$m-1$ of Theorem 2′), that

$$v(a-\beta) \geqslant -c_6\big(r(A),\, m-1,\, s(R)\big)h(R)-c \geqslant -c_6\big(r(A),\, m-1,\, s^2\big)2sh(B)-c$$

by (53). Together with (81), and observing (52), we obtain

$$v\big(B(\boldsymbol{a})\big) \geqslant -\big((1+c_8)r(A)+c_3\big)h(B)-c \geqslant -c_7h(B)-c.$$

We finally remark that we threw away the solution β of $B(\boldsymbol{\beta}) = 0$ in going from Theorem 2 to Theorem 2′. Then at the end we had to construct a solution β of $B(\boldsymbol{\beta}) = 0$. This may seem a wasteful argument. But in our inductive argument, we may have to replace B by a new B with a smaller value of $m = l(B)$. In other words, the solution β with $B(\boldsymbol{\beta}) = 0$ gets lost in the inductive argument.

References

[1] E. R. Kolchin, *Rational approximation to solutions of algebraic differential equations*, Proc. Amer. Math. Soc. 10 (1959), pp. 238–244.

[2] C. F. Osgood, *An effective lower bound on the "Diophantine Approximation" of Algebraic Functions by Rational Functions*, Mathematika 20 (1973), pp. 4–15.

[3] — *Concerning a possible "Thue-Siegel-Roth Theorem" for Algebraic differential equations* (to appear).

[4] E. G. C. Poole, *Introduction to the theory of linear differential equations*, New York 1960.

[5] K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika 2 (1955), pp. 1–20.

[6] W. M. Schmidt, *Rational approximation to solutions of linear differential equations with algebraic coefficients*, Proc. Amer. Math. Soc. 53 (1975), pp. 285–289.

[7] S. Uchiyama, *Rational approximations to algebraic functions*, J. Fac. Sci. Hokkaido Univ. Ser. 15 (1961), pp. 173–192.

UNIVERSITY OF COLORADO, Boulder
UNIVERSITÄT WIEN, Österreich

Received on 12. 10. 1975

ACTA ARITHMETICA
XXXII (1977)

Large values of Dirichlet polynomials, IV

by

M. N. Huxley (Cardiff) and M. Jutila (Turku)

1. Introduction. This paper continues [4]–[6], [8], [9]. Our object is to estimate the size of a set of pairs (s, χ) at which a Dirichlet polynomial $F(s, \chi)$ can be large. A precise statement is given in the next section where the notation is introduced. Our main tool is the reflection argument of [4], which we use in a simplified form due to Jutila [8], [9] as Lemma 8 below. It relates Dirichlet polynomials of length N to those of length about D/N, where D measures the range in which the pairs (s, χ) can lie. It is useful to have a peak function which is itself a Dirichlet polynomial: we use the H series discussed in Sections 3 and 4, which are modified Dirichlet L-functions. It is sometimes possible to use $F(s, \chi)$ itself as a peak function, as in Lemma 10 below. The L-functions can be approximated by H series of length $D^{1/2}$ (the so-called approximate functional equation), as in Lemma 14 below. Lemma 14 is implicit in the literature; we sketch the proof out of duty. Jutila [9] has a new lemma (our Lemma 7) in which $F(s, \chi)$ is raised to an even integral power, and obtains sharper results than those of [6] when $F(s, \chi)$ is very large, for instance when the exponent a of (2.23) is $4/5$.

In this paper we explore the consequences of Jutila's new lemma. Our arguments are purely combinatoric (except Lemma 14). To make the work accessible, we have summarised the main ideas of previous papers as a sequence of lemmas, stressing the combinatory rather than the analytic aspects. Our result is Theorem 2 of Section 5. It enables us to improve the zero-density theorems for Dirichlet L-functions. For instance we extend the range of the density hypotheses. Let $N(\alpha, T, \chi)$ be the number of zeros $\beta+i\gamma$ of $L(s, \chi)$ in $\beta \geqslant \alpha$, $|\gamma| \leqslant T$. Then

$$\sum_{\chi \bmod q} N(\alpha, T, \chi) \ll (qT)^{2-2\alpha+\varepsilon} \tag{1.1}$$

holds for $\alpha > 4/5$. Let an asterisk denote a sum over proper characters. Then

$$\sum_{q \leqslant Q} \sum_{\chi \bmod q}{}^{*} N(\alpha, T, \chi) \ll (Q^2T)^{2-2\alpha+\varepsilon} \tag{1.2}$$

also holds for $\alpha > 4/5$ and

$$\sum_{q\leqslant Q}\ \sideset{}{^*}\sum_{\chi \bmod q} N(\alpha, T, \chi) \ll (Q^2T^2)^{2-2\alpha+\varepsilon} \tag{1.3}$$

holds for $\alpha > 557/718 = 0.7759\ldots$ Jutila [9] had (1.1) and (1.2) for $\alpha \geqslant 21/26 = 0.8076\ldots$ and (1.3) for $\alpha \geqslant 7/9 = 0.7777\ldots$

Lemmas 18 and 12 taken together suggest that $F(s, \chi)$ and the corresponding L-function $L(s, \chi)$ cannot both be very large at the same ordinates. By a lemma of Littlewood, $L(s, \chi)$ is large only if there is a zero nearby, and the criterion for a zero ([10], Chapter 12) is that some Dirichlet polynomial should be large. Unfortunately Lemmas 12 and 18 are not powerful enough for this argument to give new bounds for L-functions.

2. Notation and conjectures. We consider finite Dirichlet series (Dirichlet polynomials) of the form

$$F(s, \chi) = \sum_{N+1}^{2N} a(n)\chi(n)n^{-s}, \tag{2.1}$$

where N is a variable positive integer, $a(n)$ are complex coefficients, χ is a variable Dirichlet character, $s = \sigma + it$ is a complex variable; any finite Dirichlet series can be divided into sums of the form (2.1) with different lengths N. We write

$$A = \max_{N<n\leqslant 2N} |a(n)|, \tag{2.2}$$

$$G = \sum_{N+1}^{2N} |a(n)|^2. \tag{2.3}$$

When different Dirichlet polynomials are distinguished by suffices, our convention is that $a_i(n)$, A_i, G_i, N_i refer to $F_i(s, \chi)$.

LEMMA 1. *Let m be a positive integer, $F_1, \ldots, F_m$ be of the form* (2.1) *and*

$$F(s, \chi) = \prod_1^m F_i(s, \chi). \tag{2.4}$$

Then for any $\varepsilon > 0$

$$G \ll N^\varepsilon \prod_1^m G_i, \tag{2.5}$$

$$A \ll N^\varepsilon \prod_1^m A_i, \tag{2.6}$$

where the constants implied in the Vinogradov symbol $\ll$ depend only on m and ε.

Proof. This is Lemma 3 of [6]; it follows since for $N < n \leqslant 2N$

$$\sum_{n_1\ldots n_m = n} 1 \ll N^\varepsilon. \tag{2.7}$$

We shall write ε for any exponent which can be taken arbitrarily small and positive. The exponents ε are not necessarily the same throughout a proof.

We are interested in the size of $F(s, \chi)$ at a set U of pairs (s, χ). Let q_0 be a positive integer, $Q \geqslant q_0$, $T \geqslant 1$ be real numbers, $D = Q^2T/q_0$, $l = \log D$. We define U to be (q_0, Q, T)-spaced if each χ is a character to some modulus $q \leqslant Q$ with $q \equiv 0 \pmod{q_0}$, for each s

$$0 \leqslant \sigma \leqslant 1/4, \tag{2.8}$$

and for any two members (s, χ), (s', χ') of U

$$|t - t'| \leqslant T, \tag{2.9}$$

$$|t - t'| \geqslant 1 \quad \text{when } \bar{\chi}\chi' \text{ is a principal character.} \tag{2.10}$$

When no ambiguity arises and

$$T \geqslant l^4 \tag{2.11}$$

we shall write (q_0, Q, T)-spaced as D-spaced. Let R be the number of elements of U. When U is (q_0, Q, T)-spaced we have $R \ll D$. We define U to be pure if either $q_0 = Q$ or if each χ is proper, and define U to be flat if $\sigma = 0$ for each pair (s, χ). We write $U^{(1)}$ for the flat set of R pairs (it, χ) corresponding to the pairs (s, χ) of U, U^2 for the set of R^2 pairs $(\bar{s} + s', \bar{\chi}\chi')$ and $U^{(2)}$ for the flat set of R^2 pairs $(it' - it, \bar{\chi}\chi')$ corresponding to all pairs (s, χ) and (s', χ') of U. A sum written over U^2 or $U^{(2)}$ is counted according to multiplicity.

We seek to relate R to the minimum value

$$V = V(F, U) = \min_U |F(s, \chi)| \tag{2.12}$$

of F on U. For each real number $k \geqslant 1$ let

$$E_k(F, U) = \sum_U |F(s, \chi)|^k, \tag{2.13}$$

and for real $M \geqslant 2$ let $B_k(M, U)$ be the least positive B for which

$$E_k(F, U) \leqslant G^{k/2}B \quad \text{if} \quad 2N \leqslant M, \tag{2.14}$$

and let $B_k^*(M, U)$ be the least positive B for which

$$E_k(F, U) \leqslant A^k N^{k/2} B \quad \text{if} \quad 2N \leqslant M. \tag{2.15}$$

We write B and B^* for B_1 and B_1^*. Since there are finitely many coefficients $a(n)$ each lying in a bounded region, each of these bounds (2.14) and (2.15) is attained by some Dirichlet polynomial. Although the definition of B in [4]–[6] is slightly different, the results of those papers hold for our present B. The simpler definition of B is permitted by Jutila's form of the reflection argument in Lemma 8 below.

A trivial remark is that, if U is the union of sets U_i, then

$$B_k(M, U) \leqslant \sum_i B_k(M, U_i). \tag{2.16}$$

In particular

$$B_k(M, U^2) \leqslant RB_k(M, U), \tag{2.17}$$

and similarly for B_k^*. If U is D-spaced, so is each U_i, and if U_i is D_i-spaced for some $D_i < D$, we may obtain a better bound for each term $B_k(M, U_i)$. This device was used in [3], the sets U_i being chosen to reduce T.

It is easy to see that

$$\max\left(R, (M/2-1)^{k/2}\right) \leqslant B_k(M, U) \leqslant R(M/2)^{k/2}, \tag{2.18}$$

and

$$(M/2-1)^{k/2} \leqslant B_k^*(M, U) \leqslant B_k(M, U) \leqslant R(M/2)^{k/2}. \tag{2.19}$$

For $k \geqslant 2$ a mean value argument shows that

$$B_k^*(M, U) \geqslant R. \tag{2.20}$$

We state Montgomery's conjectures in our notation.

Mean Values Conjecture ([10], Conjecture 9.2). *If U is pure and D-spaced and $k \geqslant 2$, then for $M \geqslant 2$ we have*

$$B_k(M, U) \ll (D+M^{k/2})(DM)^{\varepsilon}. \tag{2.21}$$

Large Values Conjecture ([10], Conjecture 9.1). *If U is pure and D-spaced then for $M \geqslant 2$ we have*

$$B_2(M, U) \ll (R+M)(DM)^{\varepsilon}. \tag{2.22}$$

Corollary. *If $F(s, \chi)$ is of the form* (2.1) *and*

$$V(F, U) = G^{1/2}N^{\alpha-1/2}. \tag{2.23}$$

then

$$R \ll N^{2-2\alpha}(DN)^{\varepsilon}. \tag{2.24}$$

For $N \geqslant D$ both conjectures follow from the theorem of the Large Sieve (Lemma 5 below). In this paper we discuss a weaker conjecture.

Density Conjecture. *If U is pure and D-spaced, $F(s, \chi)$ is of the form* (2.1) *and α is given by* (2.23), *then*

$$R \ll (D+N)^{2-2\alpha+\varepsilon}. \tag{2.25}$$

This conjecture suffices to establish the Density Hypothesis for zeros of Dirichlet L-functions (to within one of those exponents ε). The detailed deduction is given in [10], Chapter 12; it can be simplified a little when the Density Hypothesis is considered, for example by using 'outstanding zeros' as in [7]. The estimate (2.25) is required only for the range

$$(QT)^{\varepsilon} \leqslant N \leqslant (QT)^{1/2+\varepsilon}. \tag{2.26}$$

If for some α_1 the Density Conjecture holds for $\alpha > \alpha_1$ and any $\varepsilon > 0$, the corresponding Density Hypothesis for L-functions also holds for $\alpha > \alpha_1$.

3. Combinatoric lemmas. Hölder's inequality can be used to relate B_j to B_{jk}.

Lemma 2. *For any $j \geqslant 1$ and any $k > 1$*

$$B_j(M, U) \leqslant R^{1-1/k}\left(B_{jk}(M, U)\right)^{1/k}, \tag{3.1}$$

and the corresponding inequality holds between B_j^* and B_{jk}^*.

Lemma 3. *For any $j \geqslant 1$ and any $k > 1$*

$$B_{jk}(M, U) \ll \max_{U'} R'^{1-k}\left(B_j(M, U')\right)^k \log MR, \tag{3.2}$$

where the maximum is over all subsets U' of U and R' denotes the number of elements of U'. Moreover, if for fixed M, q_0, Q, T and all (q_0, Q, T)-spaced U

$$B_j(M, U) \leqslant R^{a}E \tag{3.3}$$

for some $a \geqslant 1-1/k$ and some E, then for all such U

$$B_{jk}(M, U) \ll R^{1-k+ka}E^k \min\left(\log MR, k/(1-k+ka)\right). \tag{3.4}$$

The corresponding assertions hold for B_j^* and B_{jk}^*. The implied constants depend on j and k.

Proof. Lemma 3 is proved by taking the particular $F(s, \chi)$ which attains the bound $B_{jk}(M, U)$ or $B_{jk}^*(M, U)$, and dividing U into subsets U' on which either

$$|F(s, \chi)| \leqslant G^{1/2}/R \quad (\text{or } AN^{1/2}/R) \tag{3.5}$$

or for $V' = 2^i$, i being a suitable integer in a range of length $\ll \log MR$

$$V'/2 \leqslant |F(s, \chi)| \leqslant V'. \tag{3.6}$$

The subset defined by (3.5) gives a term

$$\leqslant R'^{1-jk} \leqslant R' \leqslant R'^{1-k}\left(B_j(M, U')\right)^k \tag{3.7}$$

by (2.18). Considering B_{jk}^* we take the alternative in (3.5), and get a term

$$\leqslant R'^{1-jk} \leqslant R'^{1-k}M^{jk/2} \leqslant R'^{1-k}\left(B_j^*(M, U')\right)^k \tag{3.8}$$

by (2.19). If U' is the subset satisfying (3.6)

$$E_{jk}(F, U') \leqslant V'^{jk} R' \ll R'^{1-k}\big(R'(V'/2)^{j}\big)^{k} \ll R'^{1-k}\big(E_j(F, U')\big)^{k}, \tag{3.9}$$

which gives the main assertions of Lemma 3.

LEMMA 4. *For any $j \geqslant 1$, any integer $k > 1$ and any $\varepsilon > 0$*

$$B_{jk}(M, U) \ll B_j(M^k, U) M^{\varepsilon}, \tag{3.10}$$

where the implied constant depends on j, k and ε. The corresponding relation holds between B_{jk}^* and B_j^*:

Proof. Write $E_{jk}(F, U)$ as $E_j(F^k, U)$ and use Lemma 1 and (2.20).

Our next lemma is the theorem of the large sieve ([10], Theorem 7.5).

LEMMA 5. *If U is pure and (q_0, Q, T)-spaced then*

$$B_2(M, U) \ll D + M. \tag{3.11}$$

The logarithmic factors in [10], Theorem 7.5, are bounded for sums from $N+1$ to $2N$ rather than from 1 to N. By (2.18) Lemma 5 is best possible for $M \gg D$ or $R \gg D$.

We define an $H(M, N)$ series to be a Dirichlet series

$$H(s, \chi) = \sum_{1}^{\infty} h(n)\chi(n) n^{-s} \tag{3.12}$$

which is absolutely convergent for $\operatorname{Re} s \geqslant 0$, with real non-negative coefficients satisfying $h(n) \geqslant 1$ for $M \leqslant n \leqslant N$. All work on the Large Values Conjecture begins with Halász's Lemma.

LEMMA 6. *For any $F(s, \chi)$ of the form* (2.1) *and any $H(N, 2N)$ series $H(s, \chi)$*

$$\big(E_1(F, U)\big)^2 \leqslant G E_1(H, U^2). \tag{3.13}$$

Consequently for $M \geqslant 2$

$$\big(B(M, U)\big)^2 \leqslant M^{1/2} B^*(M, U^2). \tag{3.14}$$

Choosing $h(n)$ to be 1 or 0, we deduce (3.14) from (3.13).

Jutila [9] has recently found a lemma which combines aspects of Lemmas 4 and 6.

LEMMA 7. *Let k be a positive integer, $F(s, \chi)$ be of the form* (2.1) *and $H(s, \chi)$ be any $H(N^k, 2^k N^k)$ series. For any $\varepsilon > 0$*

$$E_{2k}(F, U^2) \ll A^{2k} N^{k\varepsilon} E_2(H, U^2). \tag{3.15}$$

The implied constant depends on k and ε.

Lemma 7 differs from Lemma 4 in the restrictions $j = 2$ and U^2 for U, but replaces F^k by $A^k H$, enabling us to choose a suitable H-series.

The series of papers [4]–[6] introduced the reflection principle: if $H(s, \chi)$ is a partial sum for $L(s, \chi)$, then by the approximate functional equation $H(s, \chi)$ is equal to a conjugate partial sum for $L(1-s, \bar{\chi})$ plus an error term. It rests on a method for establishing approximate functional equations devised by Montgomery and worked out in detail by Huxley in his doctoral thesis and [2] to estimate moments of Hecke zeta-functions. The key device is to make the length of the conjugate sum independent of s and χ. The cruder form of the method which establishes the reflection principle also gives the moments directly (to within a constant factor) without first setting up an approximate functional equation. Ramachandra [11] also noticed this short cut to the moments, and went on in [12] to rediscover the reflection principle. Jutila, after studying both versions, found a very simple form of the reflection argument ([9], Lemma 1), which we adopt as our next lemma.

LEMMA 8. *Let U be D-spaced and let $N \leqslant D$. There exists an $H(N, 2N)$ series $H(s, \chi)$ such that for any $\varepsilon > 0$*

$$H(s, \chi) \ll N^{1/2} q^{\varepsilon} \int_{-l^4}^{l^4} \Big| \sum_{1}^{D/N} \bar{\chi}(n) n^{-\frac{1}{2}+it-i\tau} \Big| d\tau + 1 \tag{3.16}$$

for all pairs (s, χ) in U^2 except those for which χ is principal and $|t| \leqslant l^4$; q denotes the modulus of χ. The implied constant depends on ε. Moreover for $k \geqslant 1$

$$E_k(H, U^2) \ll R N^k \log l + R^2 + N^{k/2} Q^{2k\varepsilon} l^{6k} B_k^*(D/N, U^{(2)}); \tag{3.17}$$

the implied constants depend on k and ε.

The reflection argument in [4] has a simpler $H(N, 2N)$ series, but requires a tedious subdivision of U^2 before we can assert (3.17). Combining Lemmas 6 and 8 gives a useful result.

LEMMA 9. *Let U be D-spaced and $M \leqslant D$. Then*

$$\big(B(M, U)\big)^2 \ll RM \log l + R^2 + M^{1/2} Q^{\varepsilon} l^6 B^*(D/M, U^{(2)}). \tag{3.18}$$

The implied constants depend on ε.

The idea of [6] can be summarised in the following lemma.

LEMMA 10. *Let $F(s, \chi)$ be of the form* (2.1) *and let $V = V(F, U)$. For any positive integers c, k, any $M \geqslant 1$ and any $\varepsilon > 0$*

$$B_k(M, U) \ll (G/V^2)^{ck/2} (MN^c)^{k\varepsilon} B_k(MN^c, U). \tag{3.19}$$

The implied constant depends on c, k and ε.

Proof. If $F_1(s, \chi)$ attains the maximum in the definition of $B_k(M, U)$, then

$$E_k(F_1 F^c, U) \geqslant E_k(F_1, U) V^{ck}. \tag{3.20}$$

The result now follows by Lemma 1.

The main result of this series of papers is Theorem 1 of [6].

THEOREM 1. *If $F(s,\chi)$ is of the form* (2.1) *with*

$$V = V(F, U) \geqslant cG^{1/2}N^{1/4}l^2 \tag{3.21}$$

for some fixed absolute $c > 0$ on a pure D-spaced set U, then for any $\varepsilon > 0$

$$R \ll GNV^{-2}l + G^3V^{-6}(DN)^{1+\varepsilon}. \tag{3.22}$$

Moreover for $1 \leqslant k \leqslant 2$

$$B_k(M, U) \ll RM^{k/4}l^{2k} + R^{1-k/2}M^{k/2}l^{k/2+1} + R^{1-k/6}(DM)^{(1+\varepsilon)k/6}. \tag{3.23}$$

The implied constants depend on k and ε.

For $N \geqslant D$ Theorem 1 follows from Lemma 5; for $N \leqslant D$ it follows from Lemmas 9, 10 and 5.

4. Lemmas involving Dirichlet L-functions. Lemma 8 really belongs here, as its proof employs L-functions. First a technical result lets us compare U with the corresponding flat set $U^{(1)}$.

LEMMA 11. *For $k \geqslant 1$ and $M \geqslant 1$ we have*

$$B_k(M, U) \ll B_k(M, U^{(1)}), \tag{4.1}$$

and if $0 \leqslant \sigma \leqslant 1/\log RM$ for each (s, χ) of U, then

$$B_k(M, U^{(1)}) \ll B_k(M, U). \tag{4.2}$$

The implied constants depend on k.

Proof. Let $F(s,\chi)$ of length N attain the bound $B_k(M, U)$. Take the contour C consisting of the semicircle centre 0 radius $1/\log N$ to the left of $\operatorname{Re} z = 0$, the semicircle centre 1, radius $1/\log N$ to the right of $\operatorname{Re} z = 1$ and the two line segments parallel to $[0, 1]$ and distant $1/\log N$ on each side, described anticlockwise. By Cauchy's integral formula and Hölder's inequality

$$2\pi i F(s,\chi) = \int_C F(it+z, \chi)(z-\sigma)^{-1}\,dz, \tag{4.3}$$

$$|F(s,\chi)|^k \ll \int_C |F(it+z,\chi)|^k |z-\sigma|^{-1}\,|dz|. \tag{4.4}$$

Summing (4.4), we have

$$\begin{aligned} E_k(F, U) &\ll \int_C \sum_{(s,\chi)\in U} |F(it+z,\chi)|^k |z-\sigma|^{-1}\,|dz| \\ &\ll e^k \int_C N^{-k\operatorname{Re} z} G^{k/2} B_k(M, U^{(1)})|z-\sigma|^{-1}\,|dz|. \end{aligned} \tag{4.5}$$

Since

$$\int_C N^{-\operatorname{Re} z}|z-\sigma|^{-1}\,|dz| \ll 1, \tag{4.6}$$

we have (4.1). A similar contour enables us to prove (4.2).

We now choose a particular $H(N, 2N)$ series

$$\begin{aligned} H(s,\chi) &= \sum_1^\infty e^{2-n/N}\chi(n)n^{-s} \\ &= \frac{e^2}{2\pi i}\int_{\operatorname{Re}(s+\omega)=\frac12} L(s+\omega,\chi)\Gamma(\omega)N^\omega\,d\omega + \varepsilon(\chi)e^2\Gamma(1-s)N^{1-s}, \end{aligned} \tag{4.7}$$

for $0 \leqslant \sigma \leqslant 1/4$, where $\varepsilon(\chi)$ is $\varphi(q)/q$ if χ is a principal character to some modulus q, 0 if not. Our next lemma follows by Lemma 11 and Hölder's inequality applied to (4.7).

LEMMA 12. *Let U be (q_0, Q, T)-spaced and let $H(s,\chi)$ be given by* (4.7). *Then for $k \geqslant 1$*

$$E_k(H, U) \ll N^k + N^{k/2}\int_{-\infty}^{\infty}\sum_{(s,\chi)\in U^{(1)}} |L(s+\tfrac12+i\tau,\chi)|^k |\Gamma(\tfrac12+i\tau)|^{k/2}\,d\tau, \tag{4.8}$$

$$E_k(H, U^2) \ll RN^k + N^{k/2}\int_{-\infty}^{\infty}\sum_{(s,\chi)\in U^{(2)}} |L(s+\tfrac12+i\tau,\chi)|^k |\Gamma(\tfrac12+i\tau)|^{k/2}\,d\tau. \tag{4.9}$$

The implied constants depend on k.

We could take the integral in (4.7) back to an abscissa depending on the particular pair (s,χ) and use bounds for $L(s,\chi)$ as in Theorem 8.4 of [10], or Littlewood's lemma on the size of $L(s,\chi)$ in a zero-free region as in [1], [5]. We state in our notation Theorems 8.2 and 8.3 of [10] for comparison.

LEMMA 13. *Let U be D-spaced, $M \geqslant 2$. Then for any $\varepsilon > 0$*

$$B_2(M, U) \ll (M + RD^{1/2}l)Q^\varepsilon. \tag{4.10}$$

The factor Q^ε may be omitted if each χ occurring in U is proper. The implied constant depends on ε.

The analytic part of Lemma 13 is (4.7) with the integral taken to $\operatorname{Re}(s+\omega) = 0$. The combinatoric part corresponds to our Lemmas 3 and 6: Theorem 8.1 of [10] is related to Lemma 6.

The next lemma uses Montgomery's technique, discussed above in the preamble to Lemma 8. We merely sketch the proof, as it is closely related to that of Lemma 8 and the moment theorems of Ramachandra [11].

LEMMA 14. *If U is flat and D-spaced, for $|\tau|\leqslant T$, $k\geqslant 1$ and $\varepsilon>0$*

$$\sum_{(s,\chi)\in U}|L(s+1/2+i\tau,\chi)|^k \ll Q^{k\varepsilon}l^{k/2}(\log l)^k B_k^*(D^{1/2},U). \tag{4.11}$$

The factor $Q^{k\varepsilon}$ may be omitted if each character χ occurring is proper. The implied constant depends on k and ε. The corresponding assertion holds for U^2, but the factor $Q^{k\varepsilon}$ cannot be omitted.

Proof. Let z denote a typical number $s+1/2+i\tau$ in (4.11), so that $\operatorname{Re}z=1/2$, and let $M=D^{1/2}/2$, $g=l^2$. The sequence

$$h(n)=\exp\left(-(n/M)^g\right) \tag{4.12}$$

for which

$$\sum_{n>3M/2}h(n)\ll\exp\left(-(3/2)^g\right)\ll 1/D, \tag{4.13}$$

are the coefficients of a Dirichlet series $H(z,\chi)$ (for which $eH(z,\chi)$ is an $H(1,M)$ series) given by the Mellin transform

$$H(z,\chi)=L(z,\chi)+\varepsilon(\chi)\Gamma\left(1+\frac{1-z}{g}\right)\frac{M^{1-z}}{1-z}+ +\frac{1}{2\pi i}\int_{\operatorname{Re}w=-g/2}L(z+w,\chi)\Gamma\left(1+\frac{w}{g}\right)\frac{M^w}{w}\,dw \tag{4.14}$$

a variant of the H series in (4.7). The terms with $n>3M/2$ in $H(z,\chi)$ are negligible by (4.13). We rearrange (4.14) as an expression for $L(z,\chi)$, raise it to the kth power and sum over pairs (s,χ). The series on the left of (4.14) and the second term on the right contribute

$$\ll l^{k/2}B_k^*(D^{1/2},U). \tag{4.15}$$

We treat the integral by means of the functional equation. Let $\chi_1 \bmod f$ be the proper character which induces $\chi \bmod q$. For any complex s we have

$$L(1-s,\chi)=(f/\pi)^{s-1/2}G(s)J(s)L(s,\overline{\chi}), \tag{4.16}$$

where $G(s)$ is a quotient of gamma-functions and

$$J(s)=\prod_{p|q}\left(1-\chi_1(p)p^{-s}\right)\left(1-\overline{\chi}_1(p)p^{-1+s}\right)^{-1}. \tag{4.17}$$

Putting $u=1-w$ we write the integral in (4.14) as

$$\int_{\operatorname{Re}u=1+g/2}\left(\frac{f}{\pi}\right)^{u-z-1/2}G(u-z)J(u-z)L(u-z,\overline{\chi})\Gamma\left(1+\frac{1-u}{g}\right)\frac{M^{1-u}}{1-u}\,du. \tag{4.18}$$

Next we put $L=\varphi_1+\varphi_2$, where, if N is the greatest even integer with $N<D^{1/2}$,

$$\varphi_1(u-z,\overline{\chi})=\sum_{n\leqslant N}\overline{\chi}(n)n^{z-u},\quad \varphi_2(u-z,\overline{\chi})=\sum_{n>N}\overline{\chi}(n)n^{z-u}, \tag{4.19}$$

and write the integral (4.18) correspondingly as I_1+I_2. Then for $\operatorname{Re}u>1/2$ we have

$$|G(u-z)|\ll\left((|\tau|+|\operatorname{Im}u|+T)/2\right)^{\operatorname{Re}u-1}, \tag{4.20}$$

and for $\operatorname{Re}u>2$

$$|\varphi_2(u-z,\overline{\chi})|\ll N^{1-\operatorname{Re}u}. \tag{4.21}$$

Hence for $|\tau|\leqslant T$

$$I_2\ll\int_{\operatorname{Re}u=1+g/2}\left(\frac{3fT}{2\pi MN}\right)^{g/2}\left|\Gamma\left(1+\frac{1-u}{g}\right)\frac{du}{1-u}\right|\ll\frac{1}{D}, \tag{4.22}$$

which is negligible. In I_1 we take the line of integration back to $\operatorname{Re}u=1+1/l$, on which

$$J(u-z)\ll\prod_{p|q}(1+p^{-1/2})^2\ll Q^{\varepsilon}, \tag{4.23}$$

and the sum over U of the kth powers of the modulus of φ_1 is bounded by (4.15) for each τ and u, and

$$\sum_U|I_1|^k\ll Q^{k\varepsilon}l^{k/2}B_k(D^{1/2},U)\left(\int\left|\Gamma\left(1+\frac{1-u}{g}\right)\frac{du}{1-u}\right|\right)^k, \tag{4.24}$$

which gives (4.11).

The proof of Lemma 8 differs as follows. Instead of taking M as above, we put $M=N$ and $2N$ and subtract in (4.14), multiplying by an appropriate constant to get an $H(N,2N)$ series. The break between φ_1 and φ_2 now comes at D/N.

Combining Lemmas 12 and 14 we have the following result.

LEMMA 15. *If $H(s,\chi)$ is given by (4.7) and U is D-spaced, then for any $k\geqslant 1$ and any $\varepsilon>0$*

$$E_k(H,U)\ll N^k+N^{k/2}Q^{k\varepsilon}l^{k/2}(\log l)^kB_k^*(D^{1/2},U^{(1)}), \tag{4.25}$$

$$E_k(H,U^2)\ll RN^k+N^{k/2}Q^{k\varepsilon}l^{k/2}(\log l)^kB_k^*(D^{1/2},U^{(2)}). \tag{4.26}$$

The implied constants depend on k and ε. The factor $Q^{k\varepsilon}$ may be omitted from (4.25) if every χ is proper.

For the proof we need only remark that the integrals in (4.8) and (4.9) are negligible from T to infinity.

Our next lemma is composed of Lemma 15 and Lemma 7.

LEMMA 16. *Let U be D-spaced, b be a positive integer. Then for any $M \geqslant 2$ and any $\varepsilon > 0$*

$$(B^*(M, U^2))^{2b} \ll R^{4b-1}(DM)^\varepsilon \big(M^b + B_2^*(D^{1/2}, U^{(1)})\big). \tag{4.27}$$

The implied constants depend on b and ε.

Proof. Let $F(s, \chi)$ of the form (2.1) attain the bound $B^*(M, U^2)$. By Lemma 7 (with $k = b$) and Hölder's inequality

$$\big(E_1(F, U^2)\big)^{2b} \leqslant R^{4b-2} E_{2b}(F, U^2) \ll R^{4b-2} N^{b\varepsilon} E_2(H, U^2). \tag{4.28}$$

We take $H(s, \chi)$ to be the H series of (4.7) with N replaced by N^b. By Lemma 15 the right hand side of (4.28) is

$$\begin{aligned} &\ll R^{4b-2} N^{b\varepsilon}\big(RN^{2b} + N^b D^\varepsilon B_2(D^{1/2}, U^{(2)})\big) \\ &\ll R^{4b-1} N^{b+b\varepsilon}\big(N^b + D^\varepsilon B_2^*(D^{1/2}, U^{(1)})\big), \end{aligned} \tag{4.29}$$

where we have used (2.17). The lemma follows by the definition of $B^*(M, U^2)$.

Our next lemma combines Lemma 16 with the reflection argument of Lemma 9.

LEMMA 17. *Let U be D-spaced, and let a and b be positive integers. If either U is pure or*

$$M^a \leqslant D \tag{4.30}$$

we have for $1 \leqslant k \leqslant 2a$ and any $\varepsilon > 0$

$$\begin{aligned} B_k(M, U) \ll R^{1-k/2a} M^{k/2}(DM)^\varepsilon + R^{1-k/4ab} D^{k/4a}(DM)^\varepsilon + \\ + R^{1-k/4ab} M^{k/4}\big(B_2(D^{1/2}, U^{(1)})\big)^{k/4ab}(DM)^\varepsilon. \end{aligned} \tag{4.31}$$

The implied constants depend on a, b and ε.

Proof. By Lemmas 2, 4 and 9 we have

$$\begin{aligned} B(M, U) &\leqslant R^{1-1/a}\big(B_a(M, U)\big)^{1/a} \ll R^{1-1/a}\big(B(M^a, U)\big)^{1/a} M^\varepsilon \\ &\ll R^{1-1/a}(DM)^\varepsilon \big(RM^a + R^2 + M^{a/2} B^*(D/M^a, U^{(2)})\big)^{1/2a}. \end{aligned} \tag{4.32}$$

Substituting for $B^*(D/M^a, U^{(2)})$ from Lemma 16 gives the case $k = 1$ of (4.31), and the general case $1 \leqslant k \leqslant 2a$ follows from the case $k = 1$ by Lemma 3.

Next we combine Lemmas 10 and 17.

LEMMA 18. *Let $F(s, \chi)$ be a fixed Dirichlet polynomial of the form (2.1), let U be pure and D-spaced and let $V = V(F, U)$. Then for any positive integers c and d and for any $\varepsilon > 0$*

$$\begin{aligned} B_2(D^{1/2}, U^{(1)}) \ll (GN/V^2)^c D^{1/2}(DN)^\varepsilon + R^{1-1/2d}(G/V^2)^c D^{1/2}(DN)^\varepsilon + \\ + R(GN^{1/2}/V^2)^{2cd/(2d-1)} D^{d/(4d-2)}(DN)^\varepsilon. \end{aligned} \tag{4.33}$$

If

$$N^{2c} \geqslant D \tag{4.34}$$

the second and third terms on the right of (4.33) may be omitted. The implied constants depend on c, d and ε.

Proof. Suppose first that

$$0 \leqslant \sigma \leqslant 1/2l \tag{4.35}$$

holds for each (s, χ) in U. Then by Lemmas 11 and 10

$$B_2(D^{1/2}, U^{(1)}) \ll B_2(D^{1/2}, U) \ll (G/V^2)^c (D^{1/2}N^c)^\varepsilon B_2(D^{1/2}N^c, U). \tag{4.36}$$

If (4.34) holds, the result is immediate from Lemma 5. If not, by Lemma 17 with $a = 1$, $b = d$, $k = 2$

$$\begin{aligned} &B_2(D^{1/2}N^c, U) \\ &\quad \ll (DN^c)^\varepsilon \big(D^{1/2}N^c + R^{1-1/2d} D^{1/2} + R^{1-1/2d} D^{1/4} N^{c/2}\big(B_2(D^{1/2}, U^{(1)})\big)^{1/2d}\big), \end{aligned} \tag{4.37}$$

and comparison of (4.36) and (4.37) gives (4.33). In general we divide U into $O(l)$ sets on each of which

$$\beta \leqslant \sigma \leqslant \beta + 1/2l \tag{4.38}$$

holds for some β, and apply the result to $F_1(s, \chi) = F(s-\beta, \chi)$ with

$$G_1 \ll GN^{-2\beta} \ll G. \tag{4.39}$$

5. A new large values theorem. In this section we state our main result and indicate how the zero-density theorems follow. The first bound (5.1) of Theorem 2 is the Theorem of Jutila [9]; the second uses Lemmas 10 and 17, and so depends on using $F(s, \chi)$ itself as a peak-function. Theorem 2 of this paper is stronger than Theorem 2 of [6] when α is large; but at $\alpha = 3/4$ (5.4) reduces to the estimate $R \ll D$ which holds whenever U is D-spaced, whereas at $\alpha = 3/4$ the result of [6] reduces to Lemma 5 and so remains nontrivial. As $b \to \infty$ (5.1) reduces to Lemma 13.

THEOREM 2. *Let U be pure and D-spaced, and let $N \geqslant 2$. For any positive integers a, b, any k in $1 \leqslant k \leqslant 2a$ and any $\varepsilon > 0$*

$$\begin{aligned} B_k(N, U) \ll R^{1-k/2a} N^{k/2}(DN)^\varepsilon + R^{1-k/4ab} D^{k/4a}(DN)^\varepsilon + \\ + R^{1-k/8ab} N^{k/4} D^{k/8ab}(DN)^\varepsilon, \end{aligned} \tag{5.1}$$

and for any positive integers c, d

$$\begin{aligned} B_k(N, U) \ll\ & R^{1-k/2a} N^{k/2}(DN)^\varepsilon + R^{1-k/4ab} D^{k/4a}(DN)^\varepsilon + \\ & + R(R^{-1} N^{ab+c} D^{1/2})^{k/(4ab+2c)}(DN)^\varepsilon + \\ & + R(R^{-1/d} N^{2ab} D)^{k/(8abd+4cd)}(DN)^\varepsilon + \\ & + RN^{1/2} D^{d/(2ab(2d-1)+2cd)}(DN)^\varepsilon. \end{aligned} \tag{5.2}$$

The terms involving d in (5.2) may be omitted if

$$N^{2c} \geqslant D. \tag{5.3}$$

If $F(s, \chi)$ is a fixed Dirichlet polynomial of the form (2.1) and α is defined by (2.23) then

$$R \ll N^{2a(1-\alpha)}(DN)^{\varepsilon} + N^{2ab(1-2\alpha)}D^{b}(DN)^{\varepsilon} + N^{2ab(3-4\alpha)}D(DN)^{\varepsilon}. \tag{5.4}$$

If moreover for some $\delta > 0$

$$N^{(4\alpha-3)(4ab+2c-2ab/d)} \gg D^{1+\delta} \tag{5.5}$$

for some sufficiently large constant in the inequality, depending on a, b, c, d and δ, then

$$R \ll N^{2a(1-\alpha)}(DN)^{\varepsilon} + N^{2ab(1-2\alpha)}D^{b}(DN)^{\varepsilon} + N^{3ab+2c-(4ab+2c)\alpha}D^{1/2}(DN)^{\varepsilon} + N^{2d(3ab+c-(4ab+2c)\alpha)}D^{d}(DN)^{\varepsilon}. \tag{5.6}$$

If (5.3) holds, the condition (5.5) is not required and we may omit the last term in (5.6). The exponent ε in (5.6) depends on δ, but may be made arbitrarily small by choice of δ. The implied constants depend on a, b and ε, and, where appropriate, also on c, d and δ.

Proof. After Lemma 17 we need only estimate $B_2^*(D^{1/2}, U^{(1)})$. Jutila's estimate (5.1) follows by Lemmas 2, 4 and 5 since

$$B_2(D^{1/2}, U^{(1)}) \ll R^{1/2}\big(B_4^*(D^{1/2}, U^{(1)})\big)^{1/2} \ll R^{1/2}D^{\varepsilon}\big(B_2^*(D, U^{(1)})\big)^{1/2} \ll R^{1/2}D^{1/2+\varepsilon}, \tag{5.7}$$

and (5.1) implies (5.4). For the second assertion it is simpler to prove (5.6), from which (5.2) can be deduced by the process used to prove Lemma 3. By Lemma 17 with $k = 1$ we have either

$$RVG^{-1/2} \leqslant B(N, U) \ll R^{1-1/2a}N^{1/2}(DN)^{\varepsilon} + R^{1-1/4ab}D^{1/4a}(DN)^{\varepsilon}, \tag{5.8}$$

which gives the first two terms in (5.6), or

$$(RVG^{-1/2})^{4ab} \leqslant \big(B(N, U)\big)^{4ab} \ll R^{4ab-1}N^{ab}(DN)^{\varepsilon}B_2^*(D^{1/2}, U^{(1)}). \tag{5.9}$$

By Lemma 18

$$R(V^4/G^2N)^{ab}(DN)^{-\varepsilon} \ll B_2(D^{1/2}, U^{(1)})(DN)^{-\varepsilon} \ll (GN/V^2)^{c}D^{1/2} + R^{1-1/2d}(G/V^2)^{c}D^{1/2} + R(GN^{1/2}/V^2)^{2cd/(2d-1)}D^{d/(4d-2)}, \tag{5.10}$$

where only the first term is needed if (5.3) holds. This gives the condition (5.5) and the last two terms in (5.6).

To prove the zero-density results we must verify (2.25) for the appropriate ranges of N and α. The consequences of Theorem 1 were worked out as Theorem 3 of [6]; we quote part as our next lemma.

LEMMA 19. *Let $F(s, \chi)$ be of the form (2.1), let U be pure and D-spaced and let α be given by (2.23). Suppose for some positive integer $a \geqslant 2$*

$$N^{a} \leqslant D \leqslant N^{a+1}. \tag{5.11}$$

Then (2.25) holds for any $\delta > 0$ and

$$\alpha \geqslant (3a-1)/(4a-2) + \delta, \tag{5.12}$$

the implied constant depending on a, δ and ε.

For large a, the condition (5.12) is

$$\alpha > 3/4 + 1/8a + O(1/a^2). \tag{5.13}$$

Using Theorem 2 we can replace (5.13) by

$$\alpha > 3/4 + 1/12a + O(1/a^2) \tag{5.14}$$

by choosing $b \sim 3a/4$, $c \sim a/2$.

LEMMA 20. *Under the hypotheses of Lemma 19, (2.25) holds for*

$$\alpha \geqslant 4/5 + \delta \quad \text{if} \quad a = 2, \tag{5.15}$$

$$\alpha \geqslant 11/14 + \delta = 0.7857\ldots \quad \text{if} \quad a = 3, \tag{5.16}$$

$$\alpha \geqslant 557/718 + \delta = 0.7759\ldots \quad \text{if} \quad a = 4, \tag{5.17}$$

and for any positive integer m

$$\alpha \geqslant 3/4 + m/(48m^2 - 40m + 7) \quad \text{if} \quad a = 4m-2, \tag{5.18}$$

$$\alpha \geqslant 3/4 + 1/(48m - 20) + \delta \quad \text{if} \quad a = 4m-1, \tag{5.19}$$

$$\alpha \geqslant 3/4 + (2m+1)/(96m^2 + 16m - 4) + \delta \quad \text{if} \quad a = 4m, \tag{5.20}$$

$$\alpha \geqslant 3/4 + (2m+1)/(96m^2 + 48m - 4) + \delta \quad \text{if} \quad a = 4m+1. \tag{5.21}$$

The implied constants depend on a, δ and ε.

Proof. These conditions follow from Theorem 2, using the following choices of b, c and d.

a	2	3	4	$4m-2$	$4m-1$	$4m$	$4m+1$
b	2	3	3	$3m-1$	$3m$	$3m+1$	$3m+1$
c	1	2	2	$2m-1$	$2m$	$2m+1$	$2m+1$
d	3	—	7	large	—	—	—

The range (2.26) corresponds to $a \geqslant 2$ in (1.1) and (1.2), and to $a \geqslant 4$ in (1.3).

References

[1] M. Forti and C. Viola, *Density estimates for the zeros of L-functions*, Acta Arith. 23 (1973), pp. 379–391.
[2] M. N. Huxley, *The large sieve inequality for algebraic number fields II: Means of moments of Hecke zeta-functions*, Proc. London Math. Soc. (3) 21 (1970), pp. 108–128.
[3] — *On the difference between consecutive primes*, Inventiones Math. 15 (1972), pp. 164–170.
[4] — *Large values of Dirichlet polynomials*, Acta Arith. 24 (1973), pp. 329–346.
[5] — *Large values of Dirichlet polynomials II*, ibid., 27 (1974), pp. 159–169.
[6] — *Large values of Dirichlet polynomials III*, ibid., 26 (1974), pp. 431–440.
[7] — *An imperfect hybrid zero-density estimate*, J. London Math. Soc. (2) 13 (1976), pp. 53–56.
[8] M. Jutila, *On large values of Dirichlet polynomials*, Colloquium on Number Theory in Debrecen 1974.
[9] — *Zero-density estimates for L-functions*, Acta Arith. 32 (1976), pp. 55–62.
[10] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Berlin–Heidelberg–New York 1971.
[11] K. Ramachandra, *A simple proof of the mean fourth power estimate for* $\zeta(\frac{1}{2}+it)$ *and* $L(\frac{1}{2}+it, \chi)$, Ann. Scuola Norm. Sup. Pisa 1 (1974), pp. 81–97.
[12] — *Some new density estimates for the zeros of the Riemann zeta-function*, Ann. Acad. Sci. Fenn. AI Math. 1 (1975), pp. 177–182.

Received on 7. 11. 1975 (790

ACTA ARITHMETICA
XXXII (1977)

ERRATA

Page, line	For	Read
210_2	$k_U^{\times} \to \boldsymbol{P}^{P_k/U}$	$k_U^{\times} \to \boldsymbol{Z}^{P_k/U}$
219^5	modulus $\mathfrak{m}\, \mathscr{H}^{(\mathfrak{m})}$	modulus $\mathfrak{m}$, $\mathscr{H}^{(\mathfrak{m})}$
222^9	(i) $\left[\frac{K'\|k}{V'}\right]\Big\|_{K(U_1\|U)}$	(i) $\left[\frac{K'\|k}{V'}\right]\Big\|_{K}$

Les volumes IV et suivants sont à obtenir chez	Volumes from IV on are available at	Die Bände IV und folgende sind zu beziehen durch	Тома IV и следующие можно получить через

Ars Polona, Krakowskie Przedmieście 7, 00-068 Warszawa

Les volumes I–III sont à obtenir chez	Volumes I–III are available at	Die Bände I–III sind zu beziehen durch	Тома I–III можно получить через

Johnson Reprint Corporation, 111 Fifth Ave., New York, N. Y.

BOOKS PUBLISHED BY THE POLISH ACADEMY OF SCIENCES INSTITUTE OF MATHEMATICS

S. Banach, Oeuvres, vol. I, 1967, 381 pp.
S. Mazurkiewicz, Travaux de topologie et ses applications, 1969, 380 pp.
W. Sierpiński, Oeuvres choisies, vol. I, 1974, 300 pp.; vol. II, 1975, 780 pp.; vol. III, 1976, 688 pp.

MONOGRAFIE MATEMATYCZNE

41. H. Rasiowa and R. Sikorski, The mathematics of metamathematics, 3 rd ed., revised, 1970, 520 pp.
43. J. Szarski, Differential inequalities, 2nd ed., 1967, 256 pp.
44. K. Borsuk, Theory of retracts, 1967, 251 pp.
45. K. Maurin, Methods of Hilbert spaces, 2nd ed., 1972, 552 pp.
50. K. Borsuk, Multidimensional analytic geometry, 1969, 443 pp.
51. R. Sikorski, Advanced calculus. Functions of several variables, 1969, 460 pp.
52. W. Ślebodziński, Exterior forms and their applications, 1970, 427 pp.
53. M. Krzyżański, Partial differential equations of second order I, 1971, 562 pp.
54. M. Krzyżański, Partial differential equations of second order II, 1971, 407 pp.
57. W. Narkiewicz, Elementary and analytic theory of algebraic numbers, 1974, 630 pp.
58. C. Bessaga and A. Pełczyński, Selected topics in infinite-dimensional topology, 1975, 353 pp.
59. K. Borsuk, Theory of shape, 1975, 379 pp.
60. R. Engelking, General topology, in print.

New series:

BANACH CENTER PUBLICATIONS

Vol. 1. Mathematical control theory, 1976, 166 pp.
Vol. 2. Mathematical foundations of computer science, in print.
Vol. 3. Mathematical models and numerical methods, in preparation.
Vol. 4. Approximation theory, in preparation.