General multiplicative functions

by

I. Z. RUZSA (Budapest)

O. Concepts and notations. Here I list (a) some definitions and notations, which are used in the paper without any more mentioning, (b) the number of definitions, which are used in more sections.

(a)

P, 2, Pn generally mean primes, p the nth prime, beginning with p1 = 2.
N, Q, R, C the set of natural, rational, real and complex numbers.
AGQ, MSR etc.: structures on these sets. The first letter shows that the operation is addition or multiplication, the third the fundamental set, the second stands for group or semigroup, MGQ etc. means the group, got by omitting the number 0.

G = (G, ε) a structure on the set G with the operation ε.

0 generally the unity of a structure.

gen K the subgroup, generated by the subset K.
g (g) the order of the element g (in a group).

C(ω), C(∞) cyclic groups.

C(ωω) quasicyclic group.
a group, isomorphic to AGQ.
demin σ, dem sup S, dem S; the lower density, the upper density, resp. the asymptotic density (if exists) of the sequence S ⊂ N.

(b)

The number in the left shows the place of the definition.

(1.1) arithmetical function,

(1.2) G-multiplicative and strongly G-multiplicative function,

(1.5) M(f), mean value of the function f,

(3.1) N(f, K, w),

(3.2) δ(f, K), δ(f, K),

(3.3) d(f, K),

(5.4) density set,

(5.5) density class,

(5.6) uniform density class,

(5.8) concentration complex and concentration group,

(3.9) concentrated and deconcentrated function,

(7.1) φ(f1, f2) and f1 → f2,

(9.9) superconcentrated function,

(9.10) f/G, factor-function.
1. Introduction. Multiplicative and additive functions were investigated in hundreds of papers. As far as I know, these terms always denoted a complex-valued arithmetical function $f$ satisfying
\[ f(nm) = f(n)f(m) \]
resp.
\[ f(nm) = f(n) + f(m) \]
for all $(n, m) = 1$. In this paper we shall show that sometimes it is more natural to regard them as particular cases of a more general concept.

(1.1) **Definition.** An arithmetical function $f$ is an arbitrary function with $\text{dom } f = \mathbb{N}$.

The most general possible concept of multiplicativity is the following.

(1.2) **Definition.** Let $G = (G, \circ)$ be a groupoid and $f: \mathbb{N} \to G$ an arithmetical function. (We must distinguish the structure $G$ from the fundamental set $G$.) We call $f$ $G$-multiplicative if
\[ f(nm) = f(n) \circ f(m) \]
for all $(n, m) = 1$. We call $f$ strongly $G$-multiplicative if (1.3) holds without any restriction on $n$ and $m$. (Generally we shall write simply $ab$ instead of $a \circ b$.)

Our main interest will be the local limit theorem, that is, statements concerning the number of natural numbers $n \leq x$ for which $f(n) = g$. For example,

(1.4) **Theorem.** If $G$ is an Abelian group and $f$ is $G$-multiplicative, then the sequence $f^{-1}(g)$ has an asymptotic density for all $g \in G$.

Observe that for functions, assuming only the values $\pm 1$, it is equivalent to the existence of the mean value
\[ M(f) = \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n). \]

This was conjectured long ago by P. Erdős and proved by E. Wirsing [11] under much more general conditions. Our theorem generalizes in another direction.

The following more complicated example gives a better illustration.

(1.6) **Corollary.** Let $f_k$, $k = 1, 2, \ldots$, be complex-valued multiplicative functions, $f_k(n) \neq 0$, and $g_k \in G$ arbitrary. The sequence of the solutions of the infinite system of equations
\[ f_k(n) = g_k \]
has an asymptotic density.

The nonzero complex numbers form a group and we can apply Theorem (1.4) to the (complete) direct product of such groups. I note that it does not remain true if we allow $f_k(n) = 0$ (albeit this does not matter if we regard only finitely many $f_k$). This shows the importance that $G$ be a group. (See also Section 2.)

At this point it is natural to ask what is the inner meaning of this theorem, that is, which sequences are proved to have a density.

(1.7) **Problem.** Given a sequence $S \subseteq \mathbb{N}$, under what conditions can one find an Abelian group $G$, a $G$-multiplicative $f$ and a $g \in G$ such that
\[ S = f^{-1}(g). \]

This way we get the purely number-theoretic content of our local limit theorem, with no "alien" concept as a group. One might hope that this leads closer to the proof, but this is not the case. Indeed, (1.7) can easily be solved.

(1.8) **Statement.** Let $S \subseteq \mathbb{N}$. The following three conditions are equivalent to each other.

(a) There exists an Abelian group $G$, a $g \in G$ and a strongly $G$-multiplicative $f$ such that $S = f^{-1}(g)$.

(b) There exists a $G_S < MGQ$ and a $0 \neq g \in Q$ such that
\[ S = gG_S \cap \mathbb{N} \]
(that is, $S$ consists of the natural numbers contained in a coset of a subgroup of the multiplicative group of the rational numbers).

(c) For arbitrary $s_1, \ldots, s_{n+1}, t_1, \ldots, t_n \in S$
\[ \prod_{i=1}^{n+1} s_i / \prod_{i=1}^{n} t_i \in S. \]

Proof in Section 6.

It does not matter that we spoke of strongly multiplicative functions. In Sections 7–8 we shall formulate and prove a theorem which states that a set of prime-powers $T$ with
\[ \sum_{p^i \in \mathcal{P}} \frac{1}{i} < \infty \]
such as the set of all $p^i$, $i \geq 2$, is always negligible.

Condition (1.10) is no good. I wonder if there will be anybody to prove directly that every sequence satisfying (1.10) has a density. The strongly multiplicative function $f$ plays the same role here as the characters in the proof of Dirichlet's theorem on the primes in arithmetical progressions.
There is one more fact emphasizing our conclusion. Erdős [3] asked the following

\[ (1.11) \quad \text{Problem. Let } S \subset \mathbb{N} \text{ be a sequence satisfying} \]

\[ \left[s_1, \ldots, s_m, t_1, \ldots, t_n \right] S, \prod s_i = \prod t_j \Rightarrow m = n. \]

What is

\[ \beta = \sup_S (\text{densup } S) ? \]

He proved \( \beta \geq 1/e \).

We can reformulate this question for functions.

\[ (1.13) \quad \text{Statement. Condition (1.12) is equivalent to the existence of a real-valued strongly additive (that is, strongly \( \Delta \mathcal{G} \text{-} \text{multiplicative}) f \) such that} \]

\[ S \subseteq f^{-1}(1). \]

Proof in Section 6.

Using (1.13) I could prove \( \beta = 1/e \) (see Section 4). Moreover, I proved this several years ago (stated without proof in Erdős–Razaa–Sárközi [4]), but nobody observed that the statement concerning strongly additive functions solved (1.11). So the real worth of conditions like (1.10) and (1.13) is that they make possible to deal with functions instead of sequences.

In Sections 2–5 we state other problems and results, and Sections 6–14 contain the proofs.

2. Algebraic and analytic questions. The existence of the density of a sequence, defined by a multiplicative function, is a typical analytic question. On the other hand, we made an algebraic generalization, regarding functions, mapping into an abstract structure. Present paper emphasizes the analytic aspects (though it does not contain a single sign of integration), but we must face some algebraic problems.

First, if \( G \) is an arbitrary groupoid, very strange things may happen: the product of two multiplicative functions may not be multiplicative, or to a multiplicative \( f \) we cannot define its "strongly multiplicative brother" \( f_b \) by \( f_b(p) = f(p) \), or maybe that there are no \( G \)-multiplicative functions at all. To avoid these pathologic cases, except of this section we suppose the following restriction.

(A) \( G \) is a commutative semigroup with unity \( e \).

\[ (2.1) \quad \text{Statement. Suppose we are given the values } \mathcal{F}(p^i) \text{ for every prime-power } (i \geq 1). \text{ If (A) holds, then there exists exactly one } G \text{-} \text{multiplicative } f \]

\[ f(p^i) = \mathcal{F}(p^i), \quad f(1) = e. \]

Of course, this is not true without \( f(1) = e \); for example, if \( G = \mathbb{M} \mathbb{R} \) and \( \mathcal{F}(p^i) = 0 \), then \( f(n) = 0 \) is also good. For sake of unicity we always suppose

\[ (B) f(1) = e. \]

Under these natural assumptions we may — and shall — define a multiplicative \( f \) by giving \( f(p^i) \) and a strongly multiplicative one by \( f(p) \).

The distance between (A) and the Abelian groups is still enormous. I cannot decide in which structures do all the \( G \)-multiplicative functions obey a local limit law (this simply means the existence of \( \text{den} f^{-1}(g) \) for all \( g \in \mathcal{G} \)), but I have some results in both directions.

\[ (2.2) \quad \text{Theorem. Let } \mathcal{D} \text{ be the multiplicative semigroup of the numbers } 0 \text{ and } 1, \quad \mathcal{D} = D_0^\alpha \text{ and } 0 = (0, 0, \ldots) \in D. \text{ There exists a } D \text{-} \text{multiplicative } f \text{ for which the sequence } f^{-1}(0) \text{ does not have a density.} \]

Let \( G \) satisfy (A), \( a, b \in \mathcal{G} \). We write \( ahb \) if both the equations \( ax = b \) and \( by = a \) are solvable in \( G \). \( H \) is a congruence-relation, so we can form the factor-semigroup \( \mathcal{G} / H \).

\[ (2.3) \quad \text{Theorem. Suppose (A) and let } \mathcal{G} \text{ be as above. If in } \mathcal{G}, \text{ every element has only a finite number of divisors, then every } G \text{-} \text{multiplicative function obeys a local limit law.} \]

For example, if \( f_1, \ldots, f_m \) are complex-valued multiplicative functions (allowed to assume 0) and \( g_1 \in \mathcal{G} \), then the sequence of the solutions of the system of equations

\[ f_i(n) = g_i \]

has a density.

There are also some results if we do not suppose (A). I can prove, that all the results, which were and will be stated for Abelian groups, are valid also in non-commutative groups. Or if \( G \) is an arbitrary finite groupoid, then all the \( G \)-multiplicative functions obey a local limit law.

Theorem (2.2) will be proved in Section 6. The other results are put off to another paper, dealing with the algebraic problems of multiplicativity.

3. The local limit laws for multiplicative functions. Let \( f : N \rightarrow G \) be an arithmetical function and \( K \in \mathcal{G} \). We introduce some notations.

\[ (3.1) \quad N(f, K, x) = \{ n \leq x, f(n) \leq K \}. \]

\[ (3.2) \quad \text{den}(f, K) = \text{densup} \ f^{-1}(K), \]

\[ (3.3) \quad \text{den}(f, K) = \text{deninf} \ f^{-1}(K) \text{ if it exists}. \]
Most frequently \( K = \{g\} \); in this case we omit the braces and write for example \( \tilde{d}(f, g) \) instead of \( \tilde{d}(f, \{g\}) \).

3.4 Definition. \( K \subset G \) is a density set for \( f \) if \( \tilde{d}(f, K) \) exists.

3.5 Definition. Let \( \mathcal{R} \) be a class of subsets of \( G \). We call \( \mathcal{R} \) a density class for \( f \) if every \( K \in \mathcal{R} \) is a density set for \( f \).

In this terms Theorem (1.4) can be formulated as \( \{g\} = \{g \in G \} \) is a density class. We prove a little more.

3.6 Definition. \( \mathcal{R} \) is a uniform density class for \( f \) if the convergence

\[
\frac{N(f, K, x)}{x} \rightarrow \tilde{d}(f, K) \quad (K \in \mathcal{R})
\]

is uniform in \( K \).

3.7 Theorem. Let \( G \) be an Abelian group. \( \mathcal{G} = \{\{x\} : x \in G\} \) is a uniform density class for every \( G \)-multiplicative function.

We can give some information about the behaviour of the values \((\tilde{d}(f, g))\).

3.8 Definition. Let \( G \) be a group and \( f \) a \( G \)-multiplicative function. We call

\[
K = \left\{ g : \left. \sum_{1 \leq \omega \leq \omega} \frac{1}{p} \leq \infty \right\}
\]

the concentration complex and

\[
G_1 = \text{gen} K
\]

the concentration group of \( f \).

3.9 Definition. Let \( G \) be a group and \( f \) \( G \)-multiplicative with the concentration group \( G_1 \). We call \( f \) concentrated if

\[
|G_1| < \infty \quad \text{and} \quad \sum_{p \in G_1} \frac{1}{p} < \infty,
\]

and deconcentrated otherwise.

3.10 Theorem. Let \( G \) be an Abelian group and \( f \) \( G \)-multiplicative. One of the following two possibilities holds.

(a) \( f \) is deconcentrated, \( \tilde{d}(f, g) = 0 \).

(b) \( f \) is concentrated, \( \tilde{d}(f, g) > 0 \) for every \( g \in \text{inf} f \) and

\[
\sum_{\omega \in \mathcal{G}} \tilde{d}(f, g) = 1.
\]

The next result is a generalization of many uniformity theorems.

3.11 Uniform Distribution Theorem. Let \( G \) be an Abelian group and \( f \) \( G \)-multiplicative with the concentration group \( G_1 \). \( \tilde{d}(f, g) \) depends only on the coset of \( G_1 \) in which \( g \) lies.

At last we state our only non-local result.

3.12 Theorem. Let \( G \) be an Abelian group and let \( \mathcal{P} \) denote the set algebra generated by all the cosets of all the subgroups of \( G \). \( \mathcal{P} \) is a density class for every \( G \)-multiplicative function.

4. Erdős’s problem and the maximum-theorems

4.1 Theorem. There exists a constant \( c > 0 \) such that for every Abelian group \( G \), \( e \neq g \in G \) and \( G \)-multiplicative \( f \) we have

\[
N(f, g, w) \leq cN(1 - c).
\]

4.2 Maximum-Theorem for Multiplicative Functions. For every Abelian group \( G \), \( e \neq g \in G \) and \( G \)-multiplicative \( f \)

\[
\tilde{d}(f, g) \leq \frac{1}{2}.
\]

Equality holds only in the following cases.

(a) \( f(2^i) = g \) for all \( 1 \leq i < \infty \); there is a prime \( q \) for which \( f(q^i) = g \) or \( e \) (depending on \( f \)) and \( f(p^i) = e \) for every other prime \( p \).

(b) \( o(g) = 2 \), \( f(p^i) = g \) or \( e \) for every prime-power and either \( f(q^i) = g \) for all \( i \geq 1 \) or

\[
\sum_{n \in G} \sum_{p \in G} \frac{1}{p} = \infty.
\]

4.3 Supremum-Theorem for strongly multiplicative functions. Let \( G \) be an Abelian group, \( f \) a strongly \( G \)-multiplicative function and

\[
p_0 = \min \{o(g) : g \in G, g \neq e\}.
\]

\( (p_0 \) is a prime or \( \infty) \).

(a) In case \( p_0 = 2 \) we have

\[
\tilde{d}(f, g) \leq \frac{1}{2}
\]

for all \( g \neq e \). Equality holds only if \( f(p^i) = g \) or \( e \),

\[
\sum_{p \in G} \frac{1}{p} = \infty.
\]

(b) In case \( p_0 > 2 \) we have for \( g \neq e \)

\[
\tilde{d}(f, g) < W_{p_0},
\]

where \( W_n \) is the maximum of the function

\[
\omega_n(x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} \quad (x \in \mathbb{R}).
\]

\( W_{p_0} \) can be arbitrarily approached. \( (\omega_n(x) = x e^{-x}, W_{\infty} = 1/e) \).
Let $S$ be a sequence satisfying Erdős’s condition (1.12). From Statement (1.13) and Theorems (4.1) and (4.3) we have

(4.4) COROLLARY. (a) There exists an absolute constant $c > 0$ such that for every $x$

$$S(x) \leq x(1 - c),$$

where

$$S(x) = \{n : n \leq x, n \in S\}.$$

(b) densup $S < 1/e$; moreover, $S$ is contained in a sequence which has a density $< 1/e$.

For real-valued additive functions (that is, for the case $G = \mathbb{Z}$) Theorems (4.1) and (4.2) were proved and (4.3) was stated without proof in a joint paper with Erdős and Sárközi [4]. For Theorem (4.1) the same proof applies, so we shall not prove it here. Theorems (4.2) and (4.3) will be proved in Sections 13 and 14.

5. Unsolved problems

(5.1) For what structures $G$ do all the $G$-multiplicative functions obey a local limit law?

In this generality probably there is no simple condition. Perhaps the (most interesting) case when $G$ is a commutative semigroup can be solved. The result, mentioned in Section 2, is very close to being best possible; it does not contain even the case when $G$ can be imbedded into a group. I have no plausible conjecture.

(5.2) Let $G$ be a group. Which subsets of $G$ are density sets for all the $G$-multiplicative functions?

I cannot solve this even in the simplest non-trivial case $G = C(\infty)$. I can show that the sets, given in Theorem (3.12), do not exhaust this class.

(5.3) Can one formulate and prove the analogues of the well-known global limit theorems?

I think nothing can be stated for arbitrary topologie groups. I failed even to find a reasonable definition for a global limit law. However this can be done without difficulty for metric groups. For example, probably an analogue of the “three-series-theorem” of Erdős and Whittaker [5] is true.

6. Inverse image sequences. The aim of this section is to prove Statements (1.8), (1.13) and Theorem (2.1). (1.8) and (1.13) are connected with the inner characterization of sequences of the form $f^{-1}(g)$; (2.1) is different, but we prove it through a similar lemma.

Proof of (1.8).

(a) $\Rightarrow \ (c)$ is trivial.

(c) $\Rightarrow \ (b)$. Let

$$G_0 = \left\{ a_1 \ldots a_n : a_i, b_j \in S \right\}, \quad G_0 = (G_0, \cdot, < MG_Q.$$

Let $r \in S$ be arbitrary. Since with an $s \in S$

$$s = r \frac{s}{r}, \quad s \in G_0,$$

hence

$$S \subset rG_0 \cap N.$$

On the other hand, let $s \in rG_0 \cap N$. $s \in rG_0 \Rightarrow s \in G_0$, so

$$s = \frac{a_1 \ldots a_n}{b_1 \ldots b_n}, \quad a_i, b_j \in S.$$ 

Now

$$s = \frac{ru_1 \ldots a_n}{b_1 \ldots b_n} \in S$$

by condition (c).

(b) $\Rightarrow \ (a)$. Let

$$\varphi: MG_Q \rightarrow MG_Q/G_0$$

be the natural homomorphism; $f = \varphi|_N$ will do.

Now we prove (1.13) in a more general form.

(6.1) STATEMENT. (a) Let $G$ be a group, $g \in G$ and $f$ a strongly $G$-multiplicative function. If $S = f^{-1}(g)$, then

$$\prod_{i=1}^{m} s_i = \prod_{j=1}^{n} t_j, \quad s_i, t_j \in S \Rightarrow n = m \pmod{o(g)}.$$  

(In case $o(g) = \infty$ we mean equality.)

(b) If $G$ is a divisible Abelian group, $g \in G$ and $S \subset N$ satisfies (6.2), then there exists a $G$-multiplicative $f$ with $S = f^{-1}(g)$.

Proof. (a) is trivial.

To prove (b) let $G_0 = \text{gen} S$, where this is meant in $MG_Q$. $G_0$ consists of the elements of the form

$$x = \frac{s_1 \ldots s_m}{t_1 \ldots t_n}, \quad s_i, t_j \in S.$$

Let for this $x$

$$\varphi_a(x) = g^{x-a}.$$
The condition (6.2) just means that this value is unique. \( \varphi_b : G_b \rightarrow G \) is a homomorphism. By a well-known theorem of Baer [1] (see also Fuchs [6]) a divisible group is injective, that is, this homomorphism can be extended to a homomorphism \( \varphi : MGQ \rightarrow G \). Since for \( s \epsilon S \), we have

\[ \varphi(s) = \varphi_b(s) = g, \]

To prove Theorem (2.2) we need

(6.3) **Lemma.** Let \( D_1 = \{0, 1\}, D_2 = (D_1, \cdot), D = D_2^\infty \) (complete direct product) and \( 0 = (0, 0, \ldots) \epsilon D \). If \( S \subset N \) satisfies

\[ \forall n \in N, \quad n \epsilon S \Rightarrow m \epsilon S, \]

then there exists a \( D \)-multiplicative \( f \) with \( S = f^{-1}(0) \).

The lemma implies Theorem (2.2) immediately, since Besicovitch [2] constructed a sequence satisfying (6.4) and having no density.

**Proof of Lemma (6.3).** A \( D \)-multiplicative \( f \) can be given via a sequence \( f_1, f_2, \ldots \) of \( D_2 \)-multiplicative functions. Let

\[ f_b(n) =
\begin{cases} 
1 & \text{if } \epsilon S \text{ and } \exists k,
0 & \text{otherwise}.
\end{cases} \]

This \( f_b \) is obviously \( D_2 \)-multiplicative. Evidently for \( s \epsilon S \), \( f_b(s) = 0 \), but if \( s \notin S \), then \( f_b(s) = 1 \).

7. **The metric space of multiplicative functions.** Throughout this and the next section we shall always speak of multiplicative functions, mapping into a fixed structure \( G \), and we shall assume conditions (A) and (B) of Section 2, without any more mentioning.

(7.1) **Definition.** Let for the multiplicative functions \( f_1, f_2 \),

\[ \varrho(f_1, f_2) = \sum_{p^k \epsilon \omega(f_1 f_2)} p^{-k}. \]

This is obviously a metrics (with the exception that it may be infinite). Convergence will always be meant in this metrics.

Our aim is to prove

(7.3) **Selection Theorem.** Suppose \( \varrho(f_1, f_2) < \infty \) and that one of the following conditions holds:

(a) \( f(2^k) = f_b(2^k) \) for all \( k \),

(b) \( f_b(2^k+1) = f(2)f_b(2^k) \) for all \( k \).

Let \( R \) be a class of subsets of \( G \), satisfying the following condition:

(7.4) \[ K \epsilon R, \quad g \epsilon G \Rightarrow g^{-1}K \epsilon R, \]

where

\[ g^{-1}K = \{ h : h \epsilon G, gh \epsilon K \}. \]

(i) If \( R \) is a density class for \( f_1 \), then so is for \( f \).

(ii) If \( R \) is a uniform density class for \( f_1 \), then so is for \( f \).

(iii) If \( d(f_1, K) = 0 \) for all \( K \epsilon R \), then \( d(f, K) = 0 \) as well.

The theorem is not true without any condition on \( f_b(2^k) \). (The counter-example is complicated.) But it remains true for arbitrary groupoids; the generalization of the proof presents only some algebraic difficulties, due to the fact that some auxiliary functions may not exist.

The theorem will be proved in the next section; here we make some preliminary investigations.

(7.5) **Statement.** For arbitrary \( f_1, f_2, K \subset G \) and \( a \geq 0 \) we have

\[ \| N(f_1, K, a) - N(f_2, K, a) \| \leq \varrho(f_1, f_2). \]

**Proof.** If \( f_1(n) \neq f_2(n) \), then \( n \) must be divisible by at least one \( p^k \) for which \( f_1(p^k) \neq f_2(p^k) \). For a fixed \( p^k \) the number of these \( n \) is \( \varrho(p^{-k}) \).

(7.6) **Statement.** For every \( f_1, f_2 \) and \( K \subset G \)

\[ \| df_1, K \| - df_2, K \| \leq \varrho(f_1, f_2), \]

and if \( df_1, K \) and \( df_2, K \) exist, then

\[ \| df_1, K \| - df_2, K \| \leq \varrho(f_1, f_2). \]

(7.7) **Statement.** If \( f_n \rightarrow f \) and \( K \subset G \), then

\[ df_n, K \rightarrow df(K), \]

and if \( df(K) \) exists then

\[ \| df(K) \| \leq \varrho(f_1, f_2). \]

(7.8) **Statement.** If \( f_n \rightarrow f \), \( K \subset G \) and \( df_n(K) \) exists for all \( n \), then \( df(K) \) exists and

\[ df(K) \rightarrow df(K). \]

(7.9) **Statement.** If \( f_n \rightarrow f \) and \( R \) is a uniform density class for \( f_n \), then \( R \) is a uniform density class for \( f \) as well.

**Proof.** We must prove

\[ a = \limsup sup_{n \rightarrow \infty} \left| \frac{N(f_n, K, a) - N(f_n, K, a)}{a} \right| = 0. \]
But from (7.5) and (7.6)
\[|N(f, K, \omega) - a(f, K)| \leq |N(f, K, \omega) - \bar{d}(f, K)| + |N(f, K) - N(f, K, \omega)| + |d(f, K) - \bar{d}(f, K)|\]
\[\leq |N(f, K, \omega) - \bar{d}(f, K)| + 2\epsilon(f, K, f).\]

By the assumption
\[a \leq 2\epsilon(f, K, f);\]
making \(n \to \infty\) we get the desired result. ■

The metrics \(\bar{d}\) is the most comfortable, but not the best one. For example, for \(\epsilon(f_1, f_2) > 1\) the statements (7.5) and (7.6) are vacuous. For sake of completeness I describe the best possible metrics \(\bar{d}\) for which (7.6) is true.

(7.10) Definition. Let \(\bar{d}(f_1, f_2)\) be the density of natural numbers that are of the form \(p^4 m, p^4 m, f_1(p^4) \neq f_2(p^4)\).

Well-known elementary methods yield
\[(7.11) \text{Statement. } \bar{d}(f_1, f_2) \text{ exists for every } f_1 \text{ and } f_2, \text{ and } \]
\[\bar{d}(f_1, f_2) = 1 - \prod_p (1 - \sum_{k \mid p^{4}m} (p - 1)p^{-k}).\]

From the definition easily follows that \(\bar{d}\) is really a metrics. (7.5) does not remain valid, but
\[(7.12) \text{Statement. For every } f_1, f_2, K = 0 \text{ and } x \geq 0 \]
\[|N(f_1, K, \omega) - N(f_2, K, \omega)| \leq x \cdot \bar{d}(f_1, f_2) + o(x),\]
where the \(o\) depends only on \(x, f_1\) and \(f_2\) and not on \(K\). ■

In Statements (7.6)-(7.9) \(\epsilon\) can be replaced by \(\bar{d}\).

\(\epsilon\) and \(\bar{d}\) are connected by the following relations, that can be got from (7.2), (7.10) and (7.11) by an easy computation.

(7.13) Statement. For every \(f_1\) and \(f_2\)
\[\bar{d}(f_1, f_2) \leq \epsilon(f_1, f_2), \]
\[1 - e^{-\epsilon(f_1, f_2)} \leq \epsilon(f_1, f_2) \leq 1 - e^{-\bar{d}(f_1, f_2)}, \]
\[\bar{d}(f_1, f_2) = 1 \text{ if and only if } \epsilon(f_1, f_2) = \infty.\]

3. The proof of the neglection theorem

(8.1) Lemma. Let \(f\) be a multiplicative function, \(p\) a prime and \(R\) a density class for \(f\), satisfying (7.4). If \(p = 2\), we suppose also condition (b) of (7.3), with \(f_0\) replaced by \(f\).

Let
\[M(K, \omega) = \{n : n \leq \omega, f(n) \epsilon K, p^4 n\}.\]

(a) For \(K \epsilon R\) there exists
\[(8.2) \quad d'(K) = \lim_{n \to \infty} \frac{M(K, \omega)}{\omega},\]
and we have
\[(8.3) \quad d(f, K) = \sum_{l=0}^{\infty} p^{-l}d'(f(p^4)^{-1}K).\]

(b) If \(K\) was a uniform density class for \(f\), then (8.2) is uniform in \(K \epsilon R\).

Proof. Let \(n = p^4 m, p^4 m, f(m) \epsilon K\) be equivalent to \(f(m) \epsilon f(p^4)^{-1}K\), so we have

\[(8.4) \quad N(f, K, \omega) = \sum_{l=0}^{\infty} M(f(p^4)^{-1}K, \omega p^{-l})\]
\[= M(K, \omega) + \sum_{l=0}^{\infty} M(f(p^4)^{-1}K, \omega p^{-l}).\]

From (8.4)
\[(8.5) \quad M(K, \omega) = N(f, K, \omega) - \sum_{l=0}^{\infty} M(f(p^4)^{-1}K, \omega p^{-l}).\]

Let first \(p > 3\) and let
\[a(K) = \limsup_{x \to \infty} \frac{M(K, \omega)}{\omega} - \liminf_{x \to \infty} \frac{M(K, \omega)}{\omega}, \quad a = \sup_{K \epsilon R} a(K).\]

Evidently \(0 \leq a(K) \leq 1, 0 \leq a \leq 1\). From (8.5), using
\[N(f, K, \omega) = \omega d(f, K) + o(\omega) \quad \text{and} \quad 0 \leq M(K, \omega) \leq \omega,\]
we get with an arbitrary \(n\)
\[a(K) \leq \sum_{l=1}^{n} a(f(p^4)^{-1}K)p^{-l} + \sum_{l=n+1}^{\infty} p^{-l}.\]

Making \(n \to \infty\)
\[a(K) \leq \sum_{l=1}^{\infty} a(f(p^4)^{-1}K)p^{-l} < \frac{a}{p - 1}.\]
so \( a \leq a/(p-1) \), which implies \( a = 0 \), which is (a). (8.3) is clear from (8.4). (Observe that a sum like

\[
\sum_{t=1}^{\infty} \frac{1}{t} M(K_t, a^p t^{-1})
\]

must be uniformly convergent, since it has the numerical majorant \( \sum t^{-1} \).

To prove (b) let

\[
b(x) = \sup_{K \in \mathcal{A}} |M(K, x) - \sigma d(K)| \quad \text{and} \quad b = \limsup_{x \to \infty} \frac{b(x)}{x}.
\]

From (8.5) and (8.3) we have

\[
|M(K, x) - \sigma d(K)| \leq |N(f, K, x) - \sigma d(f, K)| + \sum_{t=1}^{\infty} |M(f(p^t-1)K, a^p t^{-1}) - a^p t^{-1} d(K)|.
\]

Taking sup and applying the fact that \( \mathcal{A} \) was a uniform density class, we get

\[
b(x) \leq \sum_{t=1}^{\infty} b(a^p t^{-1}) + o(x).
\]

Since evidently \( b(a) \leq a \), we have with an arbitrary \( n \)

\[
b(x) \leq \sum_{t=1}^{n} b(a^p t^{-1}) + o \left( \sum_{t=n+1}^{\infty} p^{-t} + o(1) \right).
\]

From the definition of \( b \)

\[
b \leq \sum_{t=1}^{n} b(a^p t^{-1}) + \sum_{t=n+1}^{\infty} p^{-t};
\]

making \( n \to \infty \) we get \( b \leq b/(p-1) \), that is \( b = 0 \), which is (b).

If \( p = 2 \), then by condition (b) of (7.3) we have simply

\[
M(K, x) = N(f, K, x) - N \left( f, f(2)^{-1} K, \frac{x}{2} \right),
\]

which implies (a) and (b) immediately. \( \square \)

\textbf{Lemma.} Let all the letters mean the same as in (8.1). If we suppose (8.1(a)), resp. (8.1(b)), then \( \mathcal{A} \) is a density class, resp. a uniform density class for \( f \). (Here we do not suppose (7.3(b)).) The numbers \( d(f, K) \) satisfy (8.3).

This follows directly from (8.4). \( \square \)

\textbf{Proof of the theorem.} Let

\[
f(n) = \begin{cases} f(p^n), & p \leq p_n, \\ f_0(p^n), & p > p_n. \end{cases}
\]

First we show our statements for \( f_n \) by induction. For \( f \), this is the assumption. Going from \( f_n \) to \( f_{n+1} \) we first apply Lemma (8.1) for \( f = f_n \), \( p = p_{n+1} \), then Lemma (8.6) for \( f = f_{n+1} \), \( p = p_{n+1} \). (Since \( f_n \) and \( f_{n+1} \) are equal at all the prime-powers, except of the powers of \( p_{n+1} \), their \( M \)-functions are the same.)

Here the only critical point is \( n = 0 \), \( p_1 = 2 \). But in this case either \( f_1 = f_0 \) (case (7.2(a))) or (7.2(b)) holds, which justifies the application of Lemma (8.3).

Evidently \( f_n \to f \), so we get (i) from (7.8) and (ii) from (7.9). In case \( d(f_0, K) = 0 \) for all \( K \in \mathcal{A} \) we first get \( d(f_n, K) = 0 \) by (8.3) and (iii) by (7.8). \( \square \)

\textbf{9. Concentrated functions.} In this section we prove the local limit theorems for concentrated functions. We begin with the case \( |G| < \infty \). We have

\[
N(f, g, x) = \frac{1}{|G|} \sum_{x \in G} \sum_{n \in \mathbb{Z}} \phi(f(n)),
\]

where \( \phi \) runs over all the characters of \( G \).

\( \phi(n) = \chi(f(n)) \) is a complex-valued multiplicative function, assuming only the values \( e^{2\pi i kT} \), \( 0 \leq k \leq T-1 \), \( T = |G| \). Therefore we can apply the following result of Wirtinger [11] (Satz 1.2.1. and 1.2.2).

\textbf{Lemma.} Let \( \phi \) be a complex-valued multiplicative function, \( |\phi(n)| = 1 \). Suppose that there is an arc of the unit circle which contains no \( \phi(p) \). Then here exists

\[
M(\phi) = \lim \frac{1}{x} \sum_{n \leq x} \phi(n) = \prod_p \left( 1 - \frac{1}{p} \sum_{k \leq \infty} \frac{\phi(p^k)}{p^k} \right).
\]

\( M(\phi) = 0 \) if and only if either \( \phi(2^k) = -1 \) for all \( k \) or

\[
\sum_{p} \frac{1}{p} \left( 1 - \text{Re} \phi(p) \right) = \infty.
\]

\textbf{Lemma.} Let \( G \) be a finite Abelian group, \( |G| = T \), and let \( f \) be \( G \)-multiplicative. \( d(f, g) \) exists for all \( g \in G \). If the concentration group of \( f \) is \( G \), then \( d(f, g) = \frac{1}{T} \) for all \( g \in G \).

\textbf{Proof.} (9.1) and (9.2) imply that \( d(f, g) \) exists and

\[
d(f, g) = \frac{1}{T} \sum_{x} \phi(x) M(\chi(f)).
\]
To prove the second statement let us observe that the principal character \( \chi_0 = 1 \) gives \( 1/2 \). We prove that for \( \chi \neq \chi_0 \) we have \( M(\chi) = 0 \), where \( \chi = \chi(f) \). If no, then, since \( \chi(p) \neq 1 \) implies
\[
1 - \text{Re}\chi(p) \geq 1 - \cos \frac{2\pi}{p} > 0,
\]
using (9.4) we get
\[
\sum_{p \mid \chi(p) \neq 1} \frac{1}{p} < \infty.
\]
This means
\[
\sum_{p \mid \chi(p) \neq 1} \frac{1}{p} < \infty,
\]
that is, if \( K \) denotes the concentration complex of \( f \), we have \( K = \ker \chi \).

Since \( \ker \chi \) is a subgroup,
\[
\ker \chi \triangleright \text{gen} K = G, \quad \chi = \chi_0. \quad \blacksquare
\]

(9.6) \textbf{Lemma.} Let \( G \) be an Abelian group and \( f \) a concentrated \( G \)-multiplicative function.
\[
\mathcal{G} = \{ (g) : g \in G \}
\]
is a uniform density class for \( f \).

Proof. Let \( G_1 \) be the concentration group of \( f \) and
\[
f_1(p^k) = \begin{cases} f(p^k) & \text{if } f(p^k) \in G_1, \\ e & \text{otherwise}. \end{cases}
\]
We have \( g(f, f_1) < \infty \) so we can apply Lemma (9.5) and the neglection theorem. (In a finite group uniformity is fulfilled automatically.)

(9.7) \textbf{Lemma.} Let \( f \) be a concentrated function with the concentration group \( G_1 \). If \( f(p^k) \in G_1 \) for \( p \geq p_n \), then \( d(f, g) \) depends only on the coset of \( G_1 \) in which \( g \) lies.

Proof. Induction on \( n \). For \( n = 1 \) this is contained in (9.5). Suppose we know it for \( n - 1 \). Let
\[
f_n(p^k) = \begin{cases} e & \text{if } p = p_n, \\ f(p^k) & \text{otherwise}. \end{cases}
\]
We know the statement for \( f_1 \). Setting
\[
M(g, w) = |\{ m : m \leq w, f(m) = g, p_n^m \}| \quad \text{and} \quad N(f_1, g, w) = N(f_1, g, p_n^{-1}w),
\]
we have
\[
N(g, w) = N(f_1, g, w) - N(f_1, g, p_n w^{-1}).
\]

and
\[
N(f, g, w) = \sum_{k=0}^{\infty} M(g(p_n^k)^{-1}, p_n w^{-k}).
\]

From these formulas
\[
d(f, g) = \frac{2}{p} \sum_{k=0}^{\infty} d(f_1, g(p_n^k)^{-1}) p_n^{-k},
\]
and the right-hand side depends only on the coset of \( G_1 \), containing \( g \). \( \blacksquare \)

(9.8) \textbf{Lemma.} If \( f \) is concentrated with the concentration group \( G_1 \), then \( d(f, g) \) depends only on the coset of \( G_1 \), containing \( g \).

Proof. Let
\[
f_n(p^k) = \begin{cases} e & \text{if } p \geq p_n \text{ and } f(p^k) \in G_1, \\ f(p^k) & \text{otherwise}. \end{cases}
\]
Obviously \( f_n \to f \). If \( g_1 \) and \( g_2 \) are in the same coset of \( G_1 \), then by (9.7) and (7.8)
\[
d(f, g_1) - d(f, g_2) = \lim_{n \to \infty} |d(f_n, g_1) - d(f_n, g_2)| = 0. \quad \blacksquare
\]

(9.9) \textbf{Definition.} Let \( G \) be a group and \( f \) \( G \)-multiplicative. We call \( f \) superconcentrated if
\[
\sum_{p \mid \text{inf} f} p^{-k} < \infty.
\]

We can reduce the investigation of concentrated functions to superconcentrated ones.

(9.10) \textbf{Definition.} Let \( G \) be an Abelian group, \( G_1 < G \) and \( \varphi : G \to G/G_1 \) the natural homomorphism. We define the \textit{factor-function} of \( G \)-multiplicative \( f \) with respect to \( G_1 \) as
\[
f_1 = f|G_1 = \varphi(f).
\]
(That is, \( f_1(n) \) is the coset of \( G_1 \) containing \( f(n) \).)

(9.11) \textbf{Statement.} If \( f \) is concentrated with the concentration group \( G_1 \), then \( f|G_1 \) is superconcentrated. \( \blacksquare \)

By (9.8) we have
\[
d(f, g) = \frac{1}{|G_1|} d(f_1, g G_1). \quad \blacksquare
\]

2 - Acta Arithmetica XXXIII:4
(9.13) **Lemma.** If \( f \) is superconcentrated, then

\[
\frac{d(f, g)}{n_0} > 0 \quad \text{for all } g \in \text{Im } f,
\]

\[
\sum_{g \in G} \frac{d(f, g)}{n_0} = 1.
\]

Proof. Let \( g \in \text{Im } f \), that is, \( g = f(n_0) \) with some \( n_0 \). Let \( Q \) be the set of \( \text{prime-powers } p^k \) for which \( f(p^k) \neq e \) or \( p | n_0 \). Obviously

\[
\sum_{g \in Q} \frac{1}{q} < \infty.
\]

The well-known inequality of Heilbronn [9] and Rohrbach [10] easily implies that the sequence \( S \) of numbers which are divisible by no element of \( Q \) has a density

\[
\text{den}S > \prod_{q \in Q} \left(1 - \frac{1}{q}\right) > 0.
\]

Since \( n_0 S \subseteq f^{-1}(g) \), we have

\[
d(f, g) \geq \frac{\text{den}S}{n_0} > 0,
\]

which is (a).

To prove (b) let \( Q \) be the set of \( \text{prime-powers } p^k \) for which \( f(p^k) \neq e \) and let \( Q_1 \) be a finite subset of \( Q \) satisfying

(9.14)

\[
\sum_{g \in Q \setminus Q_1} \frac{1}{q} < \varepsilon.
\]

Let \( K \) denote the set of elements of \( G \) which can be written as \( f(q_1, \ldots, q_m) \), \( q_i \in Q_1, q_i \neq q_j \).

\( K \) is finite, so there exists \( d(f, K) \) and \( d(f, G \setminus K) \). If

\[
n = \prod p_i^{k_i}, \quad f(n) \in K,
\]

then we must have

\[
p_i^{k_i} \in Q \setminus Q_1
\]

for at least one \( i \).

Now because of (9.14) we have

\[
d(f, G \setminus K) < \varepsilon, \quad d(f, K) > 1 - \varepsilon;
\]

\( \varepsilon \) was arbitrary, which implies (b). \( \Box \)

Combining (9.12) and (9.13) we get

(9.15) **Statement.** Lemma (9.13) is true for every concentrated function.

10. **Deconcentrated functions.** Let \( G \) be an infinite Abelian group. By a character of \( G \) we mean a homomorphism

\[
\chi : G \to \mathbb{C}^\times; \quad |\chi(g)| = 1.
\]

The group of characters will be denoted by \( \text{char } G \).

If \( G \) is infinite, we cannot express \( N(f, g, w) \) by a finite combination of characters, but we can do the following.

Let \( \chi_1, \ldots, \chi_m \) be arbitrary characters and

\[
\phi(n) = \sum_{j=1}^m \chi_j(f(n)) \chi_j(f(n))
\]

Obviously

\[
\phi(n) = \begin{cases} m^2 & \text{if } f(n) = g, \\ 0 & \text{otherwise}, \end{cases}
\]

hence we have

\[
N(f, g, w) \leq m^{-2} \sum_{n \in S} \phi(n).
\]

We have to estimate

\[
\phi(n) = \sum_{k \leq n} \phi(n) = \sum_{k \leq n} c_k \sum_{n \in S} \psi_k(n),
\]

where

\[
c_k = \chi_k(g) \chi_k(g)
\]

and

\[
\psi_k(n) = \chi_k(f(n)) \chi_k(f(n))
\]

is a multiplicative function satisfying

\[
|\psi_k(n)| = 1.
\]

Since \( |c_k| = 1 \), for every \( g \)

\[
N(f, g, w) \leq m^{-2} \sum_{k \leq n} \sum_{n \in S} |\psi_k(n)|.
\]

We shall estimate \( \sum_{k \leq n} |\psi_k(n)| \) using the theorems of Halász. The main difficulty will be to find suitable characters, which yield a good estimate.
Halász [7] proved

(10.7) **Lemma.** Let \( \varphi \) be a complex-valued multiplicative function, \(|\varphi(n)| = 1\),

\[
\Phi(x) = \sum_{n \leq x} \varphi(n).
\]

One of the following two possibilities must hold.

(a) For arbitrary real \( \delta \)

\[
\sum_{p} \frac{1}{\log p} \left| 1 - \text{Re}\left( p^{-\varphi(p)} \right) \right| = \infty,
\]

\[
\Phi(x) = O(x).
\]

(b) There exists exactly one \( \delta = \delta(\varphi) \), for which

\[
\sum_{p} \frac{1}{\log p} \left| 1 - \text{Re}\left( p^{-\varphi(p)} \right) \right| < \infty.
\]

In this case

\[
\Phi(x) = cx^{1+\delta} L(\log x) + o(x),
\]

where \( L \) is a complex-valued function satisfying

\[
|L(x)| = 1, \quad \frac{L(u_1)}{L(u)} \to 1
\]

uniformly if \( u \to \infty \) and \( u \leq u_1 \leq 2u \).

(10.8) **Lemma.** If \( \delta(\varphi) \) and \( \delta(\psi) \) exist, then so do \( \delta(\psi) \) and \( \delta(\psi \varphi) \) as well, and

\[
\delta(\psi) = -\delta(\varphi), \quad \delta(\psi \varphi) = \delta(\varphi) + \delta(\psi).
\]

**Proof.** The first is trivial. The second is due to the inequality

\[
|a| = |b| \Rightarrow 1 - \text{Re} a \leq 2 \left( |1 - \text{Re} a| + |1 - \text{Re} b| \right).
\]

(10.9) **Lemma.** In (10.7(b))

\[
\left| \sigma \right| \leq \frac{1}{\sqrt{1 + \delta^2}}.
\]

**Proof.** Summation by parts yields (in virtue of \( L(u_1)/L(u) \to 1 \))

\[
\sum_{n \leq x} \varphi(n) n^{-\varphi} = cL(\log x)(1 + \delta)x + o(x),
\]

and the lemma follows from

\[
\left| \sum_{n \leq x} \varphi(n) n^{-\varphi} \right| \leq o(x).
\]

We need a lemma, which will be proved in the next section.

(10.10) **Lemma.** Let \( G \) be an Abelian group and \( f \) a deconcentrated \( G \)-multiplicative function. Either there is a character \( \chi \) for which \( h(\chi(f)) \) exists and differs from 0, or there is an infinite sequence \( \chi_1, \chi_2, \ldots \) of characters such that

\[
h(\chi_j(f)) \chi_j(f) \]

never exists if \( j \neq k \).

(10.11) **Lemma.** If \( f \) is deconcentrated, then

\[
\frac{N(f, g, w)}{\omega} \to 0
\]

uniformly in \( g \) when \( x \to \infty \).

**Proof.** We distinguish two cases according to the previous lemma. Suppose first that \( \chi \) is a character for which there exists

\[
h = h(\chi(f)) \neq 0.
\]

Apply (10.6) with \( X_j = X_j^\omega \).

If \( j = k \), we have the trivial estimate

\[
\left| \sum_{n \leq x} \psi(n) \right| \leq o(x).
\]

If \( j \neq k \), then by (10.7(b)), (10.8) and (10.9)

\[
\left| \sum_{n \leq x} \psi(n) \right| \leq \frac{\omega}{|j - k| m^2} + o(x) \leq \frac{\omega}{m^2} + o(x).
\]

Summing up we get

\[
N(f, g, w) \leq \frac{1}{m} \left( 1 + \frac{1}{|j|} \right) \omega + o(x);
\]

since \( m \) was arbitrary, we are ready.

Now suppose the second possibility and apply (10.6) with the \( X_j \)'s of Lemma (10.11). In case \( j = k \) we have the trivial estimate. In case \( j \neq k \) by (10.7(a))

\[
\sum_{n \leq x} \psi(n) = o(x),
\]

so we get again

\[
N(f, g, w) \leq \frac{\omega}{m} + o(x).
\]
11. Random characters. In this section we prove Lemma (10.10). We shall define a probability measure on the set of characters and then prove that if we choose \( \chi_0 \) at random, then with probability 1 either one of them satisfies the first possibility, or the whole sequence satisfies the second.

First we define a topology on \( \text{char} G \). Let the subbase of \( \chi \) consist of the sets

\[
K_\chi(g, c) = \{ \psi : |\psi(g) - \chi(g)| < c \}, \quad g \in G, \quad c > 0.
\]

This makes \( \text{char} G \) a compact Hausdorff topological group. (The space of all the functions, bounded by 1, is compact, and the characters form a closed subset.)

Now we can regard the (normalized) Haar-measure. (See, for example, Halmos [8].) This measure — which we shall denote by \( P \) — is known to be invariant under multiplication.

(11.1) **Statement.** Let \( g \in G \). If \( o(g) < \infty \), then

\[
P\left( \chi(g) = \exp \frac{2\pi ik}{o(g)} \right) = \frac{1}{o(g)}
\]

for all \( k \). If \( o(g) = \infty \), then \( \chi(g) \) is uniformly distributed in the unit circle. \( \blacksquare \)

(11.2) **Statement.** For \( g \in G \)

\[
E(\chi(g)) = \begin{cases} 
0 & \text{if } g \neq e, \\
1 & \text{if } g = e.
\end{cases}
\]

(11.3) **Lemma.** Let \( f \) be a deconcentrated \( G \)-multiplicative function. We have with probability 1

\[
\sum_{p} \frac{1}{p} (1 - \text{Re} \chi(p)) = \infty.
\]

**Proof.** Introduce the notation

\[
a(g) = \sum_{p \mid o(g)} \frac{1}{p}.
\]

We must prove

\[
\sum_{g \in G} a(g)(1 - \text{Re} \chi(g)) = \infty
\]

with probability 1.
Proof of Lemma (10.10). Observe that Lemma (11.3) just means that the event

\[ \overline{h(z(f)) = 0} \]

has probability 0. If there is a \( x \) with

\[ h(z(f)) \neq 0, \]

we are ready; if not, we know that the event

\[ \overline{h(z(f)) \text{ does not exist}} \]

happens with probability 1.

Now we choose \( z \) at will. If we have chosen \( z_1, \ldots, z_{n-1} \), we have to find a \( z_n \) such that

\[ h(z_j(f)z_n(f)), \quad 1 \leq j \leq n-1, \]

should not exist. But — since the measure \( P \) was invariant — this requirement is fulfilled by almost all \( z \).

12. Completion of proofs of Theorems (1.4), (3.7), (3.10), (3.11) and (3.12). The first four theorems are ready, we only have to gather the details. For deconcentrated functions they all follow from Lemma (10.11). For concentrated functions (3.7) and its weaker form (1.4) follow from (9.6), (3.10) from (9.13) and (3.11) from (9.8).

To prove (3.12) we require a lemma. (We leave the easy proof to the reader.)

(12.1) **Lemma.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two classes of subsets of a set \( X \). Suppose \( \mathcal{A} \) is closed under intersection and \( \mathcal{B} \) is closed under proper difference and disjoint union, that is

\[ A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}, \]

\[ (A, B \in \mathcal{B}, A \supset B) \Rightarrow A \setminus B \in \mathcal{B}, \]

\[ (A, B \in \mathcal{B}, A \cap B = \emptyset) \Rightarrow A \cup B \in \mathcal{B}. \]

If \( \mathcal{B} \supset \mathcal{A} \) then \( \mathcal{B} \) contains the set algebra generated by \( \mathcal{A} \) (if \( X \in \mathcal{A} \)).

Proof of (3.13). Let \( G \) be an Abelian group, \( f \) a \( G \)-multiplicative function, \( \mathcal{A} \) the class of all cosets of all subgroups of \( G \) and \( \mathcal{B} \) the class of all density sets for \( f \). \( \mathcal{A} \) and \( \mathcal{B} \) satisfy the conditions of Lemma (12.1), so we are ready if we can prove \( \mathcal{B} \supset \mathcal{A} \). That is, we must prove that a coset is a density set for \( f \). But let \( G_1 < G \); applying Theorem (1.4) for the factor-function

\[ f_1 = f/G_1, \]

(see the definition in (9.10)) we get the desired result.

13. Proof of Theorem (4.2). Because of Theorem (3.10) we may suppose \( f \) concentrated. Let its concentration group be \( G \). By the uniform distribution theorem (3.11) we have for all \( g \in G \)

\[ \overline{d(f, g) \leq 1/|G|}. \]

Indeed only in case \( \text{Im} f = \mathbb{G} \).

If \( |G| > 1 \), we are ready. If \( |G| = 2 \), then in (13.1) equality must hold, that is, \( \text{Im} f = \mathbb{G} \), which is contained in possibility (b).

Remains the case of superconcentrated \( f \). In this case \( f_n \to f \), where

\[ f_n(p^k) = \begin{cases} p & \text{if } p \leq P_n, \\ \varepsilon & \text{if } p > P_n. \end{cases} \]

Furthermore

\[ d(f_n, g) = \left( 1 - \frac{1}{P_n} \right)^{\sum_{k=1}^\infty f_n^{-1}(p^k) g(p^{-k})} P_n. \]

Introduce the notations

\[ S_n = \max_g d(f_n, g), \quad s_n = \max_{g \neq \varepsilon} d(f_n, g). \]

(13.2) implies

\[ S_n \leq s_{n-1}. \]

(13.4) \[ s_n \leq \left( 1 - \frac{1}{P_n} \right)^{s_{n-1}} + \sum_{k=1}^\infty S_{n-1} P_n \]

We prove \( s_n \leq 1/2 \) by induction. It is true for \( n = 0 \). If \( s_{n-1} \leq 1/2 \), then

\[ S_{n-1} = \max(s_{n-1}, d(f_{n-1}, g)) \leq \max(s_{n-1}, 1 - s_{n-1}) = 1 - s_{n-1}. \]

From (13.4)

\[ s_n \leq \left( 1 - \frac{1}{P_n} \right)^{s_{n-1}} + \frac{1}{P_n} \]

So in case \( g \neq \varepsilon \)

\[ d(f, g) = \lim d(f_n, g) \leq 1. \]

Now we have to determine the extremal functions. Suppose \( d(f, g) = 1/2 \). We have

\[ s_n \geq d(f_n, g) \to 1/2, \]

so by (13.3) \( s_n \geq 1/2 \) for all \( n \).
Regard first the case that $f$ assumes not only the values $e$ and $g$.

In this case
\[
\hat{d}(f, e) < \frac{1}{4},
\]
so for $n > n_0$
\[
\hat{d}(f_n, e) < \frac{1}{4}.
\]

This implies $S_n = s_n$; since $S_n \geq 1/3$, $s_n \leq 1/2$, we have
\[
S_n = s_n = \frac{1}{4} \quad (n > n_0).
\]

Since $\hat{d}(f_n, g) \to 1/2$ and $\hat{d}(f_n, e) \to \hat{d}(f, e) > 0$, hence for $g' \neq g$, $e$ and $n > n_0$
\[
\hat{d}(f_n, g') \leq 1 - \hat{d}(f_n, g) - \hat{d}(f_n, e) < \frac{1}{4},
\]
so $s_n = 1/2$ implies
\[
\hat{d}(f_n, g) = \frac{1}{4}.
\]

Summing up
\[
n > n_0 \Rightarrow \hat{d}(f_n, g) = \frac{1}{4}, \quad \hat{d}(f_n, g') < \frac{1}{4} \quad (g' \neq g).
\]

(This $n_0$ is the greater of the previous two — different — $n_0$'s.)

For $n > n_0$ in (13.4) equality must hold, and this is possible only if $f(p_n^k) = e$, $k = 1, 2, \ldots$ This means that the sequence stabilizes. Let $n_1$ be the greatest suffix for which $f_n \neq f_{n-1}$.

If $n = 1$, we are ready. If $n > 2$, $p_n > 2$, hence equality in (13.5) holds only under the condition
\[
s_{n-1} = \frac{1}{4},
\]
and in (13.4) only if
\[
\hat{d}(f_{n-1}, f(p_n^k)^{-1}g) = \frac{1}{4}, \quad k = 0, 1, \ldots
\]

Setting $k = 0$ we get
\[
\hat{d}(f_{n-1}, g) = \frac{1}{4}.
\]

Let $k$ be such that $f(p_n^k) = e$ (such a $k$ must exist, otherwise $f_n = f_{n-1}$).

We have
\[
\hat{d}(f_{n-1}, g') = \frac{1}{4}, \quad g' = f(p_n^k)^{-1}g \neq g.
\]

$g' \neq e$ would imply (since $\hat{d}(f_{n-1}, e) > 0$)
\[
\sum_{k \in \mathbb{Z}} \hat{d}(f_{n-1}, k) > 1;
\]
a contradiction, so $g' = e$. This means that
\[
f(p_n^k) = e \text{ or } g, \quad k = 1, 2, \ldots
\]

We got
\[
\hat{d}(f_{n-1}, e) = \hat{d}(f_{n-1}, g) = \frac{1}{4}.
\]

Therefore $\inf f_{n-1} = \{e, g\}$. There can be only one prime $p$ for which $f_{n-1}(p^k) \neq e$, since
\[
f_{n-1}(p^k) = f_{n-1}(g^m) = g, \quad p \neq q
\]
would imply
\[
f_{n-1}(p^k g^m) = g \neq e, g.
\]

(If $g' = e$, then $f_n$ would not assume values different from $e$ and $g$.) Let this prime be $p$.

\[
\hat{d}(f_{n-1}, g) = \left(1 - \frac{1}{q}\right) \sum_{p_{n-1}(k^2) = q} p^{-k} = \frac{1}{2}
\]
is possible only if $g = 2$ and $f_{n-1}(2^k) = g, k = 1, 2, \ldots$; this is case (a).

There remained the case $\inf f = \{e, g\}$.

If there is only one prime for which $f(p^k) \neq e$, we get no new solution. If we have more, we have again $g^2 = e$. Without loss of generality we can restrict ourselves to the case $g = -1, e = +1, f$ multiplicative in the usual sense. Now
\[
\hat{d}(f, g) = \frac{1}{4}
\]
is equivalent to
\[
\sum_{n \in \mathbb{Z}} f(n) = o(x).
\]

By Wirtinger's theorem (see our Lemma (9.2)) we have for some $p$
\[
1 + \sum_{k=1}^{\infty} f(p^k) = 0.
\]

This is impossible unless $p = 2, f(2^k) = -1, k = 1, 2, \ldots$, which is contained in case (b).

14. Proof of Theorem (4.3). This proof will not be a beautiful one. It is full of routine calculations, which I shall never detail. A sign (RC) is used to denote where such details are omitted. The proof of the most complicated lemmas is left to the next section.

First we settle the trivial cases. The case $p_n = 2$ follows from Theorem (4.2). Suppose $p_n > 3$.

If $f$ is deconcentrated, we have nothing to prove. If $f$ is concentrated with the concentration group $G_\infty \neq \{e\}$, then by $p_n > 3$ we have $|G_\infty| > 3$, so (13.1) gives $\hat{d}(f, g) \leq 1/3$. Since
\[
W_{n-1} \geq W_{n-1}(1) \geq 1/3 > 1/3,
\]
we are ready. Remains the case $f$ is superconcentrated.
(14.1) **Lemma.** Let \( \alpha(g) = n \), \( n \) a natural number or \( \infty \), \( P \) a set of primes and
\[
\min_{p \in P} p = M, \quad \sum_{p \in P} \frac{1}{p} = \omega.
\]
Let \( f \) denote the strongly multiplicative function defined by
\[
f(p) = \begin{cases} g & \text{if } p \in P, \\ e & \text{if } p \notin P. \end{cases}
\]
We have
\[
|d(f, g) - \omega(f)| < \frac{c}{M},
\]
with an absolute constant \( c \).
Proof in the next section. This lemma shows that \( W_0 \) can be arbitrarily approached.

(14.2) **Lemma.** Suppose we know
\[
d(f, g) < W_p
\]
for strongly \( C(p^\infty) \)-multiplicative functions \( f \) and elements \( g \) of order \( p \), \( p \) prime or \( \infty \). This implies Theorem (14.3).

**Proof.** Let \( G \) be arbitrary, \( e \neq g \in G \) and \( f \) strongly \( G \)-multiplicative. Let \( n = o(g) \) and \( p \) a prime divisor of \( n \) if \( n < \infty \) and \( \infty \) if \( n = \infty \). Let \( g_1 \in C(p^\infty) \), \( o(g_1) = p \). By (6.1) there exists a strongly \( C(p^\infty) \)-multiplicative \( f_1 \) such that
\[
f_1^{-1}(g_1) = f^{-1}(g).
\]
Now we have
\[
d(f, g) \leq d(f_1, g_1) < W_{p_1} \leq W_{p_0},
\]
since \( p_0 \leq p_1 \), and \( W \) is decreasing (RC).

By now on we restrict ourselves to groups \( C(p^\infty) \) and elements \( o(g) = p_0 \).

First we regard functions with \( f(p) = e \) with the exception of a finite set of primes, \( P \). Let
\[
P = P(P) = \{ f: f(p) \neq e \Rightarrow p \in P \},
\]
where it is to be involved that \( f \) is strongly \( G \)-multiplicative, \( G \) is a group of type \( C(p^\infty) \),
\[
\tau(f) = \{ p: p \in P, f(p) = e \},
\]
\[
s(f) = \{ p: p \in P, f(p) = g \},
\]
where \( g \) is the investigated element of order \( p_0 \).

(14.6) **Lemma.** Let \( f \in P \). At least one of the following three possibilities must hold.
(a) \( d(f, g) \leq 1/3 \),
(b) \( r(f) + s(f) = |P| \),
(c) \( \exists \) \( f^* \in P \), for which
\[
d(f^*, g) > \omega(f, g), \quad r(f^*) > r(f).
\]
Proof in the next section.

(14.7) **Lemma.** If
\[
\sup_{p \in P} d(f, g) > 1/3,
\]
then there exists
\[
\max_{f \in P} d(f, g),
\]
and it is reached only at functions satisfying \( r(f) + s(f) = |P| \).

Proof. Start with a function \( f_1 \), for which \( d(f_1, g) > 1/3 \). Iterating Lemma (14.6) we get a sequence of functions \( f_1, f_2, \ldots \) such that both \( d(f_n, g) \) and \( r(f_n) \) are strictly increasing. Since \( r(f_n) \leq |P| \), this sequence must terminate. The last function \( f^* \) satisfies
\[
r(f^*) + s(f^*) = |P|;
\]
since there are only finitely many such functions, we can select the ones with maximal \( d(f, g) \) from them.

(14.8) **Lemma.** Let \( P \) be a finite set of primes and \( f \in P \) the function for which
\[
f(p) = \begin{cases} g & \text{if } p \in P, \\ e & \text{if } p \notin P. \end{cases}
\]
Suppose
\[
d(f, h) < d(f, g)
\]
for every \( e \neq h \in G \) and \( f \neq f^* \). Then there exists another finite set \( P_1 \) of primes for which
\[
g_1 = \min_{p \in P_1} p > g_0 = \min_{p \in P} p
\]
and
\[
d(f_1, g) > d(f_1, g) + 10^{-3} q_0^2,
\]
where
\[
f_1(p) = \begin{cases} g & \text{if } p \in P_1, \\ e & \text{if } p \notin P_1. \end{cases}
\]
Proof in the next section.

(14.9) **Lemma.** For the function \( f \) of Lemma (14.8)
\[
d(f, g) < W_{p_0} - 10^{-3} q_0^{-2}.
\]

**Proof.** If \( d(f, g) \leq 1/3 \), we are ready \( (W_{p_0} \geq 1/e) \). If not, we define a sequence \( f_n \). Let \( f_1 \) be the function, given by Lemma (14.8). Given \( f_n \), let
\[
P_n = \{ p : f_n(p) \neq e \}
\]
and apply Lemma (14.7) for \( F = F(P_n) \). Let \( f'_n \) be one of the extremal functions with maximal \( r(f'_n) \). Now apply Lemma (14.8) for \( f = f'_n \); the produced function will be \( f_{n+1} \).

These functions satisfy
\[
d(f_n, g) \leq d(f'_n, g) < d(f_{n+1}, g),
\]
therefore
\[
d(f, g) > d(f, g) + 10^{-3} q_0^{-2},
\]
on the other hand,
\[
\lim \min_{f_n(0) \neq e} g = \infty,
\]
so by Lemma (14.1)
\[
\lim d(f_n, g) = \lim \min_{f_n(0) \neq e} \left( \sum_{k=0}^{n-1} \frac{1}{g} \right) \leq W_{p_0}.
\]

**Completion of the proof of the theorem.** Let \( f \) be strongly multiplicative. If \( d(f, g) \leq 1/3 \), we are ready. If not, let
\[
f_n(p) = \begin{cases} f(p) & \text{if } p \leq p_n, \\ e & \text{if } p > p_n, \end{cases}
\]
P_n given by (14.10). Applying Lemma (14.7) for \( F = F(P_n) \), we get a function \( f'_n \xi F(P_n) \) with maximal \( d(f'_n, g) \) and the greatest possible \( r(f'_n) \). Lemma (14.9) yields
\[
d(f_n, g) \leq d(f'_n, g) < W_{p_0} - 10^{-3} q_0^{-2},
\]
with
\[
q_n = \min \{ q_n : f_n(0) \neq e \}.
\]

Obviously
\[
f_n \rightarrow f, \quad d(f_n, g) \rightarrow d(f, g);
\]
if \( q_n \rightarrow \infty \), this and (14.11) immediately imply the theorem.

If \( q_n \rightarrow \infty \), then \( f_n \rightarrow E \), where \( E(n) = e \), so
\[
d(f, g) = \lim d(f_n, g) = \lim d(f'_n, g) = d(E, g) = 0,
\]
a contradiction. □

**15. Proof of Lemmas (14.1), (14.6) and (14.8)**

Proof of Lemma (14.1). First let \( n < \infty \). We can assume \( g = \exp \frac{2m}{n} \). Then we have
\[
N(f, g, x) = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} f(j) g^k = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j \in \mathbb{Z}} g^k \sum_{j \in \mathbb{Z}} f(j),
\]
from (15.1), using
\[
\log \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{g^k}{p} \right)^{-1} = \frac{g^k - 1}{p} + O(p^{-2})
\]
we get (RC)
\[
d(f, g) = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j \in \mathbb{Z}} g^k \exp(g^k - 1) x + O \left( \sum_{p \in \mathbb{Z}} p^{-1} \right).
\]
Now we get (14.2) using the formulas (RC)
\[
\sum_{p \in \mathbb{Z}} p^{-2} = O(M^{-1}), \quad \sum_{k=0}^{n-1} \sum_{j \in \mathbb{Z}} g^k \exp(g^k - 1) x = \mu_m(x).
\]
In case \( n = \infty \) we have to start with the formulas
\[
d(f, g) = \prod_{p \in \mathbb{Z}} \left( 1 - \frac{1}{p} \right) \sum_{j \in \mathbb{Z}} \frac{1}{p} \log \left( 1 - \frac{1}{p} \right) = -\frac{1}{p} + O(p^{-2}).
\]

Proof of Lemma (14.6). If (b) does not hold, then there is a \( g_{a} \in F \) such that
\[
h = f(g_{a}) \neq e, g.
\]
Now let
\[
f_{i}(g) = \begin{cases} f(g) & \text{if } g \neq g_{a}, \\ e & \text{if } g = g_{a}. \end{cases}
\]
If there exists a \( g' \in G, g' \neq e \) such that
\[
d(f_{i}, g') > d(f, g),
\]
then let η be an endomorphism of $G$ such that $\eta g' = g$. (Such an endomorphism must exist, since $G$ was a group of type $C(p^n)$ and $o(g) = p^n$.) Now $f^* = \eta f$ satisfies (c).

If there is no such $g'$, regard the following equality:

$$d(f, g) = (1 - q_0^{-1}) \sum_{i=0}^{\infty} q_0^{-i} d(f_i, gh^{-i}),$$

If $d(f_i, gh^{-i}) < d(f, g)$ for all $i$, then equality in (15.2) cannot hold unless $d(f_i, gh^{-i}) = d(f, g)$ for all $i$. In this case

$$d(f, g) = \frac{1}{2} [d(f_1, g) + d(f_1, gh^{-1}) + d(f_1, gh^{-1})] \leq \frac{1}{3},$$

since the elements $g, gh^{-1}, gh^{-2}$ must be different because $o(h) \geq p_0 \geq 3$.

This is case (a).

Remains the case $d(f_i, gh^{-i}) > d(f, g)$ for some $i$. Above we asserted that this implies $gh^{-1} = e$, that is,

$$d(f_i, e) > d(f, g).$$

Let

$$a = d(f_1, e), \quad b = d(f_1, g), \quad c = 1 - a - b.$$

Let $i = o(g)$ and let $j$ be the least natural number with $h^j = g$. Now we have

$$d(f_i, gh^{-i}) =
\begin{cases}
  a & \text{if } i = j + kt,
  b & \text{if } i = k,
  \leq e & \text{otherwise}.
\end{cases}$$

Substituting into (15.2) we get

$$d(f, g) \leq (1 - q_0^{-1}) \left[ \frac{b}{1 - q_0^{-1}} + \frac{a q_0^{-i}}{1 - q_0^{-1}} + c \left( \frac{1}{1 - q_0^{-1}} - \frac{1}{1 - q_0^{-i}} \right) \right].$$

Using the conditions

$$a + b + c = 1, \quad a > d(f, g) \geq b$$

from (15.4) we get (RC) $d(f, g) \leq 1/3$, that is again the possibility (a).

Proof of Lemma (14.8). Let

$$f_i(p) =
\begin{cases}
f(p) & \text{if } p \neq q_0,
\varepsilon & \text{if } p = q_0.
\end{cases}$$

We have, with the notation $q_0^{-1} = a$,

$$d(f, g) = (1 - q_0^{-1}) \sum_{i=0}^{\infty} a^i d(f_i, g^{-i}).$$

Now, if

$$d(f_0, e) + 2d(f_0, g) \leq 1.01,$$

then $\text{(RC)}$ $d(f, g) \leq 0.34$ and we are ready. Suppose

$$d(f_0, e) + 2d(f_0, g) > 1.01.$$

If $h \neq e, g$, then

$$d(f_0, e) + d(f_0, g) + d(f_0, h) \leq 1,$$

so

$$d(f_0, h) < d(f_0, g) - 0.01.$$

If $d(f_0, e) \leq d(f, g)$ were, then, since $d(f_0, h) < d(f, g)$ for $h \neq e$, (15.5) would imply $d(f, g) < d(f, g)$. So we must have

$$d(f_0, e) > d(f, g) > d(f_0, g).$$

Summing up, for $h \neq e, g$ we got

$$d(f, e) > d(f_0, e) > d(f_0, g) > d(f_0, h) + 0.01.$$
We have (15.8)
\[ t_0, t_1 > 0, \quad t_n < 0, \quad n = 2, 3, \ldots, \sum_{i=0}^{\infty} t_i = 0, \]
\[ t_0 + t_1 + t_3 + t_5 \leq 0. \]

We shall estimate using (15.6). In case \( i = 4 \) and \( i \geq 6 \) we use
\[ d(f_t, g^{-i}) \leq d(f_t, c). \]

In case \( i = 2, 3, 5 \) we have \( g^{-i} \neq c \), since
\[ o(g) = p \neq 1, 2, 3. \]

Furthermore in case \( i = 2 \) we have \( g^{-i} = g^{-2} \neq g \). So from (15.5), (15.6)
and (15.7)
\[ T \geq t_0 d(f_t, g) + t_1 d(f_t, c) + t_2 d(f_t, g) - 0.01 + \\ + t_3 d(f_t, g) + \sum_{i=4}^{\infty} t_i d(f_t, c) \]
\[ = (t_0 + t_1 + t_3 + t_5) d(f_t, g) - d(f_t, c) - 0.01t_2 \]
\[ \geq -0.01 t_2 = 0.01 \omega^2 \left( 1 - \omega - \frac{\omega^2}{2} \right) > 10^{-3} \omega^3, \]

since \( \omega \leq 1/2 \).

At last I propose one more problem: to find a proof of the half length.

References