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Received on 8. 9. 1975 (763)

ACTA ARITHMETICA
XXXII (1977)

Values of integer-valued multiplicative functions in residue classes

by

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1. An integer-valued arithmetical function f is said to be *weakly uniformly distributed* (mod N) [WUD (mod N)] provided the set $\{n: (f(n), N) = 1\}$ is infinite and the values of f prime to N are asymptotically uniformly distributed in residue classes (mod N) prime to N. This notion was studied in [2] in the case of polynomial-like multiplicative functions (i.e. functions f satisfying the condition $f(p^k) = W_k(p)$ for every prime p, $k = 1, 2, \ldots$ with suitable $W_k(x) \in Z[x]$) and a necessary and sufficient condition for such a function to be WUD (mod N) was found. This condition makes sense for arbitrary integer-valued multiplicative functions and it was shown in [3] that it is equivalent to the Dirichlet-weakly uniform distribution (mod N) of f, which seems to be essentially weaker than WUD (mod N).

The purpose of this note is to show that for an important class of multiplicative functions WUD (mod N) and Dirichlet-WUD (mod N) coincide and so in view of [3] a necessary and sufficient condition for f from that class to be WUD (mod N) results.

2. We shall consider integer-valued multiplicative functions f from the class F_N consisting of all functions of this type for which the series

$$\sum_{\substack{p \\ (f(p),N)>1}} \frac{1}{p}$$

converges.

We need a lemma, which for $r = 1$ is a special case of Theorem 1 of [1] whose proof carries without any change to our case, being a simple application of a theorem of E. Wirsing [4]:

Lemma. *Let for $k = 1, 2, \ldots, r$ f_k be an integer-valued additive function, $N_k \geq 2$ an integer and j_k an integer prime to N_k. Let $S = S(f_1, \ldots, f_r; N_1, \ldots, N_r; j_1, \ldots, j_r)$ be the set of all integers $n \geq 1$ for which*

$$f_k(n) \equiv j_k \pmod{N_k}$$

holds for $k = 1, 2, \ldots, r$. If now A is any set of integers closed under multiplication then the set $A \cap S$ has the natural density.

From this lemma we deduce

THEOREM I. *For any integer-valued multiplicative function f and any $N \geqslant 3$, $j \in Z$ prime to N, the density of the set*

$$\{n\colon f(n) \equiv j \pmod N\}$$

exists.

Proof. If $f \notin F_N$ then for almost all n one has $(N, f(n)) > 1$ hence the density of our set equals zero. We may thus assume that f belongs to F_N.

For $(r, N) = 1$ write

$$A_r = \{p^a\colon f(p^a) \equiv r \pmod N\}$$

and let

$$A_0 = \{p^a\colon (f(p^a), N) > 1\}.$$

Moreover denote by $\Omega_r(n)$ the number of prime powers from A_r which are unitary divisors of n. Note that the functions Ω_r are additive.

Now let $n = p_1^{a_1} \ldots p_s^{a_s}$ and let $(f(n), N) = 1$. Then no $p_i^{a_i}$ can belong to A_0 and we get

$$f(n) = f(p_1^{a_1}) \ldots f(p_s^{a_s}) \equiv \prod_{(r,N)=1} r^{\Omega_r(n)} \pmod N.$$

As the conditions $\Omega_r(n) \equiv \Omega_r(m) \pmod{\varphi(N)}$ (for all r prime to N) jointly with $\Omega_0(n) = \Omega_0(m) = 0$ imply

$$f(n) \equiv \prod_{(r,N)=1} r^{\Omega_r(n)} \equiv \prod_{(r,N)=1} r^{\Omega_r(m)} \equiv f(m) \pmod N$$

we see that for any j prime to N we have the equality

$$(1) \qquad \{n\colon f(n) \equiv j \pmod N\} = \bigcup_{\lambda \in \Lambda(j)} B_\lambda$$

where

$$\Lambda(j) = \Big\{\langle \lambda_1, \ldots, \lambda_t\rangle\colon 0 \leqslant \lambda_i < \varphi(N)\ (i = 1, \ldots, t),\ \prod_{(r,N)=1} x_r^{\lambda_r} \equiv j \pmod N\Big\},$$

$t = \varphi(N)$ and for $\lambda = \langle \lambda_1, \ldots, \lambda_t\rangle$

$$B_\lambda = \{n\colon \Omega_0(n) = 0,\ \Omega_r(n) \equiv \lambda_r \pmod{\varphi(N)}\}.$$

By the lemma all sets B_λ have a density and as the union in (1) is disjoint and has only finitely many terms our assertion results. ■

3. For functions from F_N the definition of Dirichlet-WUD (mod N) has the following form:

A function f is *Dirichlet-WUD* (mod N) provided for all j prime to N one has

$$(2) \qquad \lim_{x \to 1+0} \frac{\sum\limits_{\substack{n \\ f(n) \equiv j \pmod N}} n^{-x}}{\sum\limits_{\substack{n \\ (f(n),N)=1}} n^{-x}} = \frac{1}{\varphi(N)}.$$

As a corollary of Theorem I we obtain now

THEOREM II. *For functions belonging to F_N, WUD (mod N) and Dirichlet-WUD (mod N) are equivalent properties.*

Proof. As noted in [2] WUD (mod N) always implies Dirichlet-WUD (mod N) so assume that $f \in F_N$ and f is Dirichlet-WUD (mod N). By Theorem I the set $\{n\colon f(n) \equiv j \pmod N\}$ has for all j prime to N a density, say d_j, and one sees easily that

$$d = \sum_{(j,N)=1} d_j$$

is positive. Thus

$$\lim_{x \to 1+0} (x-1) \sum_{\substack{n \\ (f(n),N)=1}} n^{-x} = d$$

and

$$\lim_{x \to 1+0} (x-1) \sum_{\substack{n \\ f(n) \equiv j \pmod N}} n^{-x} = d_j,$$

hence the limit in (2) equals d_j/d. By assumptions all those limits are equal and so d_j is independent on j. ■

THEOREM III. *A function $f \in F_N$ is WUD (mod N) if and only if for every nonprincipal character χ (mod N) which is trivial on the multiplicative group generated by those r (mod N) for which the series*

$$\sum_{\substack{p \\ f(p) \equiv r \pmod N}} \frac{1}{p}$$

diverges there exists a prime p such that

$$\sum_{k=0}^{\infty} \chi(f(p^k)) p^{-k} = 0.$$

Proof. Combine Theorem II and the theorem of [3]. ■

4. As an application we construct now multiplicative functions $f(n) \neq n$ which are WUD (mod N) for all $N \geqslant 3$. (Note that the question whether there exists a positive multiplicative function $f(n) \neq n$ which is uniformly distributed (mod N) for all N is still unanswered.) Let L_m be the sequence of all progressions with the first term prime to the difference and let $P = \bigcup_{m=1}^{\infty} P_m$ be a partition of the set P of all primes into disjoint subsets with the property $\sum_{p\in P_m} 1/p = \infty$ $(m = 1, 2, \ldots)$. If now f is any multiplicative function such that for primes $p \in P_m$ the number $f(p)$ is a prime from L_m distinct from p and all numbers $f(q)$ for primes q less than p, then by Theorem III such a function will be WUD (mod N) for all $N \geqslant 3$.

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Received on 10. 9. 1975 (768)

ACTA ARITHMETICA
XXXII (1977)

The exponent of class groups in congruence function fields

by

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1. Introduction. For a finitely generated extension K of a field k with transcendence degree 1, the divisor class group is infinite and the null class group (the subgroup of divisor classes of degree 0) is, in general, also infinite. However, if k is finite, it is a consequence of the Riemann–Roch theorem that the number of classes of degree 0 is finite. In this, the case of congruence function fields, the order of the null class group is called the class number of the field. This null class group is analogous to the ideal class group in the case of algebraic number fields, and it plays an important role in all algebraic, arithmetic, and geometric studies of congruence function fields.

In the theory of congruence function fields, the "Riemann Hypothesis" plays an essential role. This "hypothesis" determines the real part of the zeros of the zeta function of a congruence function field. It was proved in complete generality by Andre Weil [10] after H. Hasse [7] had given a proof in the elliptic case. This result gives bounds on the class number of a congruence function field; however, while there are bounds on the order of the null class group, not much is known about its structure. The purpose here is to study the exponent of this group for congruence function fields of a particular type. These are fields K which are abelian extensions of $k(x)$, the rational function field over k, for which $\mathrm{Gal}\big(K/k(x)\big)$ has order $n_0 p^{r_0}$, where p is the characteristic of the field and n_0 is relatively prime to p; and for which the p-primary part of $\mathrm{Gal}\big(K/k(x)\big)$ is elementary abelian. The main object of this paper is to give a lower bound for the exponent of the null class group of a field of this type. A consequence of this will be that for a fixed finite field k and a fixed degree $n_0 p^{r_0}$, the exponent will approach infinity as the genus of the field goes to infinity.

It is well-known that there is a strong similarity between the theory of congruence function fields and the theory of algebraic number fields; the two together form the class of global fields, and class field theory holds for them. It would be interesting to obtain analogous results for