Elementary methods in the theory of L-functions, VII
Upper bound for $L(1, \chi)$

by

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1. If $\chi$ is a real nonprincipal character (mod $D$), then the upper bound which one can give for $L(1, \chi)$ is closely connected with the upper bound of

$$S_\chi = \max_{1 \leq \delta < \Lambda} \left| \sum_{n=2}^{b} \chi(n) \right|.$$

Using the trivial $S_\chi \leq D$ one can easily prove $L(1, \chi) \leq \log D + O(1)$, by means of the Pólya–Vinogradov inequality $S_\chi \leq O(\sqrt{D} \log D)$

$$L(1, \chi) \leq (\sqrt{\log D}) \log D$$

(see Pólya [7]) can be proved.

If $D = p$ is a prime, $\chi$ a real nonprincipal character (mod $p$), then making use of Burgess's inequality [1]

$$\left| \sum_{n=\mathcal{N}+1}^{\mathcal{N}+H} \chi(n) \right| \leq \epsilon H \quad \text{for} \quad H > p^\frac{1}{12}, \quad p > p_0(\epsilon),$$

S. Chowla [4] in 1964 proved the inequality

$$L(1, \chi_p) \leq (1 + o(1)) \log p.$$  

Burgess [2] showed in 1966 that

$$L(1, \chi_p) < 0.2456 \ldots \log p.$$  

Wirsing (unpublished) improved it to

$$L(1, \chi_p) < \frac{1}{2} (\sqrt{2} - 1 + o(1)) \log p \approx 0.207 \log p.$$  


$$L(1, \chi_p) < \frac{1}{2} \left(1 - \frac{1}{\sqrt{6}} + o(1)\right) \log p \approx 0.197 \log p.$$
Now we give an elementary proof of Stephens's result (using Burgess's inequality) generalizing it for real primitive characters, whose modulus is not necessarily prime, and improve (1.2) for real non-principal characters. Our result will follow from the following general theorem.

**Theorem 1.** If \( \theta \) is a completely multiplicative function, which takes only the values \( +1, 0, -1, x \) a real number for which
\[
\sum_{n \leq x} \theta(n) \ll x
\]
then
\[
\sum_{d \leq x} \frac{\theta(d)}{d} \ll 2 \left( 1 - \frac{1}{\sqrt{x}} + \delta \right) \log x
\]
where \( \delta = \delta(x, x) \rightarrow 0 \) if \( x \rightarrow \infty \) and \( x \rightarrow 0 \).

Theorem 1 is the best possible, because if we choose
\[
\theta(p) = \begin{cases} 
1 & \text{for } p \leq \omega^{1/2} \\
-1 & \text{for } p > \omega^{1/2} 
\end{cases}
\]
(\( p \) is a prime)
then it is easy to see that (1.8) is true with \( \epsilon = o(1) \) and that in (1.9) equality holds with \( \delta = o(1) \).

But Burgess [3] proved that if \( \chi \) is a nonprincipal character \( \text{mod } D \), then
\[
\left| \sum_{n=1}^{N+H} \chi(n) \right| \ll \epsilon H \quad \text{for} \quad H \geq D^{\delta/2}, \quad D > D_0(\epsilon)
\]
where if \( \chi \) is a primitive character then \( \tau_\chi = 1/4 \) and for an arbitrary \( \chi, \tau_\chi = 3/8 \). Thus using (1.10) we have by partial summation
\[
\left| \sum_{D^{\delta/2} < d \leq D} \frac{\chi(d)}{d} \right| \ll \epsilon \log D
\]
and using the trivial estimation \( B_\chi \leq D \) by means of Abel's inequality we get
\[
\left| \sum_{d \leq D} \frac{\chi(d)}{d} \right| \ll 1.
\]

So using Theorem 1 with \( \epsilon = D^{\delta/2} \) we have from (1.11) and (1.12)

**Theorem 2.** If \( \chi \) is a real primitive character \( \text{mod } D \), then
\[
L(1, \chi) \leq \frac{1}{2} \left( 1 - \frac{1}{\sqrt{\epsilon}} + o(1) \right) \log D
\]
if \( \chi \) is a real non-principal character \( \text{mod } D \), then
\[
L(1, \chi) \leq \frac{3}{4} \left( 1 - \frac{1}{\sqrt{\epsilon}} + o(1) \right) \log D.
\]

(1.13) is in the following sense the best possible for \( D = p \): If the least quadratic non-residue \( \text{mod } p \),
\[
N(p) \geq p^{1/4 - o(1)}
\]
then it is easy to see that in (1.13) the equality is valid. Thus any improvement of (1.13) is only possible if we improve Burgess’s theorem [1] concerning the least quadratic non-residue \( \text{mod } p \) to
\[
N(p) = O(p^\eta)
\]
with an \( \eta < \frac{1}{4\sqrt{\epsilon}} \).

The upper bound of \( L(1, \chi) \) is in connection with the class number and fundamental unit of quadratic fields. Using (1.4) S. Chowla ([4], [5]) proved that if \( p \) is a prime \( \equiv 1 \pmod{4} \), then for the class number \( h(p) \), and fundamental unit \( \epsilon > 1 \) of \( Q(\sqrt{p}) \) one has
\[
h(p) \ll (1 + o(1)) \log p
\]
and
\[
\epsilon \leq e^{(1+o(1))\sqrt{\log p}}.
\]

He also proved [3] that if \( p \) is a prime \( \equiv 3 \pmod{4} \), then for the class number \( h(-p) \) of \( Q(\sqrt{-p}) \)
\[
h(-p) \ll \left( \frac{1}{4\pi} + o(1) \right) \sqrt{p} \log p
\]
holds. If \( D \) or \( -D \), respectively is not a prime but a fundamental discriminant the best known upper bounds for class numbers of quadratic fields belonging to the discriminant \( D \) or \( -D \) respectively, are due to Landau [6], who proved the inequalities
\[
h(D) \leq (1 + o(1)) \sqrt{D} \quad (D > 0)
\]
and
\[
h(-D) \leq \left( \frac{1}{2\pi} + o(1) \right) \sqrt{-D} \log D \quad (-D < 0).
\]
Taking into account the well-known class number formulae

\begin{align}
(1.21) \quad 2h(D)\log e &= \sqrt{D}L(1, \chi) \quad (\chi(n) = \left(\frac{D}{n}\right), \quad D > 0), \\
(1.22) \quad h(-D) &= \frac{\sqrt{D}}{\pi} L(1, \chi) \quad (\chi(n) = \left(\frac{-D}{n}\right), \quad -D < -4)
\end{align}

and the inequality

\begin{align}
(1.23) \quad \varepsilon &\geq \frac{1}{2}(\sqrt{D} - 1),
\end{align}

Theorem 2 gives the following improvements of the results of S. Chowla and Landau ((1.16)-(1.20)):

**Theorem 3.** For the class number \( h(D) \) and for the fundamental real unit \( e > 1 \) of the real quadratic field belonging to the fundamental discriminant \( D > 0 \) the inequalities

\begin{align}
(1.24) \quad h(D) &\leq \frac{1}{2}\left(1 - \frac{1}{\sqrt{e}} + o(1)\right)\sqrt{D} \\
(1.25) \quad \varepsilon &\leq \frac{1}{2}\left(1 - \frac{1}{\sqrt{e}} + o(1)\right)\sqrt{D} \log D
\end{align}

hold.

**Theorem 4.** For the class number \( h(-D) \) of the imaginary quadratic field belonging to the fundamental discriminant \( D < 0 \) the inequality

\begin{align}
(1.26) \quad h(-D) &\leq \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{e}} + o(1)\right)\sqrt{D} \log D
\end{align}

holds.

2. To prove Theorem 1 first we note that if

\begin{align}
(2.1) \quad g(n) &= \sum_{d|n} \theta(d)
\end{align}

then as \( \theta(d) = O(1) \), we get

\begin{align}
(2.2) \quad \sum_{d|n} g(n) &= \sum_{d|n} \theta(d) \left[ \frac{n}{d} \right] = \omega \sum_{d|n} \frac{\theta(d)}{d} + O(\omega).
\end{align}

Let \( P, Q, T \) denote the sets of those primes, \( \leq x \), for which

\begin{align}
(2.3) \quad P = \{ p; \theta(p) = -1 \}, \quad Q = \{ q; \theta(q) = 0 \}, \quad T = \{ t; \theta(t) = 1 \}.
\end{align}

Let \( d(n) \) denote the number of divisors of \( n \).

Then we shall prove

**Lemma 1.** With the notations (2.1)-(2.3) we have

\begin{align}
(2.4) \quad \sum_{n \leq x} g(n) &= \sum_{n \leq x} d(n) - 2 \sum_{p \leq \sqrt{x}} \sum_{\eta \leq x} d\left(\frac{n}{p}\right) + 2 \sum_{p \leq \sqrt{x}} \sum_{\eta \leq x} \sum_{\eta \leq x} d\left(\frac{n}{pq}\right) + \\
&+ 2 \sum_{p \leq \sqrt{x}} \sum_{p \leq \sqrt{x}} \sum_{\eta \leq x} d\left(\frac{n}{pq}\right) - \sum_{q \leq x} \sum_{p \leq \sqrt{x}} d\left(\frac{n}{q}\right) + 2 \sum_{q \leq x} \sum_{p \leq \sqrt{x}} \sum_{\eta \leq x} d\left(\frac{n}{pq}\right)
\end{align}

Proof. Let \( c(n) d(n) \) be the sum of those terms on the right side which belong to the number \( n \) (i.e. the sum of those terms which have the form \( d(n/a) \)).

We can write \( n \) in the form \( n = ab \),

\begin{align}
(2.5) \quad m = p_1^{a_1} \cdots p_r^{a_r}, \quad b = q_1^{a_1} \cdots q_s^{a_s}, \quad a = t_1 \cdots t_r
\end{align}

where \( p_i \in P, q_i \in Q, t_i \in T \). Then we have

\begin{align}
(2.6) \quad \sigma(n) &= 1 - 2 \sum_{i=1}^r \frac{a_i}{a_i + 1} + 2 \sum_{i=1}^r \frac{a_i - 1}{a_i + 1} + 2 \sum_{i=1}^r \sum_{j=1}^r \frac{a_i a_j}{(a_i + 1)(a_j + 1)} - \\
&- \sum_{j=1}^s \frac{\beta_j}{\beta_j + 1} + 2 \sum_{j=1}^s \sum_{k=1}^s \frac{\beta_j a_k}{(\beta_j + 1)(a_k + 1)}
\end{align}

where

\begin{align}
(2.7) \quad A &= 2 \sum_{i=1}^r \frac{1}{a_i + 1} \left( a_i \sum_{j=1}^r \frac{a_j}{a_j + 1} - 1 \right) \\
(2.8) \quad B &= \sum_{j=1}^s \frac{\beta_j}{\beta_j + 1} \left( 2 \sum_{k=1}^s \frac{a_k}{a_k + 1} - 1 \right).
\end{align}

Now let us regard the following cases:

I. If \( r = 0 \), i.e. \( m = 1 \), then

\begin{align}
(2.9) \quad \sigma(n) &= 1 - \sum_{j=1}^s \frac{\beta_j}{\beta_j + 1} > \prod_{j=1}^s \frac{1}{\beta_j + 1} = \frac{1}{d(b)}.
\end{align}

If \( r \geq 1 \) then from (2.7) \( B > 0 \) and so \( \sigma(n) \gtrsim C \).

II. If \( r = 1 \), then

\begin{align}
(2.10) \quad C &= 1 - \frac{2}{a_1 + 1} = \frac{a_1 - 1}{a_1 + 1} \geq 0.
\end{align}
III. If \( r = 2 \), and \( a_1 = 1 \) or \( a_2 = 1 \), say \( a_1 = 1 \), then
\[
C = 1 - 2\left(\frac{1}{2} + \frac{1}{a_2 + 1}\right) + 2 \cdot 2 \cdot \frac{a_2}{a_2 + 1} = \frac{2(a_2 - 1)}{a_2 + 1} \geq 0.
\]

IV. If \( r = 2 \), and \( a_i \geq 2 \) for \( i = 1, 2 \) or \( r \geq 3 \), then for an arbitrary \( i \leq r \)
\[
a_i \sum_{j=1}^{r} \frac{a_j}{a_j + 1} \geq 1
\]
and so \( A \geq 0, C \geq 1 \).

On the other hand, one has
\[
(2.9) \quad g(n) = \sum_{d \mid n} \vartheta(d) = \prod_{p \mid n} \left(1 + \vartheta(p) + \cdots + \theta'(p)\right).
\]
Hence
\[
g(n) = g(a)g(b)g(m) = \begin{cases} d(a) & \text{if } m = p^2, \\ 0 & \text{if } m \neq p^2. \end{cases}
\]

So in the following cases we have:
I. \( g(n) = d(a) = \frac{d(n)}{d(b)} \leq o(n) d(n) \);
II. if \( a_i \) is odd, then \( 0 = g(n) \leq C \leq o(n) \leq o(n) d(n) \),
if \( a_i \) is even, then \( g(n) = d(a) \leq \frac{d(n)}{a_i + 1} \leq C d(n) \leq o(n) d(n) \);
III. \( 0 = g(n) \leq C \leq o(n) \leq o(n) d(n) \);
IV. \( g(n) \leq d(a) \leq d(n) \leq C d(n) \leq o(n) d(n) \).

Thus in all cases \( g(n) \leq o(n) d(n) \) which proves Lemma 1.

Now let \( S = \mathbb{P} \cup Q \), and for \( s \in S \), let
\[
(2.10) \quad s = \begin{cases} s & \text{if } s \in \mathbb{P}, \\ 2s & \text{if } s \in Q. \end{cases}
\]

Then using the well-known relations
\[
\sum_{n \leq x} \frac{d(n)}{n} = \sum_{m \leq x^{1/2}} \vartheta(m) = \frac{\vartheta}{2} \log \frac{\vartheta}{2} + O\left(\frac{\vartheta}{2}\right),
\]
\[
\sum_{p \leq x} \frac{1}{p} = o(\log x), \quad \sum_{p \leq x} \sum_{\text{prime } p' \text{ prime}} \frac{1}{pp'} = o(\log x),
\]
If we add to the right side of (2.4)
\[
\frac{1}{2} \sum_{q \leq x} \sum_{qq' \leq x} \frac{\vartheta}{qq'} \log \frac{\vartheta}{qq'} \geq 0,
\]
then we have with the notation (2.10) the following

**Corollary.** We have
\[
(2.11) \quad \sum_{n \leq x} \vartheta(n) \leq o(1) (1 - \frac{2}{\log x} - U + o(1))
\]
where
\[
(2.12) \quad U = \sum_{s \in S} \frac{\log s}{s} - \sum_{s \in S} \sum_{s' \in S} \frac{1}{ss'} \log \frac{\vartheta}{ss'}.
\]

Thus we have to estimate \( U \) from below.
Here we shall use the supposition \( \sum_{n \leq x} \vartheta(n) \leq \vartheta (x) \) from which
\[
o(1) x \geq \sum_{d \leq x} \vartheta(d) = \vartheta(x) - 2 \sum_{s | d \geq 1} \frac{1}{s} = 1 - \sum_{n \leq x} \frac{1}{n} \geq 2 \sum_{s \leq x} \frac{1}{s} \sum_{s' \leq x} \frac{1}{s'} = \vartheta(1 - 2 \sum_{s \leq x} \frac{1}{s}) - 1
\]
follows. Hence
\[
(2.13) \quad \sum_{s \leq x} \frac{1}{s} \geq \frac{1}{2} - o(1).
\]
Let
\[
S = \{s_1 < s_2 < \ldots < s_k\}
\]
and
\[
(2.14) \quad S' = \{s_1 < s_2 < \ldots < s_{k-1} \quad \sum_{i=1}^{k-1} \frac{1}{s_i} > \frac{1}{2} \quad \sum_{i=1}^{k} \frac{1}{s_i} \}
\]
If \( \sum_{s \leq x} \frac{1}{s} \leq \frac{1}{2} \) then let \( S' = S_b \).

Now we define a \( \theta'(n) \) completely multiplicative function for \( n \leq x \) with
\[
\theta'(p) = -1 \quad \text{if } p \in \mathbb{P} \cup S',
\]
\[
\theta'(q) = 0 \quad \text{if } q \not\in Q \cup S',
\]
\[
\theta'(t) = 1 \quad \text{if } t \in T \cup (S \setminus S').
\]
Thus we have from (2.9) for an arbitrary \( n \)
\[
\sum_{d|n} \theta(d) = g(n) \leq g'(n) = \sum_{d|n} \theta'(d).
\]

So we shall use the Corollary for \( \theta'(n) \) and we shall estimate the corresponding \( U' \), i.e. we shall prove

**Lemma 2.** We have
\[
U' = \sum_{s} \frac{1}{s} \log \frac{s}{s'} - 2 \sum_{s} \sum_{s' | s} \frac{1}{ss'} \log \frac{s}{s'} \geq \left( \frac{1}{6} - \frac{1}{2} + o(1) \right) \log s,
\]

where the summation in Lemma 2 runs always through \( s, s' \in S' \).

**Proof.** If in the definition of \( S' \) in (2.14) \( s_{s+1} \leq \log s' \), and \( s \neq s' \) then as \( \sum_{s} \frac{1}{s} > \frac{1}{6} \), we get
\[
U' \geq \sum_{s} \frac{1}{s} \log \left( 1 - o(1) \right) - \log s \sum_{s} \sum_{s'} \frac{1}{ss'} \geq \left( 1 - o(1) \right) \log s \left( \sum_{s} \frac{1}{s} \right) \left( 1 - \sum_{s'} \frac{1}{s'} \right) \geq \left( \frac{1}{6} - \frac{1}{6} - o(1) \right) \log s,
\]

which implies (2.16).

If \( s_{s+1} \geq \log s' \), or \( s = s' \) let
\[
\alpha = \sum_{s \leq \sqrt{y}} \frac{1}{s}, \quad \beta = \sum_{s > \sqrt{y}} \frac{1}{s}.
\]

Then as \( 1/s_{s+1} = o(1) \), we have
\[
\frac{1}{2} - o(1) \leq \alpha + \beta \leq \frac{1}{2}.
\]

With these notations the following formulæ hold
\[
D = \sum_{s \leq \sqrt{y}} \frac{1}{s} \log \frac{s}{s'} - 2 \sum_{s \leq \sqrt{y}} \sum_{s' | s} \frac{1}{ss'} \log \frac{s}{s'} \geq \sum_{s \leq \sqrt{y}} \frac{1}{s} \log \left( 1 - \sum_{s' | s} \frac{1}{s'} \right) = (1 - \alpha) \log s \cdot \alpha - (1 - \alpha) \sum_{s \leq \sqrt{y}} \frac{\log s}{s},
\]

(2.20)
\[
E = \sum_{s \leq \sqrt{y}} \frac{1}{s} \sum_{s' \leq \sqrt{y}} \frac{1}{s' \log \frac{s}{s'}} \leq \sum_{s \leq \sqrt{y}} \frac{1}{s} \sum_{s' \leq \sqrt{y}} \frac{1}{s' \log \frac{s}{s'}} = \frac{1}{2} \beta \log s \cdot \alpha - \beta \sum_{s < \sqrt{y}} \frac{\log s}{s},
\]

(2.21)
\[
F = \sum_{s \leq \sqrt{y}} \frac{1}{s} \log \frac{s}{s'} = \beta \log s - \sum_{s < \sqrt{y}} \frac{\log s}{s}.
\]

Here as \( U' = D - 2E + F \), from formulæ (2.18)–(2.21) we get
\[
U' \geq \log s \left( \alpha \left( 1 - \alpha - \beta \right) + \beta - (1 - \alpha - 2\beta) \sum_{s \leq \sqrt{y}} \frac{\log s}{s} - \sum_{s > \sqrt{y}} \frac{\log s}{s} \right)
\]
\[
= \left( \frac{1}{2} - \frac{a}{2} + o(1) \right) \log s - [\alpha + o(1)] \sum_{s \leq \sqrt{y}} \frac{\log s}{s} - \sum_{s > \sqrt{y}} \frac{\log s}{s}.
\]

On the other hand, it is easy to show that if \( S' \) is a set of primes \( s \leq y \) \( (y \neq o(1)) \), \( s = s \) or \( 2s \), and
\[
\sum_{s \leq \sqrt{y}} \frac{1}{s} = \gamma + o(1)
\]
\((\gamma \text{ a given number}) \) then the sum
\[
\sum_{s \leq \sqrt{y}} \frac{1}{s}
\]

is maximal if the set \( S' \) contains all primes in an interval \([x, y]\) and no primes less than \( x \) and for all primes \( s, s = s, s > s \). So if we use the formulæ
\[
\sum_{p \leq \sqrt{y}} \frac{\log p}{p} = \log s + o(1), \quad \sum_{p \leq \sqrt{y}} \frac{1}{p} = \log \log s + C + o(1)
\]

where \( C \) is an absolute constant, easy computation shows that if (2.23) holds then
\[
\sum_{s \leq \sqrt{y}} \frac{\log s}{s} \leq (1 - e^{-\gamma} + o(1)) \log y.
\]

Thus from (2.22) and (2.24) we have
\[
U' \geq \log s \left( \frac{1}{2} - \frac{a}{2} + \frac{a}{2} \left( 1 - e^{-\gamma} - (1 - e^{-\delta}) - o(1) \right) \right) = \theta \cdot \log s.
\]

(2.25)

\[
U' \geq \log s \left( \frac{1}{2} - \frac{a}{2} + \frac{a}{2} \left( 1 - e^{-\gamma} - (1 - e^{-\delta}) - o(1) \right) \right) = \theta \cdot \log s.
\]
Here using $\beta = \frac{1}{2} - a + o(1)$, we get with some computation that for
$0 \leq a \leq \frac{1}{2} + o(1)$

$$G = G(a, \beta) = G(a) \geq G(0) = \frac{1}{e^2} - \frac{1}{2} - o(1),$$

which proves Lemma 2.

Thus we have from formulæ (2.2), (2.15), (2.11), (2.12) and (2.16)

$$\sum_{d \leq x} \frac{\delta(d)}{d} \leq \frac{1}{\log x} \sum_{n \leq x} g(\alpha) + o(1) \leq \frac{1}{\log x} \sum_{n \leq x} g'(n) + o(1) \leq \log x \left(1 - \frac{2 \log x}{\log x} + o(1)\right) \leq \log x \left(1 - \frac{1}{\log x} + o(1)\right).$$

References


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Received on 20. 10. 1975 (781)

The factorization of $Q(L(x_1), \ldots, L(x_k))$ over a finite field where $Q(x_1, \ldots, x_k)$
is of first degree and $L(x)$ is linear

by

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1. Introduction. Let $GF(q)$ denote the finite field of order $q = p^a$ where $p$ is prime and $a \geq 1$. Let $\Gamma(p)$ denote the algebraic closure of $GF(q)$. A polynomial $Q \in GF(q; x_1, \ldots, x_k)$ is absolutely irreducible if $Q$
has no nontrivial factors over $\Gamma(p)$. Throughout this paper, the term irreducible will mean absolutely irreducible.

A polynomial with coefficients in $GF(q)$ of the form

$$L(x) = \sum_{c=0}^q a_c x^c,$$

is called a linear polynomial. The requirement that the coefficients be in $GF(q)$ insures that the operation of mapping composition for linear polynomials is commutative. Corresponding to the linear polynomial $L(x)$ we have the ordinary polynomial

$$l(x) = \sum_{c=0}^q a_c x^c.$$

We shall assume in the following that $a_0 \neq 0$; this avoids multiple factors in $L(x)$ and insures that there is a smallest integer $r$ such that $l(x)$ divides $x^r - 1$. We say that $l(x)$ has exponent $r$.

Let $Q(x_1, \ldots, x_k) = a_1 x_1 + \ldots + a_k x_k + 1$ where $[\deg a_1, \ldots, \deg a_k] = s$ (if $a \neq GF(q)$ but $a_{GF(q)}$, $1 \leq t < s$, we say that the degree of a relative to $GF(q)$ is $s$ and write $\deg a = s$). We shall assume that $\{a_1, \ldots, a_k\}$ are linearly independent over $GF(q)$; otherwise $Q(x_1, \ldots, x_k)$ can be reduced at once to a polynomial in $m$ variables by suitable first degree transformations, where $m$ is the number of elements in a maximal linearly independent subset of $a_1, \ldots, a_k$.

In this paper we describe the factorization of $Q(L(x_1), \ldots, L(x_k))$.
(We note that it is possible to have $Q(L(x_1), \ldots, L(x_k))$ reduce to a polynomial in fewer than $k$ variables even though $\{a_1, \ldots, a_k\}$ are linearly