ACTA ARITHMETICA
XXXIII (1977)

On indefinite quadratic forms in four variables

by

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1. Introduction. A well-known theorem of Meyer (for a proof and references see [10], Chapter 11) tells that if $Q(x) = Q(x_1, \ldots, x_n)$ is an indefinite integral quadratic form in at least 5 variables then the equation

$$Q(x) = 0$$

has a non-trivial solution in integers, i.e. in integers $x_1, \ldots, x_n$, not all zero. It led to the famous conjecture of Oppenheim (see, for example, [13]) that for any indefinite quadratic form $Q(x_1, \ldots, x_n)$ in $n \geqslant 5$ variables with real coefficients the inequality

$$|Q(x)| < \varepsilon \tag{1.1}$$

is non-trivially soluble for every $\varepsilon > 0$.

In the special case when the quadratic form is of the shape $\lambda_1 x_1^2 + \ldots \ldots + \lambda_5 x_5^2$ the conjecture was proved by Davenport and Heilbronn [9]. They used a modified version of the Hardy–Littlewood circle method. The corresponding result with 9 in place 5 was proved earlier by Chowla [5] who used a theorem on lattice-points in a five-dimensional ellipsoid.

In 1956–57 Davenport ([6], [7]) made first progress towards proving the conjecture in its full generality. He established the conjecture subject only to conditions on the signature of the form. The idea of the proof was based on a certain theorem of Cassels [4] and also on the modified form of the Hardy–Littlewood method. Developing this method Davenport, Ridout and Birch ([2], [3], [8], [14]) obtained various results which all together completed the proof of Oppenheim's conjecture for all $n \geqslant 21$.

If an indefinite quadratic form $Q(x_1, \ldots, x_n)$ has rational coefficients and $|Q(x)|$ assumes values arbitrary near to 0 for suitable integral values of the variables, not all 0, then $Q(x)$ actually represents 0. The following form

$$Q(x) = x_1^2 + x_2^2 - 3(x_3^2 + x_4^2)$$

does not represent zero. Therefore Oppenheim's conjecture cannot be true for arbitrary indefinite quadratic form Q with real coefficients in four or less variables. It is however conjectured that if an indefinite quadratic form $Q(x)$ in at least 3 variables with real coefficients is not proportional to an integral form, the inequality (1.1) is still soluble for every $\varepsilon > 0$ in integers, not all zero. Of course, this cannot be true for binary quadratic forms. If θ is a positive irrational number whose square root has the continued fraction expansion with bounded partial quotients then $|x^2-\theta y^2|$ does not assume values which are arbitrarily small.

The first result concerning the above conjecture was given by Oppenheim [12]. He investigated certain special forms in four variables. To illustrate his results and methods we select the simple case

$$Q(x) = x_1^2+x_2^2+x_3^2-\theta x_4^2$$

with positive irrational θ. Oppenheim used the fact that the form $x_1^2+x_2^2+x_3^2$ represents all members of the arithmetic progression $8m+1$ and then reduced the problem (1.1) to approximation

$$|8m+1-\theta x^2| < \varepsilon.$$

Next results in this direction were established by Watson [15] for other types of forms in three and four variables whose coefficients are particular quadratic irrationals. He considered ternary forms

$$Q(x, y, z) = x^2-a\theta y^2-(a\theta+1)z^2$$

and quaternary forms

$$Q(x, y, z, w) = x^2+dy^2-\psi^2(z^2+dw^2)$$

where a and d are positive integers, θ the positive solution of $\theta^2 = a\theta+1$ and $\psi \in Q(\sqrt{D})$, D non-square integer of the form u^2+dv^2 with integral u, v.

In Watson's method suitable properties of the continued fraction expansions of θ and ψ play a very important rôle. It is a very special method. It does not work for example in the case

$$Q(x) = x_1^2+x_2^2-\sqrt{2}(x_3^2+x_4^2).$$

In this paper we shall prove

THEOREM 1. *Let $Q(x) = x_1^2+x_2^2-\theta(x_3^2+x_4^2)$, where θ is a real positive irrational number. Then for every $\varepsilon > 0$ the inequality*

$$|Q(x)| < \varepsilon \tag{1.2}$$

has infinitely many solutions in integers.

The methods of linear and half-dimensional sieves are main tools in the proof of Theorem 1. We do not make use of any other non-elementary devices.

Let $b^*(m)$ denote the characteristic function of the set of integers represented properly as the sum of two squares, i.e.

$$b^*(m) = \begin{cases} 1 & \text{if } m = u^2+v^2,\ (u, v) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We shall prove the following quantitative result

THEOREM 2. *There exist absolute constants $k > 0$ and $K > 0$ such that for $|\theta-M/N| < N^{-2}$, $(M, N) = 1$, $N > (\theta+\theta^{-1})^K$ and for any $k(N)$ satisfying $N^{-k} < k(N) < 1$ we have*

$$\frac{k}{16}\,\frac{(k(N)N)^2}{\log N} < \sum_{\substack{0<n<k(N)N^2 \\ |m-\theta n|<k(N) \\ (m,n)=1}} b^*(m)b^*(n). \tag{1.3}$$

We are also able to prove the upper bound $\sum b^*(m)b^*(n) \ll \frac{(k(N)N)^2}{\log N}$ but to make paper shorter we do not give the proof.

It seems possible to prove (1.3) in a slightly more general case for an indefinite form

$$Q(x) = \varphi(x_1, x_2)+\theta\psi(x_3, x_4),$$

where φ and ψ are binary quadratic forms with rational coefficients and non-zero discriminants and θ is irrational. To make use of sieve methods in that case we need a characterization of the numbers n and m represented by forms φ and ψ in terms of residue classes of n and m as well as their prime divisors modulo discriminants of φ and ψ. This kind of characterization is known since Gauss for discriminants with one class of forms in every genus. After Linnik created his ergodic method, it became also possible to obtain a useful characterization in the general case (see [1]).

The method presented in this paper yields nothing for forms in four variables having three coefficients linearly independent over the rationals.

I would like to express my thanks to Professor Jane Pitman for calling my attention to this problem. The present version of the paper has benefited greatly from her critical remarks. I would also like to thank Dr George Greaves for stimulating discussions.

2. Auxiliary results. In this section we gather together known results concerning sieve methods which will be used in the paper.

Let $\mathscr{A}$ be a finite sequence of integers, P set of primes and for any real number $z \geq 2$

$$P(z) = \prod_{p<z,\, p\in P} p.$$

The sieve methods deal with the sifting function

$$S(\mathscr{A};P,z) = \left|\{a \in \mathscr{A};\ (a, P(z)) = 1\}\right|,$$

where $|\{\ldots\}|$ denotes cardinality of the sequence $\{\ldots\}$. To estimate $S(\mathscr{A};P,z)$ we need the following quantities

$$\mathscr{A}_d = \{a \in \mathscr{A};\ a \equiv 0 \pmod d\},$$

$$\omega(d) - \text{multiplicative function satisfying } 0 \leqslant \omega(p) < p,$$

$$W(z) = \prod_{p|P(z)} \left(1 - \frac{\omega(p)}{p}\right),$$

$$R(\mathscr{A};d) = |\mathscr{A}_d| - \frac{\omega(d)}{d} X,$$

where X is a suitable real number > 1. In fact $R(\mathscr{A};d)$ depends also on $\omega(d)$ as well as X. For the sake of simplicity we shall not indicate it. We shall use linear sieve in the form stated in [11], Theorem 8.3. We say that a multiplicative function $\omega(d)$ satisfies Halberstam–Richert's conditions, if

$$(\Omega_1) \qquad 0 < \frac{\omega(p)}{p} \leqslant 1 - \frac{1}{A_1} \text{ for } p \in P, \quad \omega(p) = 0 \text{ for } p \notin P,$$

$$(\Omega_2(\varkappa, L)) \qquad -L < \sum_{u \leqslant p < w} \frac{\omega(p)}{p} \log p - \varkappa \log \frac{w}{u} < A_2, \quad 2 \leqslant u < w,$$

where A_1, A_2 and L are suitable constants > 1 and $\varkappa > 0$.

LEMMA 1 (Halberstam–Richert). *If $\omega(d)$ satisfies conditions (Ω_1) and $(\Omega_2(1, L))$ then for $2 < s < 3$ we have*

$$(2.1) \qquad S(\mathscr{A};P,z) > W(z)X\left\{\frac{2e^C}{s}\log(s-1) + O\left(L(\log z)^{-1/14}\right)\right\} - E$$

where $C = 0.577\ldots$ is the Euler constant and

$$(2.2) \qquad E = \sum_{\substack{d|P(z)\\ d<z^s}} 3^{\nu(d)} |R(\mathscr{A};d)|.$$

The constant involved in the symbol O depends only on A_1 and A_2.

In the paper we shall use a new upper bound method which can be called "double sieve". It shall be applied twice, so we introduce here general notions and sketch the principal ideas.

Let $\mathscr{H}$ be a finite sequence of pairs of integers and P_1, P_2 sets of primes. For a real number $z \geqslant 2$ let

$$P_i(z) = \prod_{\substack{p<z\\ p\in P_i}} p, \quad i = 1, 2,$$

and define an appropriate sifting function as follows

$$S(\mathscr{H};P_1,P_2,z) = \left|\{(h_1, h_2) \in \mathscr{H};\ (h_i, P_i(z)) = 1,\ i = 1, 2\}\right|.$$

To estimate $S(\mathscr{H};P_1,P_2,z)$ we need the following quantities

$$\mathscr{H}_{d_1,d_2} = \{(h_1, h_2) \in \mathscr{H};\ h_i \equiv 0 \pmod{d_i},\ i = 1, 2\},$$

$$\omega_i(d_i) - \text{multiplicative functions satisfying } 0 \leqslant \omega_i(p) < p,\ i = 1, 2,$$

$$W_i(z) = \prod_{p|P_i(z)} \left(1 - \frac{\omega_i(p)}{p}\right), \quad i = 1, 2,$$

$$G_i(z) = \sum_{\substack{d<z\\ d|P_i(z)}} \prod_{p|d} \frac{\omega_i(p)}{p - \omega_i(p)}, \quad i = 1, 2,$$

$$R(\mathscr{H};d_1,d_2) = |\mathscr{H}_{d_1,d_2}| - \frac{\omega_1(d_1)\omega_2(d_2)}{d_1 d_2}\mathscr{X}$$

where $\mathscr{X}$ is a suitable real number > 1.

LEMMA 2. *We have*

$$S(\mathscr{H};P_1,P_2,z) \leqslant \frac{\mathscr{X}}{G_1(z)G_2(z)} + \sum_{\substack{d_i|P_i(z)\\ d_i<z^2, i=1,2}} 3^{\nu(d_1)+\nu(d_2)} |R(\mathscr{H};d_1,d_2)|.$$

The proof of Lemma 2 easily follows from the following

LEMMA 3 (A. Selberg). *Let us suppose that a multiplicative function $\omega(d)$ satisfies inequality $0 \leqslant \omega(p) < p$ for any $p|P(z)$. Then, there exists a sequence $\{\varrho_d\}$ of real numbers satisfying the conditions*

$$(2.3) \qquad \varrho_1 = 1, \quad \sum_{d|n} \varrho_d \geqslant 0 \quad \text{for any } n|P(z),$$

$$(2.4) \qquad |\varrho_d| \leqslant 3^{\nu(d)} \quad \text{for any } d|P(z),$$

$$(2.5) \qquad \varrho_d = 0 \quad \text{for } d > z^2,$$

$$(2.6) \qquad \sum_{d|P(z)} \varrho_d \frac{\omega(d)}{d} = \frac{1}{G(z)}.$$

Proof. It is implicitly given in [11], Chapter 3, Section 1. We have taken

$$\varrho_d = \sum_{[d_1,d_2]=d} \lambda_{d_1}\lambda_{d_2}.$$

The inequality (2.3) follows from

$$\sum_{d|n} \varrho_d = \Big(\sum_{d|n} \lambda_d\Big)^2.$$

The other formulae (2.4)–(2.6) follow from formulae (3.1.2), (3.1.7), (3.1.8) and (3.1.11) of [11].

Proof of Lemma 2. By Lemma 3 there exist sequences $\{\varrho_{d_i}^{(i)}\}$ of real numbers satisfying conditions (2.3)–(2.6) with ω and P replaced by ω_i and P_i respectively, $i = 1, 2$. Hence we get

$$S(\mathscr{H}; P_1, P_2, z) \leqslant \sum_{(h_1,h_2)\in\mathscr{H}} \Big(\sum_{d_1|(h_1,P_1(z))} \varrho_{d_1}^{(1)}\Big)\Big(\sum_{d_2|(h_2,P_2(z))} \varrho_{d_2}^{(2)}\Big)$$

$$= \mathscr{X}\Big(\sum_{d_1|P_1(z)} \varrho_{d_1}^{(1)} \frac{\omega_1(d_1)}{d_1}\Big)\Big(\sum_{d_2|P_2(z)} \varrho_{d_2}^{(2)} \frac{\omega_2(d_2)}{d_2}\Big) + \sum_{\substack{d_i|P_i(z)\\ i=1,2}} \varrho_{d_1}^{(1)}\varrho_{d_2}^{(2)} R(\mathscr{H}; d_1, d_2)$$

$$\leqslant \frac{\mathscr{X}}{G_1(z)G_2(z)} + \sum_{\substack{d_i|P_i(z)\\ d_i<z^2\\ i=1,2}} 3^{\nu(d_1)+\nu(d_2)} |R(\mathscr{H}; d_1, d_2)|.$$

To make use of Lemmata 1 and 2 we shall need estimates for $W(z)$ and $G(z)$.

Lemma 4 (Halberstam–Richert). *If $\omega(d)$ satisfies (Ω_1) and $(\Omega_2(\varkappa, L))$ then*

$$\frac{1}{G(z)} = W(z)e^{\varkappa C}\Gamma(\varkappa+1)\Big\{1+O\Big(\frac{L}{\log z}\Big)\Big\},$$

and

$$W(z) = \prod_p \Big(1-\frac{\omega(p)}{p}\Big)\Big(1-\frac{1}{p}\Big)^{-\varkappa} e^{-\varkappa C}(\log z)^{-\varkappa}\Big\{1+O\Big(\frac{L}{\log z}\Big)\Big\}$$

where the infinite product is convergent. The constant in the symbol O depends only on A_1, A_2 and $\varkappa$.

Proof. See [11], Lemmata (5.3) and (5.4).

We shall apply Lemmata 1 and 4 twice with $\varkappa = 1/2$ and 1. In each case the constants A_1, A_2 as well as L will be absolute.

Lemma 5 (Mertens). *For $a = \pm 1$ there exist absolute constants $c_1(a) > 0$ and $c_2(a) > 0$ such that*

$$\sum_{\substack{p\leqslant x\\ p\equiv a (\mathrm{mod}\, 4)}} \frac{1}{p} = \frac{1}{2}\log\log x + c_1(a) + O\big((\log x)^{-1}\big),$$

$$\sum_{\substack{p\leqslant x\\ p\equiv a (\mathrm{mod}\, 4)}} \frac{\log p}{p} = \frac{1}{2}\log x + O(1),$$

$$\prod_{\substack{p\leqslant x\\ p\equiv a (\mathrm{mod}\, 4)}} \Big(1-\frac{1}{p}\Big)^{-1} = (\log x)^{1/2}\big\{c_2(a)+O\big((\log x)^{-1}\big)\big\}$$

for all $x \geqslant 2$. The constant involved in the symbol O is absolute.

Lemma 6. *If x is a real non-integer number then*

$$\Big|\sum_{n=1}^{N} e^{2\pi i n x}\Big| \leqslant \frac{2}{\|x\|}$$

where $\|x\|$ is the distance of x from the nearest integer taken positively.

3. Application of Sieve Methods to Theorem 3. For the fixed coprime positive integers α, γ we can find integer numbers β, δ satisfying conditions

(3.1) $$\alpha\delta - \beta\gamma = 1,$$

(3.2) $$\alpha < \beta \leqslant 2\alpha, \quad \gamma < \delta \leqslant 2\gamma.$$

For a real number $S > 4$ let $\mathfrak{B}$ stand for the sequence of pairs

(3.3) $$\mathfrak{B} = \{(m, n) = (\alpha r + \beta t, \gamma r + \delta t)\},$$

where r, t run over all positive integers $\leqslant S$ such that

$$(r, t) \equiv (\delta-\beta, \alpha-\gamma) (\mathrm{mod}\, 4) \quad \text{and} \quad (r, t) = 1.$$

Hence, for $(m, n) \in \mathfrak{B}$ we have

(3.4) $$(m, n) \equiv (1, 1) (\mathrm{mod}\, 4),$$

(3.5) $$(m, n) = 1$$

and for $B = \max\limits_{(m,n)\in\mathfrak{B}} \{m, n\}$ we have

(3.6) $$\tfrac{1}{3}(\alpha+\gamma)S < B < 3(\alpha+\gamma)S.$$

We are now in position to formulate

Theorem 3. *There exist absolute constants $0 < \tau < 1$, $c > 10$ such that for any*

$$S > (\alpha+\gamma)^{1-\tau} + c\Big(\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}\Big)^{3/\tau}$$

we have

$$\sum_{(m,n)\in\mathfrak{B}} b^*(m)b^*(n) > \tau\frac{S^2}{\log S^2}.$$

Theorem 2 will be obtained by applying Theorem 3 with $\alpha = M$, $\gamma = N$, where M/N is a suitable approximation to θ.

We shall express the sum $\sum b^*(m)b^*(n)$ as a sifting function and then we shall use the sieve methods. In order to do this we construct from the sequence $\mathfrak{B}$ a new sequence (of the same length)

$$\mathfrak{C} = \{mn;\ (m, n) \in \mathfrak{B}\}.$$

It is known that a positive integer m is represented properly as a sum of two squares if and only if it is not divisible by 4 and by any primes from

$$P^- = \{p;\ p \equiv -1 (\mathrm{mod}\, 4)\}.$$

This proves formula

$$\sum_{(m,n)\in\mathfrak{B}} b^*(m) b^*(n) = S(\mathfrak{C}; P^-, B). \tag{3.7}$$

We see from Lemma 5 that $\omega(p) = \dfrac{2p}{p+1}$ or 0 according as $p \in P^-$ or not, satisfies Halberstam–Richert's conditions (Ω_1) and $(\Omega_2(1, L))$ with some absolute constants A_1, A_2, L. We calculate the infinite product involved in Lemma 4 as follows

$$\prod_p \left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-1} = 2\prod_p \left(1-\frac{\chi_4(p)}{p}\right)^{-1} = 2L(1, \chi_4) = \pi/2.$$

Therefore, on applying Lemma 1 with ω as above and $X = \frac{1}{2}(S/\pi)^2$ we get

LEMMA 7. *For $2 < s < 3$ and any $z \geqslant 2$ we have*

$$S(\mathfrak{C}; P^-, z) > \frac{S^2}{2\pi \log z^s}\{\log(s-1) + O((\log z)^{-1/14})\} - E_0 \tag{3.8}$$

where

$$E_0 = \sum_{\substack{d<z^s\\ d|P^-(z)}} 3^{\nu(d)} |R(\mathfrak{C}, d)|$$

and

$$R(\mathfrak{C}, d) = |\mathfrak{C}_d| - \frac{2^{\nu(d)}}{d}\prod_{p|d}\left(1+\frac{1}{p}\right)\frac{1}{2}\left(\frac{S}{\pi}\right)^2.$$

The constant in the symbol O is absolute.

By (3.7) for $d|P^-(z)$ we have

$$|\mathfrak{C}_d| = \sum_{d_1 d_2 = d} |\mathfrak{B}_{d_1, d_2}| \tag{3.9}$$

so, we can express $R(\mathfrak{C}, d)$ in terms of

$$R(\mathfrak{B}; d_1, d_2) = |\mathfrak{B}_{d_1,d_2}| - \frac{1}{d_1 d_2}\prod_{p|d_1 d_2}\left(1+\frac{1}{p}\right)^{-1}\frac{1}{2}\left(\frac{S}{\pi}\right)^2 \tag{3.10}$$

as follows

$$R(\mathfrak{C}, d) = \sum_{d_1 d_2 = d} R(\mathfrak{B}; d_1, d_2)$$

and thus

$$E_0 \leqslant \sum_{\substack{d_1 d_2 < z^s\\ d_1 d_2 | P^-(z)}} 3^{\nu(d_1 d_2)} R(\mathfrak{B}; d_1, d_2).$$

We shall deal with the last sum in the next section. Let us remark here that the estimation of $S(\mathfrak{C}; P^-, z)$ given by Lemma 7 cannot be applied directly to (3.7). It requires z being as large as B, but unfortunately we are enable to estimate E_0 successfully for such z's. We shall use Lemma 7 with s little greater than 2 and z little smaller than $B^{1/2}$. We have to estimate

$$S(\mathfrak{C}; P^-, z) - S(\mathfrak{C}; P^-, B).$$

To do this, we estimate the above difference at first by a sum of double sifting functions and then we apply Lemma 2.

Let us assume $B^{1/3} < z < B^{1/2}$ and set

$$Q = \{q = ap \leqslant B/z;\ z \leqslant p \leqslant \sqrt{B},\ p \in P^-,\ (a, 2P^-(z)) = 1\},$$

$$\mathfrak{M}^{(q)} = \{(m/q, mn);\ (m, n) \in \mathfrak{B}_{q,1}\},$$

$$\mathfrak{N}^{(q)} = \{(mn, n/q);\ (m, n) \in \mathfrak{B}_{1,q}\},$$

$$P^+ = \{p;\ p \equiv 1 (\mathrm{mod}\, 4)\}.$$

LEMMA 8. *For $B^{1/3} < z < B^{1/2}$ we have*

$$S(\mathfrak{C}; P^-, z) - S(\mathfrak{C}; P^-, B) \leqslant \sum_{q\in Q} S(\mathfrak{M}^{(q)}; P^+, P^-, z) + \sum_{q\in Q} S(\mathfrak{N}^{(q)}; P^-, P^+, z).$$

Proof. Let us suppose that the pair $(m, n) \in \mathfrak{B}$ contributes to $S(\mathfrak{C}; P^-, z) - S(\mathfrak{C}; P^-, B)$, i.e.

$$(mn, P^-(z)) = 1 \quad \text{and} \quad (mn, P^-(B)) > 1.$$

The last inequality says that either m or n, say m, is divisible by a prime from P^-. Then, by (3.4) we see that m has to be divisible by at least two primes from P^- (not necessarily distinct). Therefore

$$m = app',$$

where $p, p' \in P^-$ and $p \leqslant p'$. By $(m, P^-(z)) = 1$ we obtain

$$z \leqslant p \leqslant \sqrt{pp'} \leqslant \sqrt{m} \leqslant \sqrt{B}$$

and

$$ap = m/p' \leqslant m/z \leqslant B/z.$$

We have shown that $q = ap \in Q$. Since $m/q = p' \in P^-$ and $(mn, P^-(z)) = 1$ we see that pair $(m/q, mn)$ is counted in $S(\mathfrak{M}^{(q)}; P^+, P^-, z)$. This completes the proof of the lemma.

LEMMA 9. *For $2 \leqslant \mathfrak{z} < z$ and $q \in Q$ we have*

(3.11) $$S(\mathfrak{M}^{(q)}; P^+, P^-, \mathfrak{z}) < O\left(\frac{S^2}{q(\log \mathfrak{z})^{3/2}}\right) + E^{(q)},$$

where

$$E^{(q)} = \sum_{\substack{d_1, d_2 < \mathfrak{z}^2 \\ d_1 | P^+(\mathfrak{z}), d_2 | P^-(\mathfrak{z})}} 3^{\nu(d_1 d_2)} |R(\mathfrak{M}^{(q)}; d_1, d_2)|$$

and

(3.12) $$R(\mathfrak{M}^{(q)}; d_1, d_2) = |\mathfrak{M}^{(q)}_{d_1, d_2}| - \frac{2^{\nu(d_2)}}{d_1 d_2} \prod_{p | d_1 d_2} \left(1 + \frac{1}{p}\right)^{-1} \frac{S^2}{2 q \pi^2}.$$

The constant in the symbol O is absolute.

Proof. We apply Lemma 2 with

$$\mathscr{H} = \mathfrak{M}^{(q)}, \quad P_1 = P^+, \quad P_2 = P^-, \quad z = \mathfrak{z},$$

$$\mathscr{X} = \frac{1}{q} \prod_{p | q} \left(1 + \frac{1}{p}\right)^{-1} \frac{S^2}{2\pi^2}$$

and

$$\omega_1(p) = \begin{cases} p/(p+1) & \text{if } p \in P^+, p \nmid q, \\ 1 & \text{if } p \in P^+, p | q, \\ 0 & \text{if } p \notin P^+; \end{cases} \quad \omega_2(p) = \begin{cases} 2p/(p+1) & \text{if } p \in P^-, \\ 0 & \text{if } p \notin P^-. \end{cases}$$

One can easily show by Lemma 5 that ω_1 and ω_2 satisfy the Halberstam–Richert's conditions with absolute constants A_1, A_2, L and parameters $\varkappa = 1/2, 1$ respectively. Hence, it follows from Lemma 4

$$\frac{1}{G_1(\mathfrak{z})} \ll (\log \mathfrak{z})^{-1/2} \quad \text{and} \quad \frac{1}{G_2(\mathfrak{z})} \ll (\log \mathfrak{z})^{-1}$$

where the constant in the symbol $\ll$ is absolute. For any $d_1 | P^+(\mathfrak{z})$ and $d_2 | P^-(\mathfrak{z})$ we have $(q, d_2) = 1$ which implies

$$\frac{\omega_1(d_1)\,\omega_2(d_2)}{d_1 d_2} \mathscr{X} = \frac{2^{\nu(d_2)}}{d_1 d_2} \prod_{p | d_1 d_2} \left(1 + \frac{1}{p}\right)^{-1} \frac{1}{2q} \left(\frac{S}{\pi}\right)^2.$$

This proves (3.12) and completes the proof of the lemma.

LEMMA 10. *For $2 \leqslant \mathfrak{z} < z$ we have*

$$\sum_{q \in Q} E^{(q)} \leqslant \sum_{\substack{D_1 D_2 \leqslant B z^{-1} \mathfrak{z}^4 \\ (D_1, D_2) = 1, 2 \nmid D_1 D_2}} 4^{\nu(D_1 D_2)} |R(\mathfrak{B}; D_1, D_2)|.$$

Proof. For any $d_1 | P^+(\mathfrak{z})$ and $d_2 | P^-(\mathfrak{z})$ we have

(3.13) $$|\mathfrak{M}^{(q)}_{d_1, d_2}| = \sum_{d_2' d_2'' = d_2} |\mathfrak{B}_{q d_1 d_2', d_2''}|$$

so, we can express $R(\mathfrak{M}^{(q)}; d_1, d_2)$ in terms of (3.10) as follows

$$R(\mathfrak{M}^{(q)}; d_1, d_2) = \sum_{d_2' d_2'' = d_2} R(\mathfrak{B}; q d_1 d_2', d_2'')$$

and thus

(3.14) $$E^{(q)} \leqslant \sum_{\substack{d_1, d_2' d_2'' < \mathfrak{z}^2 \\ d_1 | P^+(\mathfrak{z}), d_2' d_2'' | P^-(\mathfrak{z})}} 3^{\nu(d_1 d_2' d_2'')} |R(\mathfrak{B}; q d_1 d_2', d_2'')|.$$

Let us consider a pair of numbers $D_1 = q d_1 d_2'$ and $D_2 = d_2''$, where d_1, d_2', d_2'' run over the same range as in (3.14) and q runs over Q. It is easily seen that d_2' and d_2'' are uniquely determined by D_1 and D_2 and $f = d_1 d_2'$ is a square-free divisor of D_1. Hence we get

(3.15) $$\sum_{q, d_1, d_2', d_2''} 3^{\nu(d_1 d_2' d_2'')} \leqslant 3^{\nu(D_2)} \sum_{f | D_1} \mu^2(f) 3^{\nu(f)} = 3^{\nu(D_2)} 4^{\nu(D_1)}.$$

We have also

(3.16) $$2 \nmid D_1 D_2, \quad (D_1, D_2) = 1 \quad \text{and} \quad D_1 D_2 \leqslant B z^{-1} \mathfrak{z}^4.$$

Therefore, by (3.14), (3.15) and (3.16) we obtain the proof of the lemma.

LEMMA 11. *For $B^{1/3} < z < B^{1/2}$ we have*

$$\sum_{q \in Q} \frac{1}{q} \ll (\log 2 B z^{-2})^{3/2} (\log z)^{-1}.$$

The constant in the symbol $\ll$ is absolute.

Proof. For $q \in Q$ we have $q = ap$, where

$$z \leqslant p \leqslant B^{1/2}, \quad a \leqslant B/pz \leqslant B z^{-2} < z \quad \text{and} \quad (a, 2P^-(z)) = 1.$$

It follows that a is divisible only by primes from P^+. Therefore

$$\sum_{q \in Q} \frac{1}{q} \leqslant \Big(\sideset{}{'}\sum_{a \leqslant B z^{-2}} \frac{1}{a}\Big)\Big(\sum_{z \leqslant p \leqslant B^{1/2}} \frac{1}{p}\Big)$$

where dash denotes that summation is taken over numbers divisible only by primes from P^+. From the Mertens' Lemma 5 we get

$$\sideset{}{'}\sum_{a} \frac{1}{a} \leqslant \prod_{\substack{p \in P^+ \\ p < B z^{-2}}} \left(1 - \frac{1}{p}\right)^{-1} \ll (\log 2 B z^{-2})^{1/2}$$

and

$$\sum_{z\leqslant p\leqslant B^{1/2}} \frac{1}{p} = \log\left(\frac{\log B}{2\log z}\right) + O\big((\log z)^{-1}\big)$$
$$= \log\left(1+\frac{\log Bz^{-2}}{2\log z}\right) + O\big((\log z)^{-1}\big) \ll \frac{\log 2Bz^{-2}}{\log z}$$

which completes the proof of the lemma.

Now Lemmata 9, 10 and 11 give

$$\sum_{q\in Q} S(\mathfrak{M}^{(q)}; P^+, P^-, \mathfrak{z}) < \sum_{\substack{D_1D_2\leqslant Bz^{-1}\mathfrak{z}^4\\ (D_1,D_2)=1,\,2\nmid D_1D_2}} 4^{\nu(D_1D_2)}|R(\mathfrak{B}; D_1, D_2)| + $$
$$+ O\left(\left(\frac{\log 2Bz^{-2}}{\log \mathfrak{z}}\right)^{3/2} \frac{S^2}{\log z}\right).$$

The same estimate may be proved for $\sum S(\mathfrak{N}^{(q)}; P^-, P^+, \mathfrak{z})$. Therefore, by Lemmata 7 and 8 we get

LEMMA 12. *For $2 < s < 3$, $2 < \mathfrak{z} < z$ and $B^{1/3} < z < B^{1/2}$ we have*

$$S(\mathfrak{C}; P^-, B) > \frac{S^2}{2\pi \log z^s}\left\{\log(s-1) + O\big((\log z)^{-1/14}\big) + \left(\frac{\log 2Bz^{-2}}{\log \mathfrak{z}}\right)^{3/2}\right\} -$$
$$-3 \sum_{\substack{d_1d_2<\max(z^S, Bz^{-1}\mathfrak{z}^4)\\ (d_1,d_2)=1,\,2\nmid d_1d_2}} 4^{\nu(d_1d_2)}|R(\mathfrak{B}; d_1, d_2)|.$$

The constant in the symbol O is absolute.

4. Central Lemma

LEMMA 13. *For any $0 < \varepsilon < 1/4$ there exists a positive constant $c(\varepsilon) > 1$ depending only on ε such that*

$$\sum_{\substack{d_1d_2<B^{1-4\varepsilon}\\ (d_1,d_2)=1,\,2\nmid d_1d_2}} 4^{\nu(d_1d_2)}|R(\mathfrak{B}; d_1, d_2)| < c(\varepsilon)\left(\frac{\alpha}{\gamma}+\frac{\gamma}{\alpha}\right)^2 (B+S^2)^{1-\varepsilon}.$$

Proof.

I. Division of the sum $|\mathfrak{B}_{d_1,d_2}|$. Using (3.1)–(3.5) and the fact that $\sum_{a|k} \mu(a)$ is 1 or 0 according as k is 1 or not we see that

$$|\mathfrak{B}_{d_1,d_2}| = \sum_{a\leqslant H} \mu(a) \sum_{r,t} 1 + \sum_{a>H} \mu(a) \sum_{r,t} 1, \tag{4.1}$$

where, in each case, r, t run over positive integers $\leqslant S$ such that

$$a|r,\quad a|t,\quad (r,t) \equiv (\delta-\beta, \alpha-\gamma) \pmod 4,\quad d_1|\alpha r+\beta t,\quad d_2|\gamma r+\delta t$$

and $1 < H \leqslant S$ is to be choosen later. The last sum can be easily estimated (on average) as follows

$$\sum_{d_1,d_2} 4^{\nu(d_1d_2)} \Big|\sum_{a>H}\sum\Big| \leqslant \sum_{a>H}\sum_{r,t}\sum_{\substack{d_1|\alpha r+\beta t\\ d_2|\gamma r+\delta t}} 4^{\nu(d_1d_2)} \tag{4.2}$$
$$< c_1(\eta) B^\eta \sum_{a>H} S^2 a^{-2} < c_1(\eta) B^\eta S^2 H^{-1},$$

where η is an arbitrary positive constant and $c_1(\eta)$ depends only on η. We shall express $\sum_{a\leqslant H}\sum$ as a trigonometric sum in the following way

$$\sum_{a\leqslant H} \mu(a) \sum_{r,t} 1 = \frac{1}{d} \sum_{a\leqslant H}\sum_{r,t}\sum_{\substack{u_i \pmod{d_i}\\ i=1,2}} e(\mu_1 r + \mu_2 t), \tag{4.3}$$

where now the range for r, t is

$$0 < r, t \leqslant S/a,\quad a(r,t) \equiv (\alpha-\beta, \gamma-\delta) \pmod 4,$$

$e(x)$ denotes $e^{2\pi i x}$, $d = d_1 d_2$ and

$$\mu_1 = a(\alpha u_1 d_2 + \gamma u_2 d_1)/d, \tag{4.4}$$

$$\mu_2 = a(\beta u_1 d_2 + \delta u_2 d_1)/d. \tag{4.5}$$

Denoting the set of all integers by Z we divide the set of pairs (u_1, u_2) $(u_i \pmod{d_i},\ i = 1, 2)$ into four classes

A	$\mu_1 \in Z,$	$\mu_2 \in Z,$
B	$\mu_1 \in Z,$	$\mu_2 \notin Z,$
C	$\mu_1 \notin Z,$	$\mu_2 \in Z,$
D	$\mu_1 \notin Z,$	$\mu_2 \notin Z.$

Using the fact that $(d_1, d_2) = 1$, $(\alpha,\gamma) = (\beta,\delta) = 1$, we note that

$$\mu_1 \in Z \Leftrightarrow d_1 | a u_1 \alpha \wedge d_2 | a u_2 \gamma, \tag{4.6}$$

$$\mu_2 \in Z \Leftrightarrow d_1 | a u_1 \beta \wedge d_2 | a u_2 \delta, \tag{4.7}$$

$$\mu_1 \in Z \wedge \mu_2 \in Z \Leftrightarrow d_1 | a u_1 \wedge d_2 | a u_2. \tag{4.8}$$

Let $A(d_1, d_2)$, $B(d_1, d_2)$, $C(d_1, d_2)$ and $D(d_1, d_2)$ stand for the parts of the sum (4.3) which correspond to the pairs (u_1, u_2) from classes A, B, C, and D respectively. Then

$$\sum_{a\leqslant H} \mu(a) \sum_{r,t} 1 = A(d_1, d_2) + B(d_1, d_2) + C(d_1, d_2) + D(d_1, d_2). \tag{4.9}$$

II. Asymptotic formula for $A(d_1, d_2)$. The main term of $|\mathfrak{B}_{d_1,d_2}|$ will be obtained from the $A(d_1, d_2)$. By (4.8) we have

$$\sum_{(u_1,u_2)\in A} 1 = (a, d_1)(a, d_2) = (a, d)$$

and thus

$$A(d_1, d_2) = \frac{1}{d}\sum_{\substack{a\leqslant H\\ 2\nmid a}} \mu(a) \sum_{(u_1,u_2)\in A}\left(\frac{S}{4a}+O(1)\right)^2 = \frac{1}{d}\sum_{\substack{a\leqslant H\\ 2\nmid a}}\mu(a)(a, d)\left\{\frac{S^2}{16a^2}+O\left(\frac{S}{a}\right)\right\}.$$

Moreover writing $\tau(d)$ for the number of positive divisors of d we have

$$\sum_{2\nmid a}\mu(a)\frac{(a, d)}{a^2} = 8\pi^{-2}\prod_{p|d}\left(1+\frac{1}{p}\right)^{-1},$$

$$\sum_{a>H}\frac{(a, d)}{a^2} \leqslant \sum_{c|d} c \sum_{a>H/c}(ac)^{-2} < 2\tau(d)H^{-1},$$

$$\sum_{a\leqslant H}\frac{(a, d)}{a} \leqslant \sum_{c|d} c \sum_{a\leqslant H/c}(ac)^{-1} < \tau(d)\log 3H.$$

Hence

$$A(d_1, d_2) = \frac{1}{d}\prod_{p|d}\left(1+\frac{1}{p}\right)^{-1}\frac{S^2}{2\pi^2} + O\left(\frac{\tau(d)}{d}(S^2H^{-1}+S\log H)\right). \tag{4.10}$$

Let us note here that

$$\sum_{d_1d_2<D} 4^{\nu(d_1d_2)}\frac{\tau(d_1d_2)}{d_1d_2} \leqslant \left(\sum_{d<D}4^{\nu(d)}\frac{\tau(d)}{d}\right)^2 \ll \log^{16} D, \qquad D\geqslant 2.$$

III. Estimates of $B(d_1, d_2)$ and $C(d_1, d_2)$. First let us consider class B. We have, by Lemma 6,

$$\sum_{r,t} e(\mu_1 r+\mu_2 t) = \left(\sum_r e(\mu_1 r)\right)\left(\sum_t e(\mu_2 t)\right) < \frac{S}{a}\,\|4\mu_2\|^{-1}$$

which gives

$$B(d_1, d_2) < \frac{S}{d}\sum_{a\leqslant H, 2\nmid a} a^{-1}\sum_{(u_1,u_2)\in B}\|4\mu_2\|^{-1}. \tag{4.11}$$

We see from (4.6) that for $(u_1, u_2)\in B$

$$u_1 = \frac{d_1}{(a\alpha, d_1)}U_1, \qquad u_2 = \frac{d_2}{(a\gamma, d_2)}U_2$$

where U_1 and U_2 run through complete sets of residues modulo $(a\alpha, d_1)$ and $(a\gamma, d_2)$ respectively, in such a way that $\mu_2\notin Z$. Substituting the above value of u_1 and u_2 to (4.5) we get

$$\mu_2 = a\left(\frac{\beta U_1}{(a\alpha, d_1)}+\frac{\delta U_2}{(a\gamma, d_2)}\right).$$

For a particular pair (U_1, U_2) it is easily seen that there is a unique integer L such that

$$4\mu_2 \equiv (L/Q)(\mathrm{mod}\,1), \qquad 0<L<Q,$$

where

$$Q = (a\alpha, d_1)(a\gamma, d_2)/(a, d).$$

Two pairs (U_1, U_2), (U_1', U_2') yield the same value of L if and only if

$$U_1\equiv U_1'\left(\mathrm{mod}\,\frac{(a\alpha, d_1)}{(a, d_1)}\right), \qquad U_2\equiv U_2'\left(\mathrm{mod}\,\frac{(a\gamma, d_2)}{(a, d_2)}\right)$$

and thus the number of pairs U_1, U_2 which yield a particular value of L is

$$(a, d_1)(a, d_2) = (a, d).$$

This leads to

$$\sum_{(u_1,u_2)\in B}\|4\mu_2\|^{-1} = (a, d)\sum_{0<L<Q}\left\|\frac{L}{Q}\right\|^{-1} = 2(a, d)\sum_{1\leqslant L<Q/2} Q/L < 2(a, d)Q\log Q \leqslant 2(a\alpha\gamma, d)\log d.$$

Finally we obtain

$$B(d_1, d_2) \leqslant S\frac{\log d}{d}\sum_{a\leqslant H}\frac{2(a\alpha\gamma, d)}{a} \leqslant S\frac{2(\alpha\gamma, d)\log d}{d}\sum_{a\leqslant H}\frac{(a, d)}{a} \leqslant 4S\frac{\tau(d)}{d}(\alpha\gamma, d)\log d\log 3H. \tag{4.12}$$

Analogously, we consider $C(d_1, d_2)$ and obtain

$$C(d_1, d_2) \leqslant 4S\frac{\tau(d)}{d}(\beta\delta, d)\log d\log 3H. \tag{4.13}$$

Let us note here that

$$\sum_{d_1d_2<D} 4^{\nu(d_1d_2)}\frac{\tau(d_1d_2)}{d_1d_2}(\alpha\beta\gamma\delta, d_1d_2)$$
$$\leqslant \Big(\sum_{c|\alpha\beta\gamma\delta} 4^{\nu(c)}\tau(c)\Big)\Big(\sum_{d_1d_2<D} 4^{\nu(d_1d_2)}\frac{\tau(d_1d_2)}{d_1d_2}\Big) < c_2(\eta)(\alpha\beta\gamma\delta)^{\eta}\log^8 D,$$

where η is an arbitrary positive number and $c_2(\eta)$ depends only on η.

IV. Division of the sum $D(d_1, d_2)$. We have, by Lemma 6,

$$\sum_{r,t} e(\mu_1 r+\mu_2 t) \leqslant \|4\mu_1\|^{-1}\|4\mu_2\|^{-1}$$

which gives

$$D(d_1, d_2) \leqslant \frac{1}{d}\sum_{\substack{a\leqslant H\\ 2\nmid a}}\sum_{(u_1,u_2)\in D}\frac{1}{\|4\mu_1\|\,\|4\mu_2\|}. \tag{4.14}$$

It is easily seen that there are unique integers L_1 and L_2 such that

$$4\mu_1 \equiv \frac{L_1}{d}(\operatorname{mod} 1), \quad 0<|L_1|<d/2,$$

$$4\mu_2 \equiv \frac{L_2}{d}(\operatorname{mod} 1), \quad 0<|L_2|<d/2,$$

so

$$\|4\mu_1\| = |L_1|/d, \quad \|4\mu_2\| = |L_2|/d$$

and thus

$$D(d_1, d_2) \leqslant d\sum_{\substack{a\leqslant H\\ 2\nmid a}}\sum_{(u_1,u_2)\in D}\frac{1}{|L_1L_2|}. \tag{4.15}$$

For arbitrary integers L_1 and L_2 let us set

$$\Delta(L_1, L_2) = (\beta L_1-\alpha L_2)(\delta L_1-\gamma L_2).$$

We divide the sum (4.15) into two sums

$$D(d_1, d_2) \leqslant d\sum_{\substack{a\leqslant H\\ 2\nmid a}}\sum_{\substack{(u_1,u_2)\in D\\ \Delta(L_1,L_2)=0}}\frac{1}{|L_1L_2|}+d\sum_{\substack{a\leqslant H\\ 2\nmid a}}\sum_{\substack{(u_1,u_2)\in D\\ \Delta(L_1,L_2)\neq 0}}\frac{1}{|L_1L_2|} \tag{4.16}$$
$$= D'(d_1, d_2)+D''(d_1, d_2).$$

V. Estimate of $D'(d_1, d_2)$. We have either $\beta L_1 = \alpha L_2$ or $\delta L_1 = \gamma L_2$. Hence, by $(\alpha, \beta) = 1$, $(\gamma, \delta) = 1$ we have either $\alpha | L_1$ and $\beta | L_2$ or $\gamma | L_1$ and $\delta | L_2$. In each case

$$\sum_{\substack{(u_1,u_2)\in D\\ \Delta(L_1,L_2)=0}}\frac{1}{|L_1L_2|} \leqslant \Big(\frac{1}{\alpha\beta}+\frac{1}{\gamma\delta}\Big)\sum_{0<|l_1|,|l_2|<d}|l_1l_2|^{-1} \leqslant 4\Big(\frac{1}{\alpha\beta}+\frac{1}{\gamma\delta}\Big)\log^2 3d$$

and thus

$$D'(d_1, d_2) \leqslant 4\Big(\frac{1}{\alpha\beta}+\frac{1}{\gamma\delta}\Big)dH\log^2 3d. \tag{4.17}$$

Let us note here that

$$\sum_{d_1d_2<D} 4^{\nu(d_1d_2)}d_1d_2 \leqslant D^2\sum_{d<D}4^{\nu(d)}\frac{\tau(d)}{d} \ll D^2\log^8 D.$$

VI. Estimate of $D''(d_1, d_2)$ (on average). Using (4.4) and (4.5) one can easily show that

$$(a\alpha, d_1)(a\gamma, d_2)|L_1 \quad\text{and}\quad (a\beta, d_1)(a\delta, d_2)|L_2, \tag{4.18}$$

$$\beta L_1 \equiv \alpha L_2 (\operatorname{mod} d_1) \quad\text{and}\quad \delta L_1 \equiv \gamma L_2 (\operatorname{mod} d_2). \tag{4.19}$$

Two pairs $(u_1, u_2)\in D$ and $(u_1', u_2')\in D$ yield the same value of L_1 if and only if

$$u_1 \equiv u_1'\Big(\operatorname{mod}\frac{d_1}{(a\alpha, d_1)}\Big), \quad u_2 \equiv u_2'\Big(\operatorname{mod}\frac{d_2}{(a\gamma, d_2)}\Big)$$

and thus the number of pairs (u_1, u_2) which yield a particular value of L_1 is

$$(a\alpha, d_1)(a\gamma, d_2).$$

This together with (4.18) and (4.19) gives

$$\sum_{\substack{(u_1,u_2)\in D\\ \Delta(L_1,L_2)\neq 0}}\frac{1}{|L_1L_2|} \leqslant \sum_{L_1,L_2}{}'\frac{(a\alpha, d_1)(a\gamma, d_2)}{|L_1L_2|} \tag{4.20}$$

where dash denotes that summation is taken over all integers L_1, L_2 satisfying

$$0<|L_1|, |L_2|<d/2,$$
$$\Delta(L_1, L_2)\neq 0, \quad d|\Delta(L_1, L_2), \tag{4.21}$$
$$(a\alpha, d_1)(a\gamma, d_2)|L_1 \quad\text{and}\quad (a\beta, d_1)(a\delta, d_2)|L_2.$$

The estimate (4.20) is rather weak. However the conditions (4.21) say that for fixed L_1, L_2 there are only few $d = d_1d_2$ for which the sum (4.20) are not empty. Therefore, we can get from (4.20) a good estimate on average

$$\sum_{\substack{d_1 d_2 < D \\ (d_1,d_2)=1, 2\nmid d_1 d_2}} 4^{\nu(d_1 d_2)} D''(d_1, d_2) \leqslant D \sum_{\substack{a \leqslant H \\ 2\nmid a}} \sum_{\substack{0<|L_1|,|L_2|<D \\ \Delta(L_1,L_2)\neq 0}} |L_1 L_2|^{-1} \sum_{d|\Delta} 4^{\nu(d)}.$$

Obviously

$$|\Delta| < (\alpha+\beta)(\gamma+\delta)D^2$$

and thus

$$\sum_{d|\Delta} 4^{\nu(d)} < c_3(\eta)(\alpha\gamma D)^{\eta},$$

where η is an arbitrary positive number and $c_3(\eta)$ depends only on η. Finally we obtain

$$\sum_{\substack{d_1 d_2 < D \\ (d_1,d_2)=1, 2\nmid d_1 d_2}} 4^{\nu(d_1 d_2)} D''(d_1, d_2) \leqslant c_3(\eta)(\alpha\gamma D)^{\eta} DH(3\log D)^2. \tag{4.22}$$

VII. Completion of the proof. To complete the proof of Lemma 13 it remains to gather together estimates (4.2), (4.10), (4.12), (4.13), (4.17) and (4.22) and to choose the parameters D, H and η optimally. We obtain

$$\sum_{\substack{d_1 d_2 < D \\ (d_1,d_2)=1, 2\nmid d_1 d_2}} 4^{\nu(d_1 d_2)} |R(\mathfrak{B}; d_1, d_2)|$$

$$\ll (\delta\alpha\beta\gamma BDS)^{\eta} \left\{ \frac{S^2}{H} + DH + D^2 H \left(\frac{1}{\alpha\beta} + \frac{1}{\gamma\delta} \right) \right\}$$

where η is an arbitrary positive number and the constant involved in the symbol $\ll$ depends only on η.

From (3.1), (3.2) and (3.6) we get

$$\alpha > B/6S\left(\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}\right) \quad \text{and} \quad \beta > B/6S\left(\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}\right)$$

and thus

$$\frac{1}{\alpha\beta} + \frac{1}{\gamma\delta} < \frac{1}{\alpha^2} + \frac{1}{\gamma^2} < 72\frac{S^2}{B^2}\left(\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}\right)^2.$$

For $D < B$ we have also

$$\alpha\beta\gamma\delta BDS < B^6.$$

Therefore our sum is

$$\ll \left(\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}\right)^2 B^{6\eta} \left(\frac{S^2}{H} + DH + \left(\frac{DS}{H}\right)^2 H \right).$$

On putting $D = B^{1-4\varepsilon}$, $H = S^{2.5\varepsilon}$ and $\eta = \varepsilon/24$ we get

$$\ll \left(\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}\right)^2 B^{\varepsilon/4}(S^{2-2.5\varepsilon} + B^{1-1.5\varepsilon}) \ll \left(\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}\right)^2 (B+S^2)^{1-\varepsilon}.$$

This completes the proof of the lemma.

5. Completion of the proof of Theorem 3. Let ε be a positive number $< 1/24$. For

$$z = B^{1-9\varepsilon/2}, \quad \mathfrak{z} = z^{1/8}, \quad s = 2\frac{1-4\varepsilon}{1-8\varepsilon}$$

all requirements of Lemma 12 are satisfied. We have

$$\max(z^s, Bz^{-1}\mathfrak{z}^4) = \max(B^{1-4\varepsilon}, B^{3/4-2\varepsilon}) \leqslant B^{1-4\varepsilon}.$$

Hence from Central Lemma and Lemma 12 we get

$$S(\mathfrak{C}; P^-, B) > \frac{S^2}{2\pi\log B}\left\{-\log(1-8\varepsilon) + O\left((\log B)^{-1/14} + \varepsilon^{3/2}\right)\right\} - 3c(\varepsilon)\left(\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}\right)^2 (B+S^2)^{1-\varepsilon}, \tag{5.1}$$

where the constant in the symbol O is absolute. For

$$S > (\alpha+\gamma)^{1-\varepsilon} \tag{5.2}$$

we obtain by (3.6) $S < B < 3S^{2-\varepsilon/(1-\varepsilon)}$, $(B+S^2)^{1-\varepsilon} \leqslant 4S^{2-\varepsilon}$ and thus by (5.1)

$$S(\mathfrak{C}; P^-, B) > \frac{S^2}{\log S}\left[\frac{8\varepsilon}{4\pi} + O(\varepsilon^{3/2}) + O\left((\log S)^{-1/14}\right) - 12c(\varepsilon)\left(\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}\right)^2 \frac{\log S}{S^{\varepsilon}}\right],$$

where the constant in the symbol O is absolute. For

$$S > \left\{c(\varepsilon)\left(\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}\right)\exp(\varepsilon^{-20})\right\}^{3/\varepsilon} \tag{5.3}$$

the term in the square bracket equals

$$\frac{2\varepsilon}{\pi} + O(\varepsilon^{3/2}),$$

where the constant in the symbol O is absolute. Hence for sufficiently small $\varepsilon > 0$ we have

$$S(\mathfrak{C}; P^-, B) \geqslant \frac{\varepsilon}{2}\,\frac{S^2}{\log S}$$

provided (5.2) and (5.3). Theorem 3 follows now by putting $\tau = \varepsilon$ and $c = \big(c(\varepsilon)\exp(\varepsilon^{-20})\big)^{3/\varepsilon}$.

6. Proof of Theorem 2. We derive Theorem 2 from Theorem 3. It will be shown that

$$k = \tau/2 \quad \text{and} \quad K = 12/\tau + 2\log_2 8c$$

fulfils the requirements of Theorem 2, where τ and c are such as in Theorem 3. Let

$$|\theta - M/N| < N^{-2}, \quad (M, N) = 1,$$
$$N > (\theta+\theta^{-1})^{K}, \quad N^{-k} < k(N) < 1.$$

If we put $\alpha = M$, $\gamma = N$ and $S = \frac{k(N)}{4}N$ then all assumptions of Theorem 3 will be satisfied. For, it is enough to show two inequalities

$$S > 2(\alpha+\gamma)^{1-\tau}, \tag{6.1}$$

$$S > 2c\left(\frac{\alpha}{\gamma}+\frac{\gamma}{\alpha}\right)^{3/\tau}. \tag{6.2}$$

Proof of (6.1). We have

$$S \geqslant \tfrac{1}{4}N^{1-k} = \tfrac{1}{4}N^{1-\tau/2}.$$

On the other hand

$$2(\alpha+\gamma)^{1-\tau} = 2\left(\frac{\alpha}{\gamma}+1\right)^{1-\tau} N^{1-\tau} < 2(2+\theta)N^{1-\tau} < 4(\theta+\theta^{-1})N^{1-\tau}.$$

Hence

$$2(\alpha+\gamma)^{1-\tau}S^{-1} < 16(\theta+\theta^{-1})N^{-\tau/2} \leqslant 16(\theta+\theta^{-1})N^{-5/K} \leqslant 16(\theta+\theta^{-1})^{-4} \leqslant 1.$$

This completes the proof of (6.1).

Proof of (6.2). We have

$$S \geqslant \tfrac{1}{4}(\theta+\theta^{-1})^{(1-\tau/2)K} \geqslant \tfrac{1}{4}(\theta+\theta^{-1})^{K/2}.$$

On the other hand

$$2c\left(\frac{\alpha}{\gamma}+\frac{\gamma}{\alpha}\right)^{3/\tau} < 2c\big(2(\theta+\theta^{-1})\big)^{3/\tau}.$$

Hence

$$2c\left(\frac{\alpha}{\gamma}+\frac{\gamma}{\alpha}\right)^{3/\tau} S^{-1} < 8c\cdot 2^{3/\tau}(\theta+\theta^{-1})^{3/\tau-K/2} < 8c\cdot 2^{6/\tau-K/2} < 1.$$

This completes the proof of (6.2).

Applying Theorem 3 we get

$$\sum_{(m,n)\in\mathfrak{B}} b^*(m)b^*(n) \geqslant k\frac{(k(N)N)^2}{16\log N}.$$

On the other hand we have

$$\sum_{(m,n)\in\mathfrak{B}} b^*(m)b^*(n) < \sum_{\substack{0<n<k(N)N^2\\ |m-\theta n|<k(N)\\ (m,n)=1}} b^*(m)b^*(n)$$

because each pair $(m, n)\in\mathfrak{B}$ satisfies the conditions

$$0 < n \leqslant (\gamma+\delta)S \leqslant 3\gamma S < k(N)N^2,$$

$$|m-\theta n| \leqslant \left|m - \frac{\alpha}{\gamma}n\right| + \left|\theta - \frac{\alpha}{\gamma}\right| n < \frac{t}{\gamma} + \frac{n}{\gamma^2} < \frac{4S}{\gamma} = k(N),$$

$$(m, n) = 1.$$

This completes the proof of Theorem 2.

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Received on 20. 6. 1975
and in revised form on 10. 3. 1976 (734)