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On character sums and the non-vanishing for $s > 0$ of Dirichlet L-series belonging to real odd characters χ

by

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1. Introduction. Let χ be a real non-principal character mod k. If

$$\sum_{n=1}^{x} \chi(n) \geqslant 0 \quad \text{for all } x \tag{1.1}$$

it follows by partial summation that

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \text{ has no real zeros in the interval } 0 < s < 1, \tag{1.2}$$

and

$$L(1,\chi) > c \quad \text{where } c \text{ is some positive absolute constant} > 2/3. \tag{1.3}$$

At the present time it is not known if there are infinitely many real primitive characters χ for which (1.2) holds. On the other hand, it has been shown that if χ is a real primitive character mod k then $\overline{\lim_{k\to\infty}} \frac{L(1,\chi)}{\log\log k} > 0$ ([2], [8]), but it is unknown if the k's for which (1.3) holds have a non-zero density in the sequence of positive integers.

The results of our numerical investigations concerning the primes $p \equiv 3 \pmod 4$ for which (1.1) holds suggest that these primes possess a positive limiting frequency in the sequence of all rational primes $\equiv 3 \pmod 4$. Our results in this connection are presented in Section 2 of this paper. In the third section we have given a brief account of related recent work and open problems on character sums. The final section consists of tables displaying pertinent computational results.

2. In this section we assume χ is a real primitive character mod k, where k is prime, and thus we may take $\chi(n)$ to be the Legendre symbol $\left(\frac{n}{k}\right)$.

First, we note that if $k = q$ is prime and $q \equiv 1 \pmod 4$ then (1.1) cannot hold because of the identity

$$\sum_{n=1}^{m}\left(\frac{n}{q}\right)+\sum_{n=1}^{q-1-m}\left(\frac{n}{q}\right)=0 \quad \text{for} \quad 0\leqslant m\leqslant q-1.$$

We investigated the condition

$$\sum_{n=1}^{x}\left(\frac{n}{q}\right)\geqslant 0 \quad \text{for} \quad x=1,2,\ldots,\frac{q-1}{2} \tag{2.1}$$

for primes $q \equiv 1 \pmod 4$ and found that (2.1) does not hold if $q \leqslant 43\,000$ except for $q = 5, 13, 37$. Elementary considerations and the identity

$$\sum_{1\leqslant n<q/6}\left(\frac{n}{q}\right)=0 \quad \text{for} \quad q\equiv 13 \pmod{24}$$

due to Johnson and Mitchell, [7], show that if (2.1) holds for q then

$$\left(\frac{2}{q}\right)=\left(\frac{5}{q}\right)=+1 \quad \text{and} \quad \left(\frac{3}{q}\right)=\left(\frac{7}{q}\right)=-1.$$

It would be interesting to determine if 37 is the last prime $\equiv 1 \pmod 4$ for which (2.1) holds.

For the remainder of this section we let p denote a prime $\equiv 3 \pmod 4$ and we concern ourselves with those p which satisfy (1.1). Let $|A|$ denote the cardinality of the set A, let $\pi(y; 4, 3)$ be the number of $p \leqslant y$ and set

$$B(y)=\left\{p \mid\ p\leqslant y \text{ and } \sum_{n=1}^{m}\left(\frac{n}{p}\right)\geqslant 0 \text{ for all } m\right\}$$

and

$$\beta(y)=\frac{|B(y)|}{\pi(y;4,3)}.$$

Our computational results on $\beta(y)$, summarized in Table I, have led us to propose the following conjectures:

CONJECTURE 1. $\lim_{y\to\infty}|B(y)| = +\infty$.

CONJECTURE 2. $\lim_{y\to\infty}\beta(y) > 0$.

It would be interesting to determine even if $\lim_{y\to\infty}\beta(y)$ exists.

Although we have been unable to obtain any lower bound for $\underline{\lim}_{y\to\infty}\beta(y)$ other than zero we can more readily obtain a non-trivial upper bound as the next theorem shows.

THEOREM 2.1.

$$\overline{\lim_{y\to\infty}}\,\beta(y)\leqslant .44.$$

Proof. Let $r_1 < r_2 < \ldots < r_k$ be all the primes $\leqslant x$, let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ where $\varepsilon_i = \pm 1$ for $1 \leqslant i \leqslant k$, and for $n = r_1^{a_1} r_2^{a_2} \ldots r_k^{a_k}$ an integer $\leqslant x$ we define $\chi_\varepsilon(n) = \varepsilon_1^{a_1}\varepsilon_2^{a_2}\ldots\varepsilon_k^{a_k}$. Note that if p is a prime such that $\left(\frac{r_i}{p}\right) = \varepsilon_i$ for $1 \leqslant i \leqslant k$ then $\chi_\varepsilon(n) = \left(\frac{n}{p}\right)$ for $n \leqslant x$. Now, set

$$W_k=\left\{\varepsilon=(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k) \mid\ \varepsilon_i=\pm 1 \text{ for } 1\leqslant i\leqslant k \text{ and } \sum_{n=1}^{m}\chi_\varepsilon(n)\geqslant 0 \text{ for all } m\leqslant x\right\}$$

and

$$P_k(y)=\left\{p \mid\ p\leqslant y \text{ and there is an } \varepsilon \text{ in } W_k \text{ such that } \left(\frac{r_i}{p}\right)=\varepsilon_i \text{ for } 1\leqslant i\leqslant k\right\}$$

$$=\left\{p \mid\ p\leqslant y \text{ and } \sum_{n=1}^{m}\left(\frac{n}{p}\right)\geqslant 0 \text{ for } m\leqslant x\right\}.$$

Clearly then

$$B(y)\subseteq P_k(y) \quad \text{for all } y \text{ and } k$$

and

$$\overline{\lim_{y\to\infty}}\,\beta(y)\leqslant \lim_{y\to\infty}\frac{|P_k(y)|}{\pi(y;4,3)} \quad \text{for all } k.$$

It follows from the prime number theorem for arithmetic progressions, the Chinese remainder theorem, and the law of quadratic reciprocity that, for each ε in W_k,

$$\left|\left\{p \mid\ p\leqslant y \text{ and } \left(\frac{r_i}{p}\right)=\varepsilon_i \text{ for } 1\leqslant i\leqslant k\right\}\right|$$

$$=\prod_{i=2}^{k}\left(\frac{r_i-1}{2}\right)\left(\frac{1}{\varphi(8r_2r_3\ldots r_k)}\,\frac{y}{\log y}+o\left(\frac{y}{\log y}\right)\right)$$

and hence that

$$\mu_k \overset{\text{def}}{=} \overline{\lim_{y\to\infty}}\,\frac{|P_k(y)|}{\pi(y;4,3)}=\lim_{y\to\infty}\frac{|P_k(y)|}{\pi(y;4,3)}=\frac{|W_k|}{2^k}.$$

Now μ_k, for any specific value of k, provides an upper bound for $\overline{\lim}_{y\to\infty}\beta(y)$; the larger the k the better the bound since $\{\mu_k\}$ is monotone decreasing.

Computation of the μ_k's is more readily accomplished in terms of the t_k's below.

Let

$$T_k(m) = \left\{\varepsilon \mid \sum_{n=1}^{i} \chi_\varepsilon(n) \geqslant 0 \text{ for } i \leqslant m-1 \text{ and } \sum_{n=1}^{m} \chi_\varepsilon(n) < 0\right\}$$

and set

$$T_{\pi(m)}(m) = T(m) \quad \text{where } \pi(m) \text{ is the number of primes } \leqslant m.$$

Thus, for example,

$$T_k(3) = \{(-1, -1, \varepsilon_3, \ldots, \varepsilon_k) \mid \varepsilon_i = \pm 1 \text{ for } 3 \leqslant i \leqslant k\},$$
$$T(7) = \{(1, -1, -1, -1), (-1, 1, -1, -1)\},$$

and

$$T(2l) = \emptyset \quad \text{for} \quad l = 0, 1, 2, \ldots$$

We have

$$t_k \stackrel{\text{def}}{=} \Big|\bigcup_{m=1}^{x} T_k(m)\Big| = \sum_{m=1}^{x} |T_k(m)| = \sum_{m=1}^{x} 2^{k-\pi(m)} |T(m)|$$

and

$$\mu_k = \frac{|W_k|}{2^k} = 1 - \frac{t_k}{2^k}.$$

At least for small m, computation of $|T(m)|$ is fairly straightforward. We have included in Table II the values of t_k and $t_k/2^k$ for $2 \leqslant k \leqslant 17$ and we may now complete the proof of Theorem 2.1 by observing that $\mu_{17} = 1 - t_{17}/2^{17} \leqslant .44$.

We close this section with a conjecture slightly stronger than Conjecture 1.

CONJECTURE 3. *Let $r_1 < r_2 < \ldots < r_m$ be the first m primes. If p is the least prime $\equiv 3 \pmod 4$ for which* $\left(\frac{r_1}{p}\right) = \left(\frac{r_2}{p}\right) = \ldots = \left(\frac{r_m}{p}\right) = +1$ *then*

$$\sum_{n=1}^{x} \left(\frac{n}{p}\right) \geqslant 0 \quad \textit{for all } x.$$

The values of $p = p(m)$ for $1 \leqslant m \leqslant 10$ are given in Table III and we have found Conjecture 3 to be valid in each of these cases.

3. In this section χ denotes a real non-principal character mod k. Recently, S. Chowla, P. Hartung, and M. J. deLeon, [5], proposed the following

HYPOTHESIS J. *To every odd real primitive character $\chi \bmod k$ there exists a real principal character $\chi_0 \bmod k'$, such that*

$$\sum_{n \leqslant x} \chi(n)\chi_0(n) \geqslant 0 \quad \textit{for all } x.$$

G. Purdy, [12], verified Hypothesis J for $k < 800000$ apart from 3 possible exceptions and his results together with those of M. Low [11], imply that $L(s, \chi) \neq 0$ for $0 < s < 1$ and χ a real odd primitive character mod k for $k < 800000$. H. L. Montgomery has suggested proving a converse to this hypothesis, namely:

If $L(s, \chi) \neq 0$ for $0 < s < 1$ then there is a character χ', induced by χ, such that

$$\sum_{n \leqslant x} \chi'(n) \geqslant 0 \quad \textit{for all } x \geqslant 0.$$

Let $\Lambda(n) = \log p$ if $n = p^a$ and $\Lambda(n) = 0$ otherwise. S. Knapowski and P. Turán, [9], have shown that if $L(s, \chi) \neq 0$ for $0 < s < 1$ and all characters $\chi \bmod k$ then

$$\sum_{\substack{n \equiv 1 \pmod k \\ n \leqslant x}} \Lambda(n) - \sum_{\substack{n \equiv l \pmod k \\ n \leqslant x}} \Lambda(n) + cx^{1/2-\varepsilon}$$

cannot be of constant sign as $x \to \infty$ for any fixed value of c, positive or negative, $\varepsilon > 0$, $(l, k) = 1$, $l \neq 1$. In the same fashion we can show that if $L(s, \chi) \neq 0$ for $0 < s < 1$ and χ is a real, odd, primitive character mod k then

(3.1) $$\sum_{n \leqslant x} \chi(n)\Lambda(n) + cx^{1/2-\varepsilon}$$

cannot be of constant sign.

Thus, if $k < 800000$ and χ odd and primitive, then expression (3.1) changes sign infinitely often as $x \to \infty$ for every fixed real value of c and $\varepsilon > 0$.

J. Littlewood, [10], showed that the assumption of the extended Riemann Hypothesis leads to

(3.2) $$L(1, \chi) = O(\log\log k)$$

and

(3.3) $$L(1, \chi) = \Omega(\log\log k).$$

S. Chowla, [4], proved that (3.3) holds without any assumption. However, to date there has been no improvement of $L(1, \chi) = O(\log k)$ and it would be of interest to be able to replace the big "O" here by a little "o".

Finally, we would like to mention results relating the size of $L(1, \chi)$ with the sums

$$\sum_{n=1}^{x} \chi(n).$$

A. Gel'fond, [6], showed that if χ is a real nonprincipal character mod k and $q = \max_{x\geqslant 1} \left|\sum_{n\leqslant x} \chi(n)\right|$ then

$$|L(1, \chi)| > \frac{c}{q \log q} \quad \text{for} \quad q > q_0.$$

E. Bombieri, [3], proved that if χ is a real non-principal character mod k and $\sum_{n\leqslant x} \chi(n) \geqslant -q$ then

$$L(1, \chi) \geqslant \frac{\pi}{16(q+1)}.$$

Table I

y	$\pi(y; 4, 3)$	$\|B(y)\|$	$\beta(y)$
9643	600	248	.41333
21191	1200	448	.37333
33391	1800	652	.36222
46171	2400	849	.35375
59167	3000	1053	.35100
72547	3600	1255	.34861
86143	4200	1455	.34643
99787	4800	1639	.34146
113567	5400	1828	.33852
127363	6000	2013	.33550
141707	6600	2203	.33379
156131	7200	2385	.33125
170603	7800	2569	.32936
175727	8000	2626	.32825

Table II

k	t_k	$t_k/2^k$	k	t_k	$t_k/2^k$
2	1	.250	10	503	.491
3	2	.250	11	1028	.502
4	6	.375	12	2106	.514
5	13	.406	13	4294	.524
6	27	.422	14	8698	.531
7	58	.453	15	17874	.545
8	119	.465	16	36457	.556
9	248	.484	17	73481	.561

Table III

m	$p(m)$ (1)	m	$p(m)$ (1)
1	7	6	1559
2	23	7	5711
3	71	8	10559
4	311	9	18191
5	479	10	31391

(1) Here $p(m)$ denotes the least prime $p \equiv 3 \pmod 4$ for which $\left(\frac{2}{p}\right) = \left(\frac{3}{p}\right) = \ldots = \left(\frac{p_m}{p}\right) = +1$, where p_m denotes the mth pirme.

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