On the constant $\beta(k)$ in Rosser's sieve

by

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Introduction. In his paper [2], Selberg gives a procedure for calculating the optimal value of the parameter $\beta(k)$ which arises in the combinatorial sieve of Barkley Rosser. The procedure works when $2k$ is a positive integer; when $2k$ is odd Selberg shows that $\beta(k)$ is algebraic, and remarks that $\beta(1)$ and $\beta(2)$ are also algebraic. In this note I prove that $\beta(k)$ is algebraic for all positive integers $k$.

For the sake of brevity I have extracted from [2] only the information necessary for this purpose; for more details see [2], and Halberstam and Richert [1].

Rosser's method leads to a differential difference equation which may be solved by Laplace transforms. The transform $K(z)$ is regular at $z = 0$, and satisfies the ordinary differential equation

\begin{equation}
\frac{d}{dz} (z K(z)) = k (1 + e^{-\beta} K(z)) + c_k e^{-\beta} U_k(z),
\end{equation}

where

\begin{equation}
U_k(z) = k \int_0^1 e^{-(t-1)z} dt - \beta^{1-k} e^{(1-\beta)z}
\end{equation}

and $c_k$ is a constant which can be determined from the boundary behaviour, once $\beta = \beta(k)$ is known. For our purpose, its value is immaterial.

To determine $\beta$, we differentiate (1) $\nu$ times for each $\nu$, $0 \leq \nu \leq 2k - 1$, and put $z = 0$; this is valid since $K$ is regular at $z = 0$. This gives the system of simultaneous equations

\begin{equation}
(\nu + 1) K^{(\nu)}(0) = 2k K^{(0)}(0) + k \sum_{r=1}^{\nu} (-1)^r \binom{\nu}{r} K^{(r-\nu)}(0) + c_k \sum_{r=2}^{\nu} (-1)^r \binom{\nu}{r} U_k^{(r-\nu)}(0),
\end{equation}
and we notice that the terms involving $K^{(2k-3)}(0)$ cancel in the last equation. We may eliminate the unknowns $K^0(0)$, $0 \leq r \leq 2k-2$ from these equations, arriving at a linear relation, with constant factor $c_k$, between the numbers $U_k(0)$, $0 \leq r \leq 2k-1$. This gives an equation for $\beta$.

Selberg's description of the procedure, which immediately follows eq. 5.7 in [1] is slightly different since he differentiates $2k$ times. The reason for this is not clear to me.

Next, we carry out the elimination. Multiply (3) by $A(n)$ and add (where the numbers $A(0)$, $A(1)$, ..., $A(2k-1)$ are to be determined).

We get

(4)

$$c_k \sum_{s=0}^{2k-1} A(s) \frac{(-1)^r}{r!} U_k^{(r)}(0) = \sum_{s=0}^{2k-1} A(n) \left[ (v+1-k)K^0(0) - \left. \frac{d}{dt} \right|_{t=v} K^0(0) \right]$$

$$= \sum_{s=0}^{2k-1} \left[ (s+1-k)A(s) - k \sum_{r=0}^{2k-1} (-1)^r \frac{r!}{(2k-1-r)!} A(n) \right] K^0(0).$$

We select any non-zero value for $A(2k-1)$, and determine the remaining $A's$ from the equations

(5)

$$(s+1-k)A(s) = k \sum_{r=0}^{2k-1} (-1)^{r-s} \frac{r!}{(2k-1-r)!} A(n), \quad 0 \leq s \leq 2k-2.$$

This gives

(6)

$$\frac{1}{k} \sum_{s=0}^{2k-1} (s+1-k)A(s) U_k^0(0) = \sum_{s=0}^{2k-1} \sum_{r=0}^{2k-1} (-1)^s \frac{r!}{(2k-1-r)!} A(n) U_k^0(0)$$

$$= \sum_{s=0}^{2k-1} A(n) \sum_{r=0}^{2k-1} (-1)^s \frac{r!}{(2k-1-r)!} U_k^{(r)}(0) = 0$$

in view of (4) and (5). If we choose $A(2k-1) = 1$, the numbers $A(n)$ are rational functions of $k$, over $Q$. Moreover, if $k$ is an integer, $U_k^0(0)$ is a rational function of $\beta$ over $Q$ except when the case $s = k-1$ when it involves logarithms. However, the term involving $U_k^{(k-1)}(0)$ drops out of (6), so that $\beta(k)$ is algebraic for all positive integers $k$.

I am very grateful to the referee for pointing out that $\beta(k)-1$ is in fact the largest real positive root of the polynomial

(7)

$$g(z) = \sum_{s=0}^{2k-1} (-1)^{s-1} A(s) z^s.$$