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On the constant $\beta(k)$ in Rosser's sieve

by

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Introduction. In his paper [2], Selberg gives a procedure for calculating the optimal value of the parameter $\beta(k)$ which arises in the combinatorial sieve of Barkley Rosser. The procedure works when $2k$ is a positive integer; when $2k$ is odd Selberg shows that $\beta(k)$ is algebraic, and remarks that $\beta(1)$ and $\beta(2)$ are also algebraic. In this note I prove that $\beta(k)$ is algebraic for all positive integers k.

For the sake of brevity I have extracted from [2] only the information necessary for this purpose: for more details see [2], and Halberstam and Richert [1].

Rosser's method leads to a differential difference equation which may be solved by Laplace transforms. The transform $K(z)$ is regular at $z = 0$, and satisfies the ordinary differential equation

$$\frac{d}{dz}\big(zK(z)\big) = k(1+e^{-z})K(z)+c_k e^{-z}U_k(z), \tag{1}$$

where

$$U_k(z) = k\int_{\beta}^{\beta+1}\frac{e^{-(t-1)z}\,dt}{(t-1)^k}-\beta^{1-k}e^{(1-\beta)z} \tag{2}$$

and c_k is a constant which can be determined from the boundary behaviour, once $\beta = \beta(k)$ is known. For our purpose, its value is immaterial.

To determine β, we differentiate (1) ν times for each ν, $0 \leqslant \nu \leqslant 2k-1$, and put $z = 0$; this is valid since K is regular at $z = 0$. This gives the system of simultaneous equations

$$(\nu+1)K^{(\nu)}(0) = 2kK^{(\nu)}(0)+k\sum_{r=1}^{\nu}(-1)^r\binom{\nu}{r}K^{(\nu-r)}(0)+ \\ +c_k\sum_{r=0}^{\nu}(-1)^r\binom{\nu}{r}U_k^{(\nu-r)}(0), \tag{3}$$

and we notice that the terms involving $K^{(2k-1)}(0)$ cancel in the laste quation. We may eliminate the unknowns $K^{(\nu)}(0)$, $0 \leqslant \nu \leqslant 2k-2$ from these equations, arriving at a linear relation, with constant factor c_k, between the numbers $U_k^{(\nu)}(0)$, $0 \leqslant \nu \leqslant 2k-1$. This gives an equation for β.

Selberg's description of the procedure, which immediately follows eq. 5.7 in [1] is slightly different since he differentiates $2k$ times. The reason for this is not clear to me.

Next, we carry out the elimination. Multiply (3) by $A(\nu)$ and add (where the numbers $A(0), A(1), \ldots, A(2k-1)$ are to be determined). We get

$$(4)\qquad c_k \sum_{\nu=0}^{2k-1} A(\nu) \sum_{r=0}^{\nu} (-1)^r \binom{\nu}{r} U_k^{(\nu-r)}(0)$$
$$= \sum_{\nu=0}^{2k-1} A(\nu)\Big[(\nu+1-k)K^{(\nu)}(0) - k\sum_{r=0}^{\nu}(-1)^r\binom{\nu}{r}K^{(\nu-r)}(0)\Big]$$
$$= \sum_{s=0}^{2k-2}\Big[(s+1-k)A(s) - k\sum_{\nu=s}^{2k-1}(-1)^{\nu-s}\binom{\nu}{s}A(\nu)\Big]K^{(s)}(0).$$

We select any non-zero value for $A(2k-1)$, and determine the remaining A's from the equations

$$(5)\qquad (s+1-k)A(s) = k\sum_{\nu=s}^{2k-1}(-1)^{\nu-s}\binom{\nu}{s}A(\nu), \qquad 0 \leqslant s \leqslant 2k-2.$$

This gives

$$(6)\qquad \frac{1}{k}\sum_{s=0}^{2k-1}(s+1-k)A(s)\,U_k^{(s)}(0) = \sum_{s=0}^{2k-1}\sum_{\nu=s}^{2k-1}(-1)^{\nu-s}\binom{\nu}{s}A(\nu)\,U_k^{(s)}(0)$$
$$= \sum_{\nu=0}^{2k-1}A(\nu)\sum_{r=0}^{\nu}(-1)^r\binom{\nu}{r}U_k^{(\nu-r)}(0) = 0$$

in view of (4) and (5). If we choose $A(2k-1) = 1$, the numbers $A(\nu)$ are rational functions of k, over Q. Moreover, if k is an integer, $U_k^{(s)}(0)$ is a rational function of β over Q except in the case $s = k-1$ when it involves logarithms. However, the term involving $U_k^{(k-1)}(0)$ drops out of (6), so that $\beta(k)$ is algebraic for all positive integers k.

I am very grateful to the referee for pointing out that $\beta(k)-1$ is in fact the largest real positive root of the polynomial

$$(7)\qquad g(z) = \sum_{s=0}^{2k-1}(-1)^{s-1}A(s)z^s.$$

This is useful for computing β since the coefficients $A(s)$ can readily be found from (5). Notice the $g(z)$ is the unique monic polynomial such that

$$(8)\qquad \{zg(z)\}' = k\{g(z)+g(z+1)\}.$$

Now we prove that $g(\beta-1) = 0$. From (6), we have

$$0 = \frac{1}{k}\sum_{s=0}^{2k-1}(s+1-k)A(s)\,U_k^{(s)}(0)$$
$$= \int_{\beta}^{\beta+1}\sum_{s=0}^{2k-1}(s+1-k)(-1)^s A(s)(t-1)^{s-k}dt +$$
$$+\frac{1}{k}\beta^{1-k}\sum_{s=0}^{2k-1}(s+1-k)(-1)^{s-1}A(s)(\beta-1)^s,$$

by the definition (2) of $U_k(z)$. Writing this in terms of g, we obtain

$$0 = \int_{\beta}^{\beta+1} -\frac{d}{dt}\left[\frac{g(t-1)}{(t-1)^{k-1}}\right]dt + \frac{(\beta-1)^k}{k\beta^{k-1}}\left[\frac{d}{dt}\,\frac{g(t-1)}{(t-1)^{k-1}}\right]_{t=\beta}$$
$$= \frac{g(\beta-1)}{(\beta-1)^{k-1}} - \frac{g(\beta)}{\beta^{k-1}} + \frac{(\beta-1)^k}{k\beta^{k-1}}\left[\frac{kg(\beta)}{(\beta-1)^k}\right] = \frac{g(\beta-1)}{(\beta-1)^{k-1}}$$

using (8). This completes the proof. Finally, I have computed g for $k = 2, 2.5, 3$. The polynomials are

$$z^3-6z^2+9z-8/3,$$
$$z^4-10z^3+30z^2-\frac{85}{3}z+\frac{55}{12},$$
$$z^5-15z^4+75z^3-145z^2+90z-18/5.$$

When $k = 5/2$, this agrees with Selberg's calculation. I find that $\beta(2) = 4.833\ldots$ and $\beta(3) = 7.919\ldots$; my value of $\beta(2)$ is slightly larger than Selberg's.

References

[1] H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, 1974.
[2] A. Selberg, *Sieve methods*, Proc. Sympos. Pure Math. 20 (1971), pp. 311–315.

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