ACTA ARITHMETICA
XXXIII (1977)

On the size of prime factors of integers

by

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1. Introduction. For n a positive integer and $y > 1$ a real number we define

$$d(y,n) = \max\{d \mid d|n;\ p|d \text{ and } p \text{ prime} \Rightarrow p < y\}$$

and $\gamma(y,n) = \big(\log d(y,n)\big)/\log y$.

In this paper we make more precise some results of Erdös [4] on the size of $\gamma(p,n)$ where p is a prime factor of n. For real $u \geq 0$ define

$$\varphi(u,n) = \sum_{\substack{p|n \\ \gamma(p,n)>u}} 1,$$

we then have the following "Turán's method" result:

THEOREM 1. *For $\varphi(u,n)$ as defined above and $\tau(u)$ defined below we have*

(i) $$\sum_{n\leq x} \varphi(u,n) = x\big(1+o(1)\big)\tau(u)\log_2 x + o(x),$$

(ii) $$\sum_{n\leq x} \big(\varphi(u,n)-\tau(u)\log_2 n\big)^2 = x\Big\{o\Big(\big(\tau(u)\log_2 x\big)^2\Big) + O\big(1+\tau(u)\log_2 x\big)\Big\}$$

uniformly in u as $x \to \infty$.

Here and elsewhere in this paper $\log_k n$ denotes the k-fold iterated logarithm.

The function $\tau(u)$ is defined as follows. Let $\varrho(u)$ be the real valued function defined by the following properties

(1) $$\begin{cases} \varrho(u) = 0 \ (u < 0); \ \varrho(u) = 1 \ (0 \leq u \leq 1), \\ u\varrho'(u) = -\varrho(u-1) \ (u > 1); \ \varrho(u) \text{ is continuous for } u > 0. \end{cases}$$

De Bruijn [3] has studied the asymptotic behaviour of $\varrho(u)$ in some detail and in particular has shown that

(2) $$\varrho(u) = \exp\{-u\log u - u\log_2 u + O(u)\} \quad \text{as} \quad u \to \infty.$$

We define

$$\tau(u) = e^{-\gamma}\int_u^\infty \varrho(v)\,dv.$$

For large u, $\tau(u)$ behaves very like $\varrho(u)$, in fact we have

(3) $$\tau(u) = \big(e^{-\gamma}+o(1)\big)(u+1)\varrho(u+1) \quad \text{as} \quad u\to\infty.$$

A proof of (3) goes as follows

$$\tau(u) = e^{-\gamma}\int_{u+1}^\infty \varrho(v-1)\,dv$$

$$= -e^{-\gamma}\int_{u+1}^\infty v\varrho'(v)\,dv \quad \text{by (1)}$$

$$= -e^{-\gamma}[v\varrho(v)]_{u+1}^\infty + e^{-\gamma}\int_{u+1}^\infty \varrho(v)\,dv$$

which gives

$$\tau(u)-\tau(u+1) = e^{-\gamma}(u+1)\varrho(u+1).$$

Summing we get

$$\tau(u) = e^{-\gamma}(u+1)\varrho(u+1)+e^{-\gamma}\sum_{k=2}^\infty (u+k)\varrho(u+k).$$

It can easily be verified from De Bruijn's asymptotic formula for $\varrho(u)$ [3] that $\varrho(u+1)\leqslant \varrho(u)/u$ and (3) follows.

Put

$$P(n) = \max_{p|n}\gamma(p,n).$$

Erdös [4] has shown that for almost all n (i.e. on a sequence with asymptotic density 1)

(4) $$P(n) = \big(1+o(1)\big)\log_3 n/\log_4 n.$$

Using Theorem 1 we can obtain a more precise result than this. For $x > e^e$ we define $\xi(x)$ to be the root of

$$\tau(\xi)\log_2 x = 1.$$

THEOREM 2. *For almost all integers n*

$$P(n) = \xi(n)+o(1).$$

Proof. It follows from (2) and (3) that for any $\varepsilon > 0$

$$\tau\big(\xi(x)+\varepsilon\big)\log_2 x\to 0 \quad \text{and} \quad \tau\big(\xi(x)-\varepsilon\big)\log_2 x\to\infty \quad \text{as} \quad x\to\infty$$

but this, combined with the slow rate of growth of $\xi(n)$ and with Theorem 1,

implies that for almost all n

$$\varphi\big(\xi(n)+\varepsilon, n\big) = 0 \quad \text{and} \quad \varphi\big(\xi(n)-\varepsilon, n\big)\to\infty$$

and the result follows.

In the same paper Erdös outlined a proof that there exists a continuous function $\varphi(u)$ such that for fixed u and almost all n $\varphi(u,n) = \big(1+o(1)\big)\varphi(u)\log_2 n$. It follows at once from Theorem 1 that this result holds with $\varphi(u) = \tau(u)$ and in fact we have

THEOREM 3. *For almost all integers n*

$$\sup_{u\geqslant 0}\left|\frac{\varphi(u,n)}{\log_2 n}-\tau(u)\right|\to 0.$$

Proof. Let $\varepsilon > 0$, it is enough to show that for almost all n and all $u\geqslant 0$

$$|\varphi(u,n)/\log_2 n-\tau(u)| < \varepsilon.$$

Choose an integer $N > 0$ and a real number $A > 0$ to satisfy $\tau(A) < \varepsilon/4$ and $A/N < \varepsilon/2$. By Theorem 1 (i)

$$\sum_{n\leqslant x}\varphi(A,n)/\log_2 n \leqslant \tau(A)x+o(x)\leqslant(\varepsilon/3)x \quad \text{for large } x,$$

and so we can certainly say that for almost all n and for $u\geqslant A$

$$|\varphi(u,n)/\log_2 n-\tau(u)| < \varepsilon.$$

Next, by Theorem 1 (ii), we can say that for almost all n and for integer k with $0\leqslant k\leqslant N$

$$\left|\frac{\varphi(kA/N,n)}{\log_2 n}-\tau\left(\frac{kA}{N}\right)\right| < \frac{\varepsilon}{2}.$$

For $0\leqslant u < A$ there is some k such that $kA/N\leqslant u < (k+1)A/N$ and we have

$$|\varphi(u,n)/\log_2 n-\tau(u)| < \frac{\varepsilon}{2}+\tau(kA/N)-\tau\big((k+1)A/N\big) < \frac{\varepsilon}{2}+\frac{A}{N}\leqslant\varepsilon.$$

We can also determine the average value of $\gamma(p,n)$ for almost all n

$$\frac{1}{\log_2 n}\sum_{p|n}\gamma(p,n) = -\frac{1}{\log_2 n}\int_0^\infty u\,d\varphi(u,n)$$

integrating by parts

$$= \int_0^\infty \frac{\varphi(u,n)}{\log_2 n}\,du,$$

and it is not hard to show using Theorem 3 and Theorem 1 (i) that for almost all n

$$\int_0^\infty \frac{\varphi(u,n)}{\log_2 n}\,du = (1+o(1))\int_0^\infty \tau(u)\,du = (1+o(1))f'(0) = 1+o(1),$$

with $f(z)$ as defined in § 2. Hence for almost all integers n

$$\frac{1}{\log_2 n}\sum_{p|n}\gamma(p,n) = (1+o(1)).$$

If $y > w \geqslant 0$ are real numbers we use the notation $\sum_n^{w,y} a(n)$ to denote the sum of $a(n)$ only over n with no prime factors $< w$ or $\geqslant y$. We abbreviate $\sum^{0,y}$ by $\sum^y$. To prove Theorem 1 we have to estimate sums of the form

$$\sum_{a>y^u}^{w,y}\frac{1}{a}.$$

This could probably be done using the methods of de Bruijn ([1] and [2]), but instead we use a different method using Fourier transforms and a result borrowed from probability theory (Lemma 3). This has the advantage of being fairly self contained and also enables us to get better error terms than I was able to get using de Bruijn's methods. In § 2 we prove some lemmas and Theorem 1 (i), then in § 3 we complete the proof of Theorem 1.

2. For real u and real $y > 0$ we define

$$F_y(u) = \prod_{p<y}\left(1-\frac{1}{p}\right)\sum_{a\leqslant y^u}^{y}\frac{1}{a},$$

then for each y $F_y(u)$ is a distribution function with support in $(0,\infty)$. For complex z with $\operatorname{Re} z < \log y$ we define

$$f_y(z) = \int_0^\infty e^{uz}\,dF_y(u).$$

If we write the integral as a sum we see that

$$f_y(z) = \prod_{p<y}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{1-z/\log y}}\right)^{-1} \tag{5}$$

and so $f_y(z)$ is analytic for $\operatorname{Re} z < \log y$.

Finally we define

$$f(z) = e^{-\gamma}\int_0^\infty e^{uz}\varrho(u)\,du. \tag{6}$$

LEMMA 1.

$$f(z) = \exp\left\{\int_0^z \frac{e^s-1}{s}\,ds\right\}.$$

Proof.

$$f(z) = e^{-\gamma}\int_0^\infty e^{(u-1)z}\varrho(u-1)\,du$$

$$= -e^{-\gamma}e^{-z}\int_1^\infty e^{uz}u\varrho'(u)\,du \quad \text{by (1)}$$

$$= -e^{-\gamma}e^{-z}\frac{d}{dz}\int_0^\infty e^{uz}\varrho'(u)\,du \quad \text{as } \varrho'(u) = 0 \text{ for } 0<u<1$$

$$= -e^{-\gamma}e^{-z}\frac{d}{dz}\left\{[e^{uz}\varrho(u)]_0^\infty - z\int_0^\infty e^{uz}\varrho(u)\,du\right\}$$

$$= e^{-z}(zf'(z)+f(z)),$$

i.e.

$$\frac{f'(z)}{f(z)} = \frac{e^z-1}{z}.$$

Hence

$$f(z) = c\exp\left\{\int_0^z \frac{e^s-1}{s}\,ds\right\} = cg(z) \text{ say}.$$

Integrating (6) by parts gives

$$zcg(z) = zf(z) = e^{-\gamma}[e^{zu}\varrho(u)]_0^\infty - e^{-\gamma}\int_0^\infty e^{zu}\varrho'(u)\,du.$$

If we put $z = it$ and let $t\to\infty$ the integral on the right tends to 0 and we get

$$e^{-\gamma} = c\lim_{t\to\infty} itg(it) = ce^{-\gamma} \quad \text{(see [3] for example)}$$

and so $c = 1$.

If we put $z = 0$ in (6) we see that

$$e^{-\gamma}\int_0^\infty \varrho(u)\,du = 1$$

and so if we define $F(u) = e^{-\gamma}\int_{-\infty}^{u}\varrho(w)\,dw$ then $F(u)$ is a distribution function.

LEMMA 2. *For any complex number* z

(i) $$\lim_{y\to\infty} f_y(z) = f(z).$$

In addition for real t *and integer* $y \geqslant 3$ *with* $0 \leqslant t \leqslant \log y$

(ii) $$|f_y(it)-f(it)| = O(\log^{-1} y)\frac{t}{t+1},$$

(iii) $$\left|\frac{f_y(1+it)}{f_y(1)} - \frac{f(1+it)}{f(1)}\right| = O(\log^{-1} y)\frac{t}{t+1},$$

(iv) $$|f_y(1)-f(1)| = O(\log^{-1} y).$$

Proof. Let $\sigma = \operatorname{Re} z$ and $\log y > 2\sigma$. We will show that

(7) $$\frac{f'_y(z)}{f_y(z)} = \frac{e^z-1}{z} + O\left(\frac{e^\sigma}{\log y}\right).$$

If we integrate (7) and observe that $f_y(0) = f(0) = 1$ we see that $\log f_y(z) \to \log f(z)$ as $y\to\infty$ and (i) follows. The proofs of (ii), (iii) and (iv) are very similar and so we just prove (iii). Write

$$g_y(t) = \frac{f_y(1+it)f(1)}{f(1+it)f_y(1)}$$

then

$$\frac{g'_y(t)}{g_y(t)} = \frac{1}{i}\left(\frac{f'_y(1+it)}{f_y(1+it)} - \frac{f'(1+it)}{f(1+it)}\right) = O(\log^{-1} y) \quad \text{by (7).}$$

If we put

$$G_y(t) = \int_0^t \frac{g'_y(x)}{g_y(x)}\,dx$$

then

$$G_y(t) = O(t\log^{-1} y) \quad \text{and} \quad g_y(t) = \exp G_y(t).$$

Hence

$$g_y(t)-1 = G_y(t)\{(\exp G_y(t)-1)/G_y(t)\} = O(t\log^{-1} y)$$

for $t \leqslant \log y$. Multiplying by $f(1+it)/f(1)$ we get

$$\frac{f_y(1+it)}{f_y(1)} - \frac{f(1+it)}{f(1)} = tf(1+it)\,O(\log^{-1} y)$$

and (iii) follows as $tf(1+it)$ is easily shown to be bounded.

We now prove (7).

$$\frac{f'_y(z)}{f_y(z)} = \sum_{p<y}\frac{\log p}{\log y}p^{(z/\log y)-1}\left(1-p^{(z/\log y)-1}\right)^{-1}$$
$$= \frac{1}{\log y}\sum_{p<y}p^{(z/\log y)-1}\log p + O\left(\frac{1}{\log y}\sum_{p<y}\frac{\log p}{p^{2(1-\sigma/\log y)}}\right)$$
$$= \frac{1}{\log y}\sum_{p<y}p^{(z/\log y)-1}\log p + O\left(\frac{e^\sigma}{\log y}\right),$$

as the sum in brackets is convergent for $2\sigma < \log y$. If we write the sum above as an integral we get

$$\frac{f'_y(z)}{f_y(z)} = \frac{1}{\log y}\int_1^y x^{(z/\log y)-1}\log x\cdot d[\mathrm{li}\,x] + \frac{1}{\log y}\int_1^y x^{(z/\log y)-1}\log x\cdot d[\pi(x)-\mathrm{li}\,x] + O\left(\frac{e^\sigma}{\log y}\right),$$

where $\mathrm{li}\,x$ is the logarithmic integral

$$\lim_{\delta\to 0}\left(\int_0^{1-\delta}\frac{1}{\log t}\,dt + \int_{1+\delta}^{x}\frac{1}{\log t}\,dt\right)$$

and $\pi(x)$ is the number of primes $\leqslant x$. The first integral above is equal to

$$\frac{1}{\log y}\int_1^y x^{(z/\log y)-1}\,dx = \frac{e^z-1}{z}$$

as required.

If the second integral is integrated by parts it equals

$$\frac{1}{\log y}\left[x^{(z/\log y)-1}\log x\left(\pi(x)-\mathrm{li}\,x\right)\right]_1^y - \frac{1}{\log y}\int_1^y\left(\pi(x)-\mathrm{li}\,x\right)x^{(z/\log y)-2}O(\log x)\,dx$$
$$\ll \frac{e^\sigma}{\log y}\left\{\frac{\log y}{y}\left(\pi(y)-\mathrm{li}\,y\right) + \int_1^y\left(\pi(x)-\mathrm{li}\,x\right)\frac{\log x}{x^2}\,dx\right\},$$

but by the prime number theorem $|\pi(x)-\mathrm{li}\,x| \ll \dfrac{x}{(\log x)^2}$ and the result follows.

The next lemma is due to Esseen ([5], Theorem 2a). We have replaced the condition $|G'(u)| \leqslant M$ by $\big(G(u)-G(v)\big)/(u-v) \leqslant M$ but the proof is the same.

LEMMA 3. *If $G(u)$ and $H(u)$ are two distribution functions,*

$$g(t) = \int_{-\infty}^{\infty} e^{iut}\,dG(u), \qquad h(t) = \int_{-\infty}^{\infty} e^{iut}\,dH(u)$$

and $\big(G(u)-G(v)\big)/(u-v) \leqslant M$ for all u and v then

$$|G(u)-H(u)| \ll \int_0^T \frac{|g(t)-h(t)|}{t}\,dt + \frac{M}{T}$$

for all u and all $T > 0$.

We are now ready to obtain our estimate for $F_y(u)$.

LEMMA 4. *For integer $y \geqslant 2$ and real $u \geqslant 0$*

$$\prod_{p<y}\Big(1-\frac{1}{p}\Big)\sum_{a>y^u}^{y}\frac{1}{a} = \tau(u)+O\Big(\frac{e^{-u}|\log_2 y|}{\log y}\Big).$$

Proof. For $y = 2$ and $u > 0$ the left hand side is zero but the error term is greater than the main term. For $y \geqslant 3$ write

$$c_y = \int_0^{\infty} e^v dF_y(v) = f_y(1), \qquad c = \int_0^{\infty} e^v dF(v) = f(1),$$

$$G_y(u) = \int_{-\infty}^{u} e^v dF_y(v) \quad \text{and} \quad G(u) = \int_{-\infty}^{u} e^v dF(v).$$

Then $c_y^{-1}G_y(u)$ and $c^{-1}G(u)$ are distribution functions and

$$\int_0^{\infty} e^{iut}\,d\big(c_y^{-1}G_y(u)\big) = c_y^{-1}\int_0^{\infty} e^{u(1+it)}\,dF_y(u) = \frac{f_y(1+it)}{f_y(1)},$$

similarly

$$\int_0^{\infty} e^{it}\,d\big(c^{-1}G(u)\big) = \frac{f(1+it)}{f(1)}.$$

$G(u)$ clearly satisfies the condition of Lemma 3 with $M = \sup e^u\varrho(u)$, and so by Lemma 2 (iii) and Lemma 3

$$|c_y^{-1}G_y(u)-c^{-1}G(u)| \ll \frac{1}{\log y}\int_0^{\log y}\frac{dt}{t+1}+\frac{1}{\log y} \ll \frac{\log_2 y}{\log y}.$$

Lemma 2 (iv) implies that $c_y^{-1} = c+O(\log^{-1}y)$ and so, for $y \geqslant 3$ and $u \geqslant 0$,

(8) $$G_y(u) = G(u)+O(\log_2 y/\log y).$$

Now

$$\prod_{p<y}\Big(1-\frac{1}{p}\Big)\sum_{a>y^u}^{y}\frac{1}{a} = \int_u^{\infty} dF_y(v) = \int_u^{\infty} e^{-v}dG_y(v)$$

integrating by parts

$$= [e^{v}G_y(v)]_u^{\infty}+\int_u^{\infty} e^{-v}G_y(v)\,dv$$

$$= [e^{-v}G(v)]_u^{\infty}+\int_u^{\infty} e^{-v}G(v)\,dv+O(e^{-u}\log_2 y/\log y) \quad \text{by (8)}$$

$$= \int_u^{\infty} e^{-v}dG(v)+O(e^{-u}\log_2 y/\log y) = \tau(u)+O(e^{-u}\log_2 y/\log y)$$

as required.

Our final lemma in this section is due to de Bruijn [1].

LEMMA 5. *Let $\Phi(x, y)$ denote the number of integers $\leqslant x$ all of whose prime factors are $\geqslant y$, then*

$$\Phi(x, y)-1 = x\prod_{p<y}\Big(1-\frac{1}{p}\Big)\psi(x, y)$$

where

$$\psi(y^u, y) = 1+O(e^{-ua}) \quad \textit{for} \quad y \geqslant 2,\ u \geqslant 0$$

and a an absolute positive constant.

Proof. For $u \geqslant 1$ this follows at once from [1], 1.16. For $0 \leqslant u < 1$ the result is trivial as $\Phi(x, y) = 1$ if $x < y$.

We are now ready to prove Theorem 1 (i). If we write

$$\delta(u, p, n) = \begin{cases} 1 & \text{if} \quad p|n \text{ and } \gamma(p, n) > u, \\ 0 & \text{otherwise} \end{cases}$$

then

$$\sum_{n\leqslant x}\varphi(u, n) = \sum_{n\leqslant x}\sum_{p\leqslant x}\delta(u, p, n) = \sum_{p\leqslant x}\sum_{n\leqslant x}\delta(u, p, n) = \sum_{p\leqslant x}\sum_{a>p^u}^{p}\Phi\Big(\frac{x}{ap}, p\Big)$$

$$= \sum_{p\leqslant x} \sum_{a>p^u}^{p} \left(\Phi\left(\frac{x}{ap}, p\right) - 1\right) + N(u, x)$$

say, where $N(u, x)$ is the number of $n \leqslant x$ whose largest prime factor is $< n^{1/(1+u)}$. It is not hard to see that

$$(9) \qquad N(u, x) = \begin{cases} o(x) & \text{if} \quad u \geqslant \log_4 x, \\ O(x) = o\big(\chi\tau(u)\log_2 x\big) & \text{if} \quad u \leqslant \log_4 x. \end{cases}$$

Applying Lemma 5 we get

$$\sum_{n\leqslant x} \varphi(u, n) = x \sum_{p<x} \frac{1}{p} \prod_{p'<p} \left(1 - \frac{1}{p}\right) \sum_{a>p^u}^{p} \frac{1}{a} \psi\left(\frac{x}{ap}, p\right) + N(u, x).$$

If we abreviate $\sum_{p'<p} \left(1 - \frac{1}{p'}\right)$ by π_p we can write

$$(10) \qquad \sum_{n\leqslant x} \varphi(u, n) = xS_1 + xS_2 + N(u, x)$$

where

$$S_1 = \sum_{p\leqslant x} \frac{1}{p} \pi_p \sum_{a>p^u}^{p} \frac{1}{a} \quad \text{and} \quad S_2 = \sum_{p\leqslant x} \frac{1}{p} \pi_p \sum_{a>p^u}^{p} \frac{1}{a} \left(\psi\left(\frac{x}{ap}\right) - 1\right).$$

By Lemma 4

$$(11) \quad S_1 = \tau(u) \sum_{p\leqslant x} \frac{1}{p} + O(1) e^{-u} \sum_{p\leqslant x} \frac{\log_2 p}{p \log p} = \big(1 + o(1)\big)\tau(u)\log_2 x + O(e^{-u}).$$

If we consider separately the cases $u \geqslant \log_4 x$ and $u \leqslant \log_4 x$ we see at once that

$$(12) \qquad e^{-u} = o(1) + o\big(\tau(u)\log_2 x\big).$$

We now deal with S_2. $\psi(x, y)$ is bounded for all x, y and so we have

$$S_2 \ll S_3 + S_4 + S_5,$$

where, writing $x' = x^{1/(u+1+2\log_2 x)}$,

$$S_3 = \sum_{x'\leqslant p\leqslant x} \frac{1}{p} \pi_p \sum_{a>p^u}^{p} \frac{1}{a}, \quad S_4 = \sum_{p\leqslant x} \frac{1}{p} \pi_p \sum_{a>p^{u+\log_2 x}}^{p} \frac{1}{a}$$

and

$$S_5 = \sum_{p<x'} \frac{1}{p} \pi_p \sum_{p^u<a\leqslant p^{u+\log_2 x}}^{p} \frac{1}{a} e^{-a\log_2 x}.$$

$S_5 \ll \log_2 x/(\log x)^a = o(1)$ and it follows from Lemma 4 that $S_4 \ll \log_2 x/\log x = o(1)$. Also by Lemma 4

$$S_3 \ll \tau(u)\log(u + 1 + 2\log_2 x) + \sum_{x'\leqslant p<x} \frac{\log_2 p}{p\log p} = \tau(u)o(\log_2 x) + o(1)$$

as required.

We have shown that $S_2 = o\big(\tau(u)\log_2 x\big) + o(1)$ and this in conjunction with (9), (10), (11) and (12) completes the proof of Theorem 1 (i).

3. Theorem 1 (ii). Before we can complete the proof of Theorem 1 we need two more lemmas.

LEMMA 6. *For integer y, w with $y > w \geqslant 2$ and for all $u \geqslant 0$ we have*

$$\prod_{w\leqslant p<y} \left(1 - \frac{1}{p}\right) \sum_{a>y^u}^{w,y} \frac{1}{a} = \tau(u) + O\left(\frac{\log(v+1)}{v}\right)$$

with $v = (\log y)/\log w$.

Proof. If we define

$$F_{w,y}(u) = \prod_{w\leqslant p<y} \left(1 - \frac{1}{p}\right) \sum_{a\leqslant y^u}^{w,y} \frac{1}{a},$$

then $F_{w,y}(u)$ is a distribution function with characteristic function $f_y(it)/f_w(it/v)$. In the light of Lemma 3 it will be enough to show that

$$\int_0^v \left| \frac{f_y(it)}{f_w(it/v)} - f(it) \right| \frac{dt}{t} = O\left(\frac{\log(v+1)}{v}\right).$$

Clearly $f_w(it/v) = 1 + O(t/v)$ and $f_w(it/v)^{-1} = O(1)$ for $t \leqslant v$ and all z. Hence

$$\int_0^v \left| \frac{f_y(it)}{f_w(it/v)} - f(it) \right| \frac{dt}{t} \ll \int_0^v |f_y(it) - f_w(it/v)f(it)| \frac{dt}{t}$$

$$\ll \int_0^v |f_y(it) - f(it)| \frac{dt}{t} + \frac{1}{v} \int_0^v |f(it)t| \frac{dt}{t+1}$$

$$\ll \frac{1}{v} \int_0^v \frac{dt}{t+1} \ll \frac{\log(v+1)}{v}$$

by Lemma 2 (ii) and because $tf(it)$ is bounded.

LEMMA 7. *For all real u, w with $w > u > 0$ we have*

$$\sum_{k=1}^{\infty}\frac{1}{k}\Big\{F(u)-F\Big(u-\frac{w}{k}\Big)\Big\}=O\Big(\log\frac{w}{u}\Big)+O(1).$$

Proof.

$$\sum_{k=1}^{\infty}\frac{1}{k}\Big\{F(u)-F\Big(u-\frac{w}{k}\Big)\Big\}=\int_{1}^{\infty}\frac{1}{x}\Big\{F(u)-F\Big(u-\frac{w}{x}\Big)\Big\}dx+O(1)$$

$$=\int_{1}^{\infty}\int_{u-w/x}^{u}\frac{\varrho(v)}{x}\,dv\,dx+O(1)$$

changing the order of integration

$$=\int_{0}^{u}\varrho(v)\int_{1}^{w/(u-v)}x^{-1}\,dx\,dv+O(1)$$

$$=\int_{0}^{u}\log\Big(\frac{w}{u-v}\Big)\varrho(v)\,dv+O(1)$$

$$=\int_{0}^{u}\log\frac{w}{v}\,\varrho(u-v)\,dv+O(1)$$

$$\ll e^{-u}\int_{0}^{1}\log\frac{w}{v}\,e^{v}\,dv+e^{-u}\int_{1}^{u}\log\frac{w}{v}\,e^{v}\,dv+O(1)$$

using the fact that $\varrho(v)\ll e^{-v}$ and $\log\frac{w}{v}\geqslant 0$. The first integral above is $\ll e^{-u}\log w+O(1)$

$$=e^{-u}\log\frac{w}{u}+e^{-u}\log u+O(1)\ll\log\frac{w}{u}+O(1)$$

as required. Integrating the second integral by parts we get

$$e^{-u}\Big[\log\frac{w}{v}\,e^{v}\Big]_{1}^{u}+e^{-u}\int_{1}^{u}\frac{1}{v}\,e^{v}\,dv=\log\frac{w}{v}-e^{-u}\log w+O(1)\ll\log\frac{w}{u}+O(1)$$

as required.

The proof of Theorem 1 will follow easily if we prove that

$$\sum_{n\leqslant x}\varphi(u,n)^{2}=x\Big\{\big(1+o(1)\big)\big(\tau(u)\log_{2}x\big)^{2}+O\big(\tau(u)\log_{2}x\big)+O(1)\Big\}. \tag{13}$$

Now

$$\sum_{n\leqslant x}\varphi(u,n)^{2}=\sum_{n\leqslant x}\sum_{p\leqslant x}\sum_{q\leqslant x}\delta(u,p,n)\,\delta(u,q,n) \tag{14}$$

$$=2\sum_{p<q\leqslant x}\sum_{n\leqslant x}\delta(u,p,n)\,\delta(u,q,n)+\sum_{n\leqslant x}\varphi(u,n)$$

$$=2\sum_{p<q\leqslant x}\sum_{a>p^{u}}^{p}\sum_{b>q^{u/ap}}^{p,q}\Phi\Big(\frac{x}{abpq},q\Big)+xO\big(\tau(u)\log_{2}x+1\big)$$

$$=2x\sum_{p<q\leqslant x}\frac{1}{pq}\pi_{p}\sum_{a>p^{u}}^{p}\frac{1}{a}\pi_{q}/\pi_{p}\sum_{b>q^{u/ap}}^{p,q}\frac{1}{b}\psi\Big(\frac{x}{abpq},q\Big)+E,$$

where E denotes a function of x and u bounded by the error term in (13).

First we deal with the term arising from the error term in Lemma 6, and show that

$$\sum_{p<q\leqslant x}\frac{1}{pq}\pi_{p}\sum_{a>p^{u}}^{p}\frac{1}{a}\log\Big(\frac{\log q}{\log p}+1\Big)\frac{\log p}{\log q}=O\big(\tau(u)\log_{2}x\big)+O(1). \tag{15}$$

The sum above is clearly bounded by

$$\sum_{p\leqslant x}\frac{1}{p}\pi_{p}\sum_{a>p^{u}}^{p}\frac{1}{a}\sum_{1\leqslant v\leqslant\log x/\log p}\frac{\log(v+1)}{v}\sum_{p^{v}<q\leqslant p^{v+1}}\frac{1}{q}$$

$$\ll\sum_{p\leqslant x}\frac{1}{p}\pi_{p}\sum_{a>p^{u}}^{p}\frac{1}{a}\sum_{v=1}^{\infty}\frac{\log(v+1)}{v^{2}}=O\big(\tau(u)\log_{2}x+1\big)$$

as in the proof of part (i).

We next deal with the sum where $q^{u}/ap<b\leqslant q^{u}$. Because of Lemma 6 and (15) we can replace

$$\pi_{q}/\pi_{p}\sum_{q^{u}/ap<b\leqslant q^{u}}^{p,q}\frac{1}{b}\quad\text{by}\quad F\Big(u-\frac{\log a}{\log q}-\frac{\log p}{\log q}\Big)-F(u),$$

and so it is enough to get a bound for

$$\sum_{p<q\leqslant x}\frac{1}{pq}\pi_{p}\sum_{a<p^{u}}^{p}\frac{1}{a}\Big\{F\Big(u-\frac{\log a}{\log q}-\frac{\log p}{\log q}\Big)-F(u)\Big\}.$$

If we replace the sum over a by an integral and take the sum over q inside we get

$$\sum_{p<x}\frac{1}{p}\int_u^\infty \sum_{p<q\leqslant x}\Big\{F\Big(u-(w+1)\frac{\log p}{\log q}\Big)-F(u)\Big\}dF_p(w)$$

$$\leqslant \sum_{p<x}\frac{1}{p}\int_u^\infty \sum_{k=1}^{\log x}\sum_{p^k<q\leqslant p^{k+1}}\frac{1}{q}\Big\{F\Big(u-\frac{w+1}{k}\Big)-F(u)\Big\}dF_p(w)$$

$$\ll \sum_{p<x}\frac{1}{p}\int_u^\infty \sum_{k=1}^{\infty}\frac{1}{k}\Big\{F\Big(u-\frac{w+1}{k}\Big)-F(u)\Big\}dF_p(w)$$

$$\ll \sum_{p<x}\frac{1}{p}\int_u^\infty \log\frac{w+1}{u}\,dF_p(w)+\sum_{p<x}\frac{1}{p}\int_u^\infty dF_p(w)$$

by Lemma 7. The second term above is just

$$\sum_{p<x}\frac{1}{p}\pi_p\sum_{a>p^u}^{p}\frac{1}{a}=O\big(\tau(u)\log_2 x+1\big).$$

If $G_p(w)$ is defined as in the proof of Lemma 4 then the first term above equals

$$(16)\qquad \sum_{p<x}\frac{1}{p}\int_u^\infty \log\frac{w+1}{u}e^{-w}dG_p(w).$$

If we integrate by parts, apply (8), and then reverse the partial integration we find that (16) equals

$$\sum_{p<x}\frac{1}{p}\int_u^\infty \log\frac{w+1}{u}e^{-w}dG(w)+O\Big(\sum_{p<x}\frac{\log_2 p}{p\log p}\Big)$$
$$=\log_2 x\int_u^\infty \log\frac{w+1}{u}\,dF(w)+O(1).$$

Now

$$\int_u^\infty \log\frac{w+1}{u}\,dF(w)=-\int_u^\infty \log\frac{w+1}{u}\,d\tau(w)$$
$$=-\Big[\log\frac{w+1}{u}\tau(u)\Big]_u^\infty+\int_u^\infty\frac{\tau(w)}{w+1}dw\ll\frac{1}{u}\tau(u)+\int_u^\infty\varrho(w+1)\qquad\text{by (3)}$$
$$\ll\tau(u)$$

as required.

We are now left with

$$2x\sum_{p<q\leqslant x}\frac{1}{pq}\pi_p\sum_{a>p^u}^{p}\frac{1}{a}\pi_q/\pi_p\sum_{b>q^u}^{p,q}\frac{1}{b}\psi\Big(\frac{x}{abpq},q\Big)=S_1+S_2\ \text{say},$$

where

$$S_1=2x\sum_{p<q\leqslant x}\frac{1}{pq}\pi_p\sum_{a>p^u}^{p}\frac{1}{a}\pi_q/\pi_p\sum_{b>q^u}^{p,q}\frac{1}{b}$$

and

$$S_2=2x\sum_{p<q\leqslant x}\frac{1}{pq}\pi_p\sum_{a>p^u}^{p}\frac{1}{a}\pi_q/\pi_p\sum_{b>q^u}^{p,q}\frac{1}{b}\Big(\psi\Big(\frac{x}{abpq}\Big)-1\Big).$$

By (15) we can replace $\pi_q/\pi_p\sum_{b>q^u}^{p,q}\frac{1}{b}$ in S_1 by $\tau(u)$ and we get

$$S_1=2x\tau(u)\sum_{p<q\leqslant x}\frac{1}{pq}\pi_p\sum_{a>p^u}^{p}\frac{1}{a}+E$$
$$=2x\big(\tau(u)\big)^2\sum_{p<q\leqslant x}\frac{1}{pq}+O\Big(2x\tau(u)\sum_{p<q\leqslant x}\Big(\frac{\log_2 p}{qp\log p}\Big)\Big)+E$$
$$=x\big(\tau(u)\log_2 x\big)^2+E,$$

which gives us the main term.

S_2 can be dealt with in the same way as its counterpart in Theorem 1. We write

$$x'=x^{1/(2u+2+3\log_2 x)}$$

and divide the sum up as follows:

(i) $x'<q\leqslant x$,

(ii) $a>p^{u+\log_2 x}$ or $b>q^{u+\log_2 x}$,

(iii) $q<x'$, $p^u<a\leqslant p^{u+\log_2 x}$, $q^u<b\leqslant q^{u+\log_2 x}$.

We then find that $S_2=xo\big((\tau(u)\log_2 x)^2\big)+o(x)$ and this completes the proof of (13).

Theorem 1 (ii) now follows easily.

$$\sum_{n\leqslant x}\big(\varphi(u,n)-\tau(u)\log_2 n\big)^2$$
$$=\sum_{n\leqslant x}\varphi(u,n)^2-2\tau(u)\sum_{n\leqslant x}\varphi(u,n)\log_2 n+\tau(u)^2\sum_{n\leqslant x}(\log_2 n)^2$$
$$=x\Big\{o\big((\tau(u)\log_2 x)^2\big)+O\big(\tau(u)\log_2 x\big)+O(1)\Big\}+$$
$$+2\tau(u)\sum_{n\leqslant x}\varphi(u,n)(\log_2 x-\log_2 n)$$

but the sum on the right is easily shown to be $o(x)$ and the result follows.

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Received on 18. 8. 1975 (7

ACTA ARITHMETICA
XXXIII (1977)

On character sums and the non-vanishing for $s > 0$ of Dirichlet L-series belonging to real odd characters χ

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1. Introduction. Let χ be a real non-principal character mod k. If

$$\sum_{n=1}^{x} \chi(n) \geqslant 0 \quad \text{for all } x \tag{1.1}$$

it follows by partial summation that

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \text{ has no real zeros in the interval } 0 < s < 1, \tag{1.2}$$

and

$$L(1, \chi) > c \quad \text{where } c \text{ is some positive absolute constant} > 2/3. \tag{1.3}$$

At the present time it is not known if there are infinitely many real primitive characters χ for which (1.2) holds. On the other hand, it has been shown that if χ is a real primitive character mod k then $\overline{\lim\limits_{k\to\infty}} \frac{L(1, \chi)}{\log\log k} > 0$ ([2], [8]), but it is unknown if the k's for which (1.3) holds have a non-zero density in the sequence of positive integers.

The results of our numerical investigations concerning the primes $p \equiv 3 \pmod 4$ for which (1.1) holds suggest that these primes possess a positive limiting frequency in the sequence of all rational primes $\equiv 3 \pmod 4$. Our results in this connection are presented in Section 2 of this paper. In the third section we have given a brief account of related recent work and open problems on character sums. The final section consists of tables displaying pertinent computational results.

2. In this section we assume χ is a real primitive character mod k, where k is prime, and thus we may take $\chi(n)$ to be the Legendre symbol $\left(\frac{n}{k}\right)$.