ACTA ARITHMETICA
XXXIII (1977)

Elementary methods in the theory of L-functions, VIII
Real zeros of real L-functions

by

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1. In papers I [4], II [5], IV [6] and V [7] of this series we discussed the problem of real zeros of real L-functions, and among other things we proved the theorems of Hecke [1], Landau [1], Page [3], Siegel [8], Walfisz [10], and Tatuzawa [9]. The aim of this paper is to prove these theorems in a new, simple, unique way. Differently from the former papers of this series, we shall also use some complex analysis, namely the notion of analytic continuation (and Cauchy inequality for the derivative, but this is avoidable). Without this, we do not use any special knowledge from number theory or analysis(1). We note that until now there was no unique treatment for these problems (though on the one hand the theorems of Hecke and Siegel, on the other hand the theorems of Landau and Page were proved similarly). The former proofs are also more complicated (especially Tatuzawa's theorem) and they use deep results of complex analysis. It is interesting that without any hand computation, we also improve the best constants in case of Landau's and of Page's theorem. (Regarding Page's theorem this improvement is also contained in paper V [7], where it is proved in another way.)

2. THEOREM 1 (Siegel–Walfisz). *For an arbitrary* $\varepsilon > 0$ *there is an ineffective constant* $c(\varepsilon)$ *such that for a real primitive character* $\chi \pmod D$, $L(s, \chi)$ *has no zero in the interval* $[1 - c(\varepsilon)D^{-\varepsilon}, 1]$.

THEOREM 2 (Hecke). *If* χ *is a real non-principal character* $\pmod D$, *and* $L(s, \chi)$ *has no zero in the interval* $[1-\beta, 1]$ *with* $0 < \beta \leqslant 1/\log D$, *then*

$$L(1, \chi) > \big(1 + o(1)\big)e^{-3/2}\beta.$$

THEOREM 3 (Siegel). *For an arbitrary* $\varepsilon > 0$, *there is an ineffective constant* $c'(\varepsilon)$ *such that for a real primitive character* $\chi \pmod D$

$$L(1, \chi) > c'(\varepsilon)D^{-\varepsilon}.$$

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(1) Except for the class number formula in Theorem 7 to show the consequence of Theorem 6 for the class number of imaginary quadratic fields.

Theorems 1 and 3 are also true for real non-principal characters $\chi \pmod D$ because if χ is induced by the primitive $\chi_1 \pmod{D_1}$, then $L(s,\chi)$ and $L(s,\chi_1)$ have the same zeros for $\operatorname{Re} s > 0$; $D_1 \leqslant D$, further

$$L(1,\chi) = L(1,\chi_1)\prod_{p|D}\left(1-\frac{\chi_1(p)}{p}\right) \gg \frac{L(1,\chi_1)}{\log\log D}.$$

THEOREM 4 (Page). *If χ is a real non-principal character* $\pmod D$, *then $L(s,\chi)$ has at most one, simple zero in the interval*

$$\left[1-\frac{c}{\log D};\,1\right]$$

where $c = 1+o(1)$ if we do not use the Pólya–Vinogradov inequality; and $c = 2+o(1)$ if we use it.

(This was proved with these constants in paper V, where we noticed that with Burgess inequality we get even $c = 4+o(1)$. The former best result, $c = 0.28$ is due to Miech [2].)

THEOREM 5 (Landau). *If $\chi_1 \neq \chi_2$ are real primitive characters* $\pmod{D_1}$ *and* $\pmod{D_2}$ *resp. and for real δ_1, δ_2*

$$L(1-\delta_1,\chi_1) = L(1-\delta_2,\chi_2) = 0$$

then

$$\max(\delta_1,\delta_2) > \frac{c'}{\log D_1 D_2},$$

where $c' = \frac{1}{2}+o(1)$ if we do not use the Pólya–Vinogradov inequality; $c' = 1+o(1)$ if we use it.

(The former best result $c' = 0.1$ is due to Miech [2].)

THEOREM 6 (Tatuzawa). *If $0<\varepsilon\leqslant 1/5$, χ is a real primitive character* $\pmod D$, *where $D \geqslant D_0$ (absolute effective constant) then*

$$L(s,\chi)\neq 0 \quad \text{for} \quad s\in[1-\varepsilon/7D^{\varepsilon},1] \tag{2.1}$$

and

$$L(1,\chi) > \varepsilon/35D^{\varepsilon}, \tag{2.2}$$

with the possible exception not more then one D, and one real primitive character $\pmod D$.

THEOREM 7 (Tatuzawa). *If $0<\varepsilon\leqslant 1/5$, $-D<0$ is a fundamental discriminant, $D > D_0$ (absolute effective constant), then for the class number*

$h(-D)$ of the imaginary quadratic field with discriminant $-D$

$$h(-D) > \frac{\varepsilon}{35\pi}D^{1/2-\varepsilon} \tag{2.3}$$

with the possible exception of at most one fundamental discriminant.

If $h\geqslant 1$, $D \geqslant \big(2000h(\log h+10)\big)^2$ and $D > D_0$ (absolute effective constant) then

$$h(-D) > h \tag{2.4}$$

with the possible exception of at most one fundamental discriminant.

3. First we shall prove 3 lemmata.

LEMMA 1. *Let θ_i $(i = 1,2,\ldots,j)$ be number-theoretic functions, for which $\theta_i(n) = O(1)$ and for an arbitrary n*

$$\Big|\sum_{d=1}^{n}\theta_i(d)\Big| \leqslant A_i. \tag{3.1}$$

Let further

$$L_i(1) = L(1,\theta_i) = \sum_{n=1}^{\infty}\frac{\theta_i(n)}{n}, \qquad \prod_{i=1}^{j}A_i = A,$$

$$f(m) = \sum_{d_0d_1\ldots d_j=m}\theta_1(d_1)\ldots\theta_j(d_j). \tag{3.2}$$

Then for an arbitrary $y\geqslant A$ we have

$$\sum_{m\leqslant y}f(m) = y\left\{\prod_{i=1}^{j}L_i(1)+O\left(\left(\frac{A}{y}\right)^{1/(j+1)}\log^{j-1}y\right)\right\} \tag{3.3}$$

(where the constant in the O symbol is depending only on j).

Proof. We shall prove by complete induction with respect to the number j, that for $z<y$

$$\sum_{m\leqslant y}\ \sum_{\substack{m=d_0d_1\ldots d_j\\ d_j>z}}\theta_1(d_1)\ldots\theta_j(d_j) \ll A_j\log^{j-1}y\cdot\frac{y}{z}. \tag{3.4}$$

(3.4) is true for $j=1$, as in this case, applying Abel's inequality, from (3.1) we get:

$$\sum_{m\leqslant y}\ \sum_{\substack{m=d_0d_1\\ d_1>z}}\theta_1(d_1) = \sum_{z<d_1\leqslant y}\theta_1(d_1)\sum_{\substack{d_0|m\\ m\leqslant y}}1 \ll A_1\frac{y}{z}. \tag{3.5}$$

Further if (3.4) is true for $j-1$, then

$$\sum_{m\le y}\ \sum_{m=d_0d_1\dots d_j,\,d_j>z}\theta_1(d_1)\dots\theta_j(d_j) \tag{3.6}$$

$$=\sum_{d_1\le y/z}\theta_1(d_1)\sum_{n\le y/d_1}\ \sum_{n=d_0d_2\dots d_j,\,d_j>z}\theta_2(d_2)\dots\theta_j(d_j)$$

$$\ll\sum_{d_1\le y/z}A_j\frac{y}{d_1z}\log^{j-2}\frac{y}{d_1}\ll A_j\frac{y}{z}\log^{j-1}y,$$

i.e. (3.4) is true also for j, thus (3.4) is proved. For the proof of (3.3) we note that by Abel's inequality from (3.1) we have

$$\sum_{d>z}\frac{\theta_i(d)}{d}\ll\frac{A_i}{z} \tag{3.7}$$

and thus

$$L_i(1)=\sum_{d=1}^{A_i}\frac{\theta_i(d)}{d}+\sum_{d>A_i}\frac{\theta_i(d)}{d}\ll\log A_i+O(1)\ll\log A_i. \tag{3.8}$$

We shall prove (3.3) also by complete induction according to j. (3.3) is true for $j=1$, as in consequence of (3.4) and (3.7) we have

$$\sum_{m\le y}\ \sum_{m=d_0d_1}\theta_1(d_1)=\sum_{d_1\le\sqrt{yA_1}}\theta_1(d_1)\sum_{\substack{d_1\mid m\\ m\le y}}1+\sum_{m\le y}\ \sum_{\substack{m=d_0d_1\\ d_1>\sqrt{yA_1}}}\theta_1(d_1) \tag{3.9}$$

$$=\sum_{d_1\le\sqrt{yA_1}}\theta_1(d_1)\frac{y}{d_1}+O\Big(\sum_{d_1\le\sqrt{yA_1}}1\Big)+O\Big(A_1\frac{y}{\sqrt{yA_1}}\Big)$$

$$=yL_1(1)+O\Big(y\frac{A_1}{\sqrt{A_1y}}\Big)+O(\sqrt{A_1y})+O(\sqrt{A_1y}).$$

Further if (3.3) is true for $j-1$, then applying (3.4), (3.7) and (3.8) we get with a $z\le y$ to be choosen later

$$\sum_{m\le y}\ \sum_{m=d_0d_1\dots d_j}\theta_1(d_1)\dots\theta_j(d_j) \tag{3.10}$$

$$=\sum_{d_j\le z}\theta_j(d_j)\sum_{\substack{n=d_0d_1\dots d_{j-1}\\ n\le y/d_j}}\theta_1(d_1)\dots\theta_{j-1}(d_{j-1})+\sum_{m\le y}\ \sum_{\substack{m=d_0d_1\dots d_j\\ d_j>z}}\theta_1(d_1)\dots\theta_j(d_j)$$

$$=\sum_{d_j\le z}\theta_j(d_j)\frac{y}{d_j}\prod_{i=1}^{j-1}L_i(1)+O\Big(\sum_{d_j\le z}\Big(\prod_{i=1}^{j-1}A_i^{1/j}\Big)\Big(\frac{y}{d_j}\Big)^{1-1/j}\log^{j-2}\frac{y}{d_j}\Big)+$$

$$+O\Big(A_j\log^{j-1}y\cdot\frac{y}{z}\Big)$$

$$=y\Big(\prod_{i=1}^{j-1}L_i(1)\Big)L_j(1)+O\Big(y\prod_{i=1}^{j-1}L_i(1)\frac{A_j}{z}\Big)+$$

$$+O\Big(y\prod_{i=1}^{j-1}A_i^{1/j}\Big(\frac{z}{y}\Big)^{1/j}\log^{j-1}y\Big)+O\Big(A_j\log^{j-1}y\cdot\frac{y}{z}\Big)$$

$$=y\Big\{\prod_{i=1}^{j}L_i(1)+O\Big(\log^{j-1}y\Big(\frac{A_j}{z}+\Big(\frac{\prod_{i=1}^{j-1}A_iz}{y}\Big)^{1/j}\Big)\Big)\Big\}.$$

Now choosing $z=\Big(yA_j^j\big(\prod_{i=1}^{j}A_i\big)^{-1}\Big)^{1/(j+1)}$ $(\le y)$ we get (3.3) for j, which proves Lemma 1.

Now we prove 2 lemmata for Dirichlet series whose coefficients $f(m)$ satisfy an equality of type (3.3).

LEMMA 2. *Let* $F(s)=\sum_{m=1}^{\infty}f(m)m^{-s}$ *where*

$$H(y)=\sum_{m\le y}f(m)=y\big(a+\nu(y)\big)=y\Big(a+O\Big(\Big(\frac{A}{y}\Big)^{1/(j+1)}\log^{j-1}y\Big)\Big) \tag{3.11}$$

with an $a\ne 0$, $A>0$, $j\ge 1$ *and for an arbitrary* $y\ge A$.

Then $F(s)$ *can be analitically continued in the half-plane* $\sigma=\operatorname{Re}s>1-1/(j+1)$ *except a single pole in* $s=1$, *and we have for* $\sigma>1-1/(j+2)$

$$\sum_{m\le x}\frac{f(m)}{m^s}=F(s)+\frac{a}{1-s}x^{1-s}+O\Big(|s|\Big(\frac{A}{x}\Big)^{1/(j+1)}x^{1-\sigma}\log^{j-1}x\Big) \tag{3.12}$$

(*where the constant in* O *symbol depends only on* j).

Proof. For $\sigma>1$ we can write

$$F(s)=\sum_{m\le x}\frac{f(m)}{m^s}+\sum_{m>x}\frac{f(m)}{m^s}. \tag{3.13}$$

Using (3.11) we have for $\sigma>1$

$$(3.14)\qquad \sum_{m>x}\frac{f(m)}{m^s}=-\frac{H(x)}{x^s}+s\int_x^\infty\frac{H(t)}{t^{s+1}}\,dt$$

$$=-\frac{ax}{x^s}+s\int_x^\infty\frac{at}{t^{s+1}}\,dt-\frac{\nu(x)x}{x^s}+s\int_x^\infty\frac{\nu(t)t}{t^{s+1}}\,dt$$

$$=\frac{a}{s-1}x^{1-s}-\nu(x)x^{1-s}+s\int_x^\infty\frac{\nu(t)}{t^s}\,dt.$$

Here the first term is analytic in the whole plane except a single pole in $s=1$, the second term is analytic in the whole plane, and the third one is by (3.11) analytic for $\sigma>1-1/(j+1)$ and for $\sigma>1-1/(j+2)$

$$(3.15)\qquad \int_x^\infty\frac{\nu(t)}{t^s}\,dt=O(A^{1/(j+1)}\log^{j-1}x\cdot x^{-1/(j+1)+1-\sigma})$$

holds. Lemma 2 now follows from (3.11), (3.13), (3.14) and (3.15).

Now applying the Cauchy inequality for the function

$$G(s)=F(s)+\frac{a}{1-s}x^{1-s}-\sum_{m\leqslant x}\frac{f(m)}{m^s}$$

with $r=1/(j+2)(j+3)$ we get

LEMMA 3(²). *If the function $F(s)$ satisfies the conditions of Lemma 2, then for $x\geqslant A$, $\sigma\geqslant 1-1/(j+3)$ we have*

$$(3.16)\qquad -\sum_{m\leqslant x}\frac{f(m)\log m}{m^s}$$

$$=F'(s)+\frac{a}{1-s}x^{1-s}\left(\frac{1}{1-s}-\log x\right)+O\left(|s|\left(\frac{A}{x}\right)^{1/(j+1)}\log^{j-1}x\cdot x^{1-\sigma}\right)$$

(*where the constant in the O symbol depends only on j*).

4. Henceforth let χ,χ_1 and χ_2 $(\chi_1\neq\chi_2)$ be real non-principal (and in case of χ_1 and χ_2 primitive) characters $(\bmod D)$, $(\bmod D_1)$ and $(\bmod D_2)$ resp. Let

$$F_1(s)=F(s,\chi)=\zeta(s)L(s,\chi),$$
$$F_2(s)=F(s,\chi_1,\chi_2)=\zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2).$$

Then Lemma 1 (and so Lemmas 2 and 3) are valid for $F_i(s)=\sum_{m=1}^{\infty}f_i(m)m^{-s}$

(²) We can get Lemma 3 without using the Cauchy inequality, directly from (3.11) applying the same method as in Lemma 2.

in case of $F(s,\chi)$ with $A=D$, $j=1$, and in case of $F(s,\chi_1,\chi_2)$ with $A=(D_1D_2)^2$, $j=3$. Further $f_i(1)=1$, $f_i(m)\geqslant 0$ and $f_i(l^2)\geqslant 1$. This follows from the Euler product considering

$$\log F(s,\chi)=\sum_p\sum_{r=1}^{\infty}\frac{1+\chi(p^r)}{p^{rs}r}$$

and

$$\log F(s,\chi_1,\chi_2)=\sum_p\sum_{r=1}^{\infty}\frac{\big(1+\chi_1(p^r)\big)\big(1+\chi_2(p^r)\big)}{p^{rs}r}$$

where the coefficients are for odd r non-negative, and for even r positive.

Now we turn to the proofs of Theorems 1–7.

Proof of Theorem 1. Let $0<\varepsilon<1/5$ and χ_1 be a real primitive character $(\bmod D_1)$ for which with a real $\gamma\leqslant\varepsilon/3$, $L(1-\gamma,\chi_1)=0$. (If such χ_1 does not exist, Theorem 1 is obviously true.) Henceforth let this character χ_1 (depending only on ε) be fixed. Let χ_2 be any real primitive character $(\bmod D_2)$ $(\chi_2\neq\chi_1)$ for which $L(1-\delta,\chi_2)=0$ with a real $\delta\leqslant\gamma$. (If such χ_2 does not exist, Theorem 1 is obviously true.) Let us regard

$$F(s)=F(s,\chi_1,\chi_2)=\sum_{m=1}^{\infty}f(m)m^{-s}$$

for real $s\geqslant 1-1/20$. If we set $x=A^{3/2}=(D_1D_2)^3$ then the error term in Lemma 2 is $o(1)$, and as $f(1)=1$, $f(m)\geqslant 0$, if for a real τ with $0<\tau\leqslant 1/20$, $F(1-\tau)=0$ then we have by Lemma 2

$$(4.1)\qquad \big(1+o(1)\big)\sum_{m\leqslant x}\frac{f(m)}{m^{1-\tau}}=\frac{a}{\tau}x^{\tau}.$$

Applying this with $\tau=\delta$ and $\tau=\gamma$ we have by $\delta\leqslant\gamma$

$$(4.2)\qquad \big(1+o(1)\big)\frac{a}{\gamma}x^{\gamma}=\sum_{m\leqslant x}\frac{f(m)}{m^{1-\gamma}}\geqslant\sum_{m\leqslant x}\frac{f(m)}{m^{1-\delta}}=\big(1+o(1)\big)\frac{a}{\delta}x^{\delta}$$

and so

$$(4.3)\qquad \delta\geqslant\frac{\big(1+o(1)\big)\gamma}{x^{\gamma}}\geqslant\frac{\big(1+o(1)\big)\gamma}{(D_1D_2)^{3\cdot\frac{\varepsilon}{3}}}=\frac{\big(1+o(1)\big)\gamma}{D_1^{\varepsilon}}\cdot\frac{1}{D_2^{\varepsilon}}=\frac{c(\varepsilon)}{D_2^{\varepsilon}}.\ \blacksquare$$

Proof of Theorem 2. Let us apply Lemma 2 for

$$F(s)=F(s,\chi)=\sum_{m=1}^{\infty}f(m)m^{-s}$$

with $x=A^{3/2}=D^{3/2}$, $s=1-\beta$, where $0<\beta\leqslant 1/\log D\leqslant 1/10$. Then the error term is $o(1)$; thus in consequence of our assumption and $L(1)\geqslant 0$

we have $L(1-\beta) > 0$ and so $F(1-\beta) < 0$. Thus considering $f(1) = 1$, $f(m) \geqslant 0$ we have

$$1+o(1) \leqslant \big(1+o(1)\big)\sum_{m\leqslant x}\frac{f(m)}{m^{1-\beta}} = F(1-\beta)+\frac{a}{\beta}x^{\beta} \tag{4.4}$$

$$< \frac{a}{\beta}x^{\beta} = \frac{a}{\beta}D^{\frac{3}{2}\beta} \leqslant \frac{e^{3/2}a}{\beta} = \frac{e^{3/2}L(1,\chi)}{\beta}. \ \blacksquare$$

Theorem 3 is the immediate consequence of Theorems 1 and 2.

Proof of Theorems 4 and 5. If the function $F(s)$ satisfies the conditions of Lemma 3, further $a > 0$, $f(m) \geqslant 0$ and $f(4) \geqslant 1$, then $F(s)$ has at most one, simple zero in the interval $\left[1-\frac{1-o(1)}{\log A}, 1\right]$.

Namely, if we choose $x = A^{1+\varepsilon}$ where $\varepsilon > 0$, $A > A_0(\varepsilon, j)$ and if $1-1/\log x \leqslant s \leqslant 1$, then the error term in Lemma 3 is $o(1)$, so we have by (3.16):

$$F'(s) = -\sum_{m\leqslant x}\frac{f(m)\log m}{m^s} - \frac{a}{1-s}x^{1-s}\left(\frac{1}{1-s}-\log x\right)+o(1) \tag{4.5}$$

$$\leqslant -\frac{\log 4}{4}+o(1) < 0.$$

Applying this for $F(s) = F(s,\chi)$ we get Theorem 4 (in case of the trivial $A = D$ with $c = 1+o(1)$; in case of the Pólya–Vinogradov inequality $A = 2\sqrt{D}\log D$ with $c = 2+o(1)$) for $F(s) = F(s, \chi_1, \chi_2)$ we get Theorem 5 (with the constants $c' = \frac{1}{2}+o(1)$ and $c' = 1+o(1)$ resp.)

Proof of Theorem 6. Let D_1 be the minimal modulus ($\geqslant D_0$ absolute constant) for which there exists a real primitive character χ_1 with a Siegel zero $1-\gamma$, where $\gamma < \varepsilon/7D_1^{\varepsilon}$. Let $D_2 \geqslant D_1$ another modulus, χ_2 a real primitive character (mod D_2) ($\chi_2 \neq \chi_1$) for which $L(1-\delta, \chi_2) = 0$. Let $D = D_1D_2$, $x = D^3$ ($\leqslant D_2^6$). We shall use that $\tau x^{-\tau}$ is maximal for $\tau = 1/\log x$, and is monotonically decreasing for $\tau > 1/\log x$. We shall further use that by Theorem 5 if $D_2 \geqslant D_0$, $D_1 \geqslant D_0$ then

$$\max(\gamma, \delta) > \frac{1}{3\log D}. \tag{4.6}$$

Thus in case of $\delta \geqslant \gamma$ we have

$$\delta > \frac{1}{3\log D} > \frac{1}{2e\log D} \geqslant \frac{\varepsilon/2}{2D^{\varepsilon/2}} > \frac{\varepsilon}{7D_2^{\varepsilon}}. \tag{4.7}$$

Let us assume $\delta \leqslant \gamma$. Then

$$\frac{\varepsilon}{6} > \gamma > \frac{1}{3\log D} = \frac{1}{\log x}. \tag{4.8}$$

Now let us regard $F(s) = F(s, \chi_1, \chi_2)$. From (4.3) for $D_2 \geqslant D_1 \geqslant D_0$ using (4.8) and $x^{1/6} \leqslant D_2$ we get

$$\delta \geqslant \frac{\big(1+o(1)\big)\gamma}{x^{\gamma}} \geqslant \frac{6}{7}\cdot\frac{\gamma}{x^{\gamma}} \geqslant \frac{6}{7}\,\frac{\varepsilon/6}{x^{\varepsilon/6}} \geqslant \frac{\varepsilon}{7D_2^{\varepsilon}}. \ \blacksquare \tag{4.9}$$

As $\frac{\varepsilon}{7D^{\varepsilon}} \leqslant \frac{1}{7e\log D} < \frac{1}{\log D}$ and $e^{3/2} < 5$ on applying Theorem 2 (2.2) follows directly from (4.9).

Proof of Theorem 7. Applying Dirichlet's class number formula

$$h(-D) = \frac{\sqrt{D}}{\pi}L(1,\chi_D) \qquad \left(\chi_D(n) = \left(\frac{-D}{n}\right), D > 4\right)$$

from (2.2) we get (2.3).

If $D \geqslant [2000h(\log h+10)]^2$ and $D \geqslant D_0$ then applying (2.3) with $\varepsilon = [2(\log h+10)]^{-1}$ with the possible exception of at most one fundamental discriminant $-D$, we get

$$h(-D) \geqslant \frac{1}{70\pi(\log h+10)}\sqrt{D}^{\,1-\frac{1}{\log h+10}} \tag{4.10}$$

$$\geqslant \frac{1}{70\pi(\log h+10)}2000^{0.9}h^{1-\frac{1}{\log h}}(\log h+10)^{1-\frac{1}{\log h+10}}$$

$$= \frac{2000^{0.9}}{70\pi e(\log h+10)^{\frac{1}{\log h+10}}}h \geqslant \frac{2000^{0.9}}{70\pi e\sqrt[10]{10}}h > h. \ \blacksquare$$

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Received on 30. 12. 1975 (803

Acta Arithmetica
XXXIII (1977)

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