left( D(x, p), P^{\alpha_2} \right) + \left( 0, P^{-1} F(0) H(0) E(0) \right)
\]

where

\[
D(x, p) = c_1 x_1 + \ldots + c_{2k} x_{2k} + A(x', p') + x_{2k+1} \left( c_{2k+1} x_{2k+1} + \ldots + c_{2k+3} x_{2k+3} \right) + \ldots + x_{2k+3} \left( c_{2k+3} x_{2k+3} + \ldots + c_{2k+6} x_{2k+6} \right).
\]

\( A \) and \( B \) are given by (4.13) and (4.16), and the summations are over

\[
P_{2k+1} < a_{2k+1} < 2P_{2k+1}^{-1}, \quad P_{2k+2} < a_{2k+2} < 2P_{2k+2}^{-1} \quad (i \leq 22),
\]

\[
p_{i+1} < a_{n+1} \quad (19 \leq i \leq 24),
\]

\[
p_{i} < q, \quad 9 \leq i < 32,
\]

and \( a_{2k+1} < 2P_{2k+1}^{-1} \quad (65 \leq i < 90) \).

By (7.1), the multiple sum in (7.5) can be written in the form

\[
\sum_{q | n \prod q} q^{-\alpha_2} \sum_{a \in \mathbb{R}(q, a)} \prod_{i=1}^{n} S_i(q, a) \Xi_1(a, P^{\alpha_2}).
\]

For each variable \( x_i \), we can replace the sum over the \( x_i \) by an expression of the form

\[
P_i q^{-1} S_i(q, a m_i^p) + O(q)
\]

(\( \lambda_i \) is 1/2 when \( i \geq 92 \) and \( 3/8 \) when \( i = 91 \), and \( P_i \) is \( P \) when \( i \geq 91 \)) unless \( 37 \leq i \leq 48 \) or \( 77 \leq i \leq 82 \). In which case we obtain an expression of the form

\[
P_i q^{-1} \left( S_i(q, a m_i^p) - \frac{1}{P_i} S_i(q, a (m_i P_i)^p) \right) + O(q).
\]

The number \( m_i \) is either 1 or a product of prime numbers. In view of (6.6) and (5.6) the contribution from the \( O \) terms can be accommodated in the error term in (7.5). Since the number of different prime divisors of \( q \) is \( O(q) \) we can, for each \( q \) in the range, neglect those \( P \) for which at least one element divides \( q \). But then for the \( P \) that remain we can replace all the \( m_i \) and \( m_i P_i \) by one. We can then add back those \( P \) we neglected, the total error in doing so being easily accommodated in the error term of (7.5).
Now we observe that
\[ P_i^k = \sum_{r_i^k < x_i < r_i^{k+1}} 1 + O(1) = \sum_{x_i < r_i^k} 1 + O(1) \]
and
\[ P_i^k P_i^{k-1} = \sum_{r_i^k < x_i < r_i^{k+1}} 1 + O(1) . \]
The \( O \) terms are again easily accommodated in the error term in (7.5) and the resulting main term is
\[ J(P)P^{-s}E(0)H(0)\varphi(0)E(0) \sum_{q < P^{\alpha}} q^{-s} \sum_{\sigma=1 \atop (\sigma, 2) = 1} \prod_{l=1}^{s} S_{\sigma}(q, \alpha) . \]

By (6.6) we can complete the summation to infinity with a negligible error term, and this gives the lemma.

8. Completion of the proof. By (3.26), (4.21), (5.3) and (5.12),
\[ \mathcal{N}(P) = \int \frac{P(a)H(a)\varphi(a)E(a)d\alpha}{\xi} + O(P^{-s})P^{-s}E(0)H(0)\varphi(0)E(0) \]
and by (6.9) and Lemma 14
\[ \int \frac{P(a)H(a)\varphi(a)E(a)d\alpha}{\xi} = (J(P)\Xi + O(P^{-s})P^{-s}E(0)H(0)\varphi(0)E(0) . \]
The hypothesis that the congruence condition is satisfied ensures that by standard arguments \( \Xi > 1 \). Thus \( \mathcal{N}(P) \to \infty \) as \( P \to \infty \) which proves the theorem.

References