Finally we prove the estimate (2). First we assume that $t < d^{1/2}$. Let $q > \varepsilon^{-1}$ be a fixed positive integer. The condition $t > 2$ implies $\varphi(t) \geqslant 2$ and so if $t \leqslant t_0(q)$ the theorem gives

$$\gamma(d, p) \ll d^{1/\varphi(t)} \ll d^{1/2}.$$

If $t > t_0$ then $\gamma(d, p) \ll \varphi(t) d^{1/q} \ll d^{1/2+\varepsilon}$ as required.

Tietäväinen [8] has shown that if $2d$ different residue classes can be represented as the sum of w dth powers, then

$$\gamma(d, p) \ll w \log d.$$

It follows easily from the Cauchy–Davenport Theorem ([2] or [8]) that we can represent $2d$ residue classes as the sum of $2d/t$ dth powers and thus

$$\gamma(d, p) \ll dt^{-1} \log d.$$

This proves the result at once for $t \geqslant d^{1/2}$.

I am grateful to Dr. Maurice Dodson for suggesting a number of improvements in the presentation of this paper.

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Received on 23. 8. 1975
and in revised form on 15. 10. 1975 (755)

ACTA ARITHMETICA
XXXII (1977)

Elementary methods in the theory of L-functions, V
The theorems of Landau and Page

by

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1. Landau [4] proved in 1918 that if the L-functions belonging to real primitive characters $\chi_1 \pmod{D_1}$ and $\chi_2 \pmod{D_2}$ ($\chi_1 \neq \chi_2$) respectively, have $1-\delta_1$ and $1-\delta_2$ real zeros, respectively, then

$$\max(\delta_1, \delta_2) > \frac{c}{\log D_1 D_2}, \tag{1.1}$$

where c is an absolute constant. This fact was used by Landau only to prove that the negative fundamental discriminants for which the class number $h(-D)$ of the imaginary quadratic field belonging to the discriminant $-D$

$$h(-D) = o\left(\frac{\sqrt{D}}{\log D}\right) \tag{1.2}$$

are very rare. Namely combining Hecke's theorem (see also Landau [4]) with (1.1) one has immediately the inequality

$$\max\left(\frac{h(-D_1)}{\sqrt{D_1}}, \frac{h(-D_2)}{\sqrt{D_2}}\right) > \frac{c'}{\log D_1 D_2}. \tag{1.3}$$

Page [6] proved (1.1) for the case $\chi_1 = \chi_2$, i.e. he showed that an L-function belonging to a real non-principal character $\chi \pmod{D}$ has at most one, simple zero in the interval

$$\left[1 - \frac{c''}{\log D}, 1\right] \tag{1.4}$$

where c'' is an absolute constant.

The mentioned results of Page and Landau concerning the real zeros of real L-functions together with the results – concerning the zeros of complex L-functions and the complex zeros of real L-functions – of Gronwall [3] and Titchmarsh [9] were used by Page [6] to get better results for the distribution of primes in arithmetic progressions.

Now we shall prove Page's theorem and (using the basic idea of the original proof) Landau's theorem in a relatively simple way, using only elements of the real analysis. Known proofs for Landau's and Page's theorem use either the functional equation for L-functions (see [4], [6]) or some general theorems of the theory of complex functions concerning the zeros of analytic functions (see Prachar [8], Satz 6.1 and 6.4).

So we shall prove

THEOREM 1. *If the L-functions belonging to the real primitive characters $\chi_1 \pmod{D_1}$ and $\chi_2 \pmod{D_2}$ respectively (where $D_1, D_2 \geqslant D_0$ effective constant and $\chi_1 \neq \chi_2$) have $1-\delta_1$ and $1-\delta_2$ real zeros, respectively, then the inequality*

$$\max(\delta_1, \delta_2) > \frac{1}{227 \log D_1 D_2} \tag{1.5}$$

holds.

THEOREM 2. *An L-function belonging to the real non-principal character $\chi \pmod{D}$ has at most one, simple real zero in the interval*

$$\left[1-\frac{2+o(1)}{\log D}, 1\right]. \tag{1.6}$$

This was proved earlier by R. J. Miech [5] only for the interval $[1-0.28/\log D, 1]$.

If we use Burgess's inequality [1] that for a primitive character $\chi \pmod{D}$

$$\left|\sum_{n=N+1}^{N+H} \chi(n)\right| \leqslant C(r, \varepsilon) H^{1-\frac{1}{r}} D^{\frac{r+1}{4r^2}+\varepsilon} \tag{1.7}$$

(where r denote an arbitrary natural number, ε an arbitrary positive number and $C(r, \varepsilon)$ a constant depending on r and ε) we have Theorem 2 with the interval

$$\left[1-\frac{4+o(1)}{\log D}, 1\right]. \tag{1.8}$$

(Directly we get (1.8) only for real primitive characters, but if $\chi \pmod{D}$ is induced by the primitive character $\chi_1 \pmod{D_1}$ then $L(1-\delta, \chi) = 0$ implies $L(1-\delta, \chi_1) = 0$ and so

$$\delta \geqslant \frac{4+o(1)}{\log D_1} \geqslant \frac{4+o(1)}{\log D}.) \tag{1.9}$$

Though the use of Burgess's inequality involves only a little technical difficulty we give here the proof in the more simple version using the Pólya–Vinogradov inequality. (If we use Burgess's inequality, in Lemma 3 the error term will be $o(1)$ on choosing

$$x = D^{1/4+\varepsilon} \cdot \tau^{-8}$$

and thus we get the interval (1.8).)

2. To prove Theorem 1 we need the following

LEMMA 1. *If χ is a real non-principal character* $\pmod{D}$

$$L(s) = L(s, \chi), \quad 0 < \tau < 0.01$$

hen the inequality

$$-\frac{L'}{L}(1+\tau) < \frac{\zeta'}{\zeta}(1+\tau) + 9 \log D + \frac{6}{5\tau} \tag{2.1}$$

holds.

If further

$$L(1-\delta, \chi) = 0, \quad 0 < \delta < \tfrac{1}{4},$$

then the inequality

$$-\frac{L'}{L}(1+\tau) < \frac{\zeta'}{\zeta}(1+\tau) + 3 \log D + \frac{\delta}{\tau}\left(6 \log D + \frac{6}{5\tau}\right) \tag{2.2}$$

holds.

To prove Lemma 1 we shall use

LEMMA 2. *If χ is a real non-principal character* $\pmod{D}$, $g(n) = \sum_{d|n} \chi(d)$, *then for an arbitrary τ, with $0 < \tau < \frac{1}{2}$, there exists a c_τ, such that for $x \geqslant 3D$, $L(s) = L(s, \chi)$, the relations*

$$\sum_{n \leqslant x} \frac{g(n)}{n^{1-\tau}} = \left(c_\tau - \frac{1}{\tau}\right) L(1-\tau) + \frac{L(1) x^\tau}{\tau} + \frac{4\vartheta x^\tau \sqrt{D}}{\tau \sqrt{x}} \tag{2.3}$$

and

$$\sum_{n \leqslant x} \frac{g(n)}{n^{1+\tau}} = \zeta(1+\tau) L(1+\tau) - \frac{L(1) x^{\tau}}{\tau} + \frac{4\vartheta \sqrt{D}}{\tau \sqrt{x}} \tag{2.4}$$

hold, where ϑ denotes a real number, possibly different on various appearances, such that $|\vartheta| \leqslant 1$.

The first part of Lemma 2, (2.3), is Lemma 0 of [7] (with the only modification that in the proof z must be chosen $z = \sqrt{3Dx}$; (2.4) one can prove analogously to (2.3)).

A consequence of (2.4) for $x\to\infty$ is the equality

$$\sum_{n=1}^{\infty}\frac{g(n)}{n^{1+\tau}}=\zeta(1+\tau)L(1+\tau). \tag{2.5}$$

In consequence of

$$g(n)=\prod_{p^{a}\|n}\big(1+\chi(p)+\ldots+\chi^{a}(p)\big)\geqslant 0$$

we have

$$-\frac{L'}{L}(1+\tau)-\frac{\zeta'}{\zeta}(1+\tau) \tag{2.6}$$

$$=\frac{\sum\limits_{n=1}^{\infty}\frac{g(n)\log n}{n^{1+\tau}}}{\sum\limits_{n=1}^{\infty}\frac{g(n)}{n^{1+\tau}}}\leqslant\frac{\sum\limits_{n\leqslant D^3}\frac{g(n)\log n}{n^{1+\tau}}}{\sum\limits_{n\leqslant D^3}\frac{g(n)}{n^{1+\tau}}}+\frac{\sum\limits_{n>D^3}\frac{g(n)\log n}{n^{1+\tau}}}{\sum\limits_{n=1}^{\infty}\frac{g(n)}{n^{1+\tau}}}\leqslant 3\log D+\frac{B}{C}$$

where

$$B=\sum_{n>D^3}\frac{g(n)\log n}{n^{1+\tau}},\qquad C=\sum_{n=1}^{\infty}\frac{g(n)}{n^{1+\tau}}. \tag{2.7}$$

A consequence of (2.4) for $k\geqslant 3$ is the equality

$$\sum_{D^k<n\leqslant D^{k+1}}\frac{g(n)}{n^{1+\tau}}=\frac{L(1)}{\tau}\left(\frac{1}{D^{k\tau}}-\frac{1}{D^{(k+1)\tau}}\right)+\frac{8\vartheta\sqrt{D}}{\tau\sqrt{D^k}}. \tag{2.8}$$

Thus using the easy elementary lower bound of A. O. Gelfond [2]

$$L(1)\geqslant\frac{1}{\sqrt{D}\log^2 D}$$

we have

$$B=\sum_{k=3}^{\infty}\sum_{n=D^k+1}^{D^{k+1}}\frac{g(n)\log n}{n^{1+\tau}} \tag{2.9}$$

$$\leqslant\sum_{k=3}^{\infty}(k+1)\log D\left\{\frac{(D^{\tau}-1)L(1)}{\tau D^{(k+1)\tau}}+\frac{8\sqrt{D}}{\tau\sqrt{D^k}}\right\}$$

$$=\frac{\log D}{\tau}\left\{\frac{L(1)}{D^{3\tau}}\left(4+\frac{1}{D^{\tau}-1}\right)+\frac{32}{\sqrt{D}(\sqrt{D}-1)}+\frac{8}{\sqrt{D}(\sqrt{D}-1)^2}\right\}$$

$$<\frac{\log D}{\tau}\left\{\frac{L(1)}{D^{3\tau}}\left(5+\frac{1}{\tau\log D}\right)\right\}=\frac{L(1)}{\tau D^{3\tau}}\left(5\log D+\frac{1}{\tau}\right).$$

Using (2.8), the inequality $L(1)\geqslant 1/\sqrt{D}\log^2 D$ and in case of a real zero $1-\delta>\frac{1}{2}$ (2.3) too, by $\tau<0.01$ we get the inequalities

$$C\geqslant\sum_{n>D^3}\frac{g(n)}{n^{1+\tau}}=\frac{L(1)}{\tau D^{3\tau}}+\frac{4\vartheta\sqrt{D}}{\tau\sqrt{D^3}}\geqslant\frac{5L(1)}{6\tau D^{3\tau}} \tag{2.10}$$

and

$$C\geqslant\sum_{n\leqslant D^3}\frac{g(n)}{n^{1+\tau}}\geqslant\frac{1}{D^{3\tau+3\delta}}\sum_{n\leqslant D^3}\frac{g(n)}{n^{1-\delta}} \tag{2.11}$$

$$=\frac{1}{D^{3\tau+3\delta}}\left(\frac{L(1)D^{3\delta}}{\delta}+\frac{4\vartheta D^{3\delta}\sqrt{D}}{\delta\sqrt{D^3}}\right)$$

$$\geqslant\frac{1}{D^{3\tau+3\delta}}\cdot\frac{5L(1)D^{3\delta}}{6\delta}=\frac{5L(1)}{6\delta D^{3\delta}}.$$

So from (2.9), (2.10), and (2.11) we get the inequalities

$$\frac{B}{C}\leqslant\frac{6}{5}\left(5\log D+\frac{1}{\tau}\right) \tag{2.12}$$

and

$$\frac{B}{C}\leqslant\frac{6\delta}{5\tau}\left(5\log D+\frac{1}{\tau}\right) \tag{2.13}$$

which together with (2.6) prove Lemma 1.

Now using the fact that $\chi_1\chi_2$ is a non-principal character mod D_1D_2 we can apply (2.1) for χ_1 and χ_2 and (2.2) for $\chi_1\chi_2$ and so for a real τ with $0<\tau<0.01$ we have

$$0\leqslant\sum_{n=1}^{\infty}\Lambda(n)\big(1+\chi_1(n)\big)\big(1+\chi_2(n)\big)n^{-1-\tau} \tag{2.14}$$

$$=-\frac{\zeta'}{\zeta}(1+\tau)-\frac{L'}{L}(1+\tau,\chi_1)-\frac{L'}{L}(1+\tau,\chi_2)-\frac{L'}{L}(1+\tau,\chi_1\chi_2)$$

$$\leqslant 2\frac{\zeta'}{\zeta}(1+\tau)+3(\log D_1+\log D_2)+9\log D_1D_2+$$

$$+\frac{6}{\tau}(\delta_1\log D_1+\delta_2\log D_2)+\frac{6}{5\tau^2}(\delta_1+\delta_2)+\frac{6}{5\tau}.$$

Here for $0<\tau<0.01$ one has

$$2\frac{\zeta'}{\zeta}(1+\tau)+\frac{6}{5\tau}\leqslant-2\cdot 0.98\frac{1}{\tau}+\frac{6}{5\tau}<-\frac{3}{4\tau}. \tag{2.15}$$

Put $D = D_1 D_2$, $\delta = \max(\delta_1, \delta_2)$ then (2.14) implies the inequality

$$\frac{3}{4\tau} \leqslant 12\log D + \frac{6}{\tau}\delta\log D + \frac{12\delta}{5\tau^2}. \tag{2.16}$$

Now choosing $\tau = (40\log D)^{-1}$ (2.16) is equivalent to the inequality

$$30\log D < 12\log D + 240\log D(\delta\log D) + 3840\log^2 D\cdot\delta. \tag{2.17}$$

Hence

$$\delta > \frac{1}{227\log D}. \ \blacksquare \tag{2.18}$$

3. To prove Theorem 2 we need the following

LEMMA 3. *Let χ be a real non-principal character* $(\mathrm{mod}\, D)$,

$$A = 2\sqrt{D}\log D, \quad L(s) = L(s, \chi), \quad g(n) = \sum_{d\mid n}\chi(d).$$

Then for an arbitrary real τ, with $0 < \tau < \frac{1}{3}$ there is such a c_τ, and such a c'_τ, $|c'_\tau| < 3$, that for $x \geqslant A$ the equality

$$\sum_{n\leqslant x}\frac{g(n)\log n}{n^{1-\tau}} = \left(\frac{1}{\tau} - c_\tau\right)L'(1-\tau) + \left(\frac{1}{\tau^2} + c_\tau\right)L(1-\tau) - \frac{L(1)x^\tau}{\tau}\left(\frac{1}{\tau} - \log x\right) + \frac{6\vartheta x^\tau\log x\sqrt{A}}{\tau^2\sqrt{x}} \tag{3.1}$$

holds.

For proving Lemma 3 we shall use the following lemma of the real elementary analysis.

LEMMA. *For an arbitrary τ with $0 < \tau < \frac{1}{3}$ there is such a c_τ, and such a c'_τ, $|c'_\tau| < 3$, that for $u \geqslant 1$ the relations*

$$\sum_{m\leqslant u}\frac{1}{m^{1-\tau}} = c_\tau + \frac{1}{\tau}(u^\tau - 1) + \frac{\vartheta}{u^{1-\tau}} < \frac{u^\tau}{\tau} \tag{3.2}$$

and

$$\sum_{m\leqslant u}\frac{\log m}{m^{1-\tau}} = c'_\tau + \frac{1}{\tau^2} - \frac{u^\tau}{\tau}\left(\frac{1}{\tau} - \log u\right) + \frac{\vartheta\log u}{u^{1-\tau}} \tag{3.3}$$

hold.

Now let z be a number, to be chosen later for which $1 \leqslant z \leqslant x$. Then

$$\sum_{n\leqslant x}\frac{g(n)\log n}{n^{1-\tau}} = \sum_{d\leqslant x}\chi(d)\sum_{m\leqslant x/d}\frac{\log md}{(md)^{1-\tau}} = \sum_{d\leqslant z}{}_1 + \sum_{z<d\leqslant x}{}_2. \tag{3.4}$$

Here using Lemma 4 we have

$$\begin{aligned}\sum\nolimits_1 &= \sum_{d\leqslant z}\frac{\chi(d)\log d}{d^{1-\tau}}\sum_{m\leqslant x/d}\frac{1}{m^{1-\tau}} + \sum_{d\leqslant z}\frac{\chi(d)}{d^{1-\tau}}\sum_{m\leqslant x/d}\frac{\log m}{m^{1-\tau}} \\ &= \sum_{d\leqslant z}\frac{\chi(d)\log d}{d^{1-\tau}}\left\{c_\tau + \frac{1}{\tau}\left(\frac{x^\tau}{d^\tau} - 1\right) + \frac{\vartheta d^{1-\tau}}{x^{1-\tau}}\right\} + \\ &\quad + \sum_{d\leqslant z}\frac{\chi(d)}{d^{1-\tau}}\left\{c'_\tau - \frac{1}{\tau^2}\frac{x^\tau}{d^\tau} + \frac{1}{\tau^2} + \frac{1}{\tau}\log\frac{x}{d}\cdot\frac{x^\tau}{d^\tau} + \frac{\vartheta d^{1-\tau}\log(x/d)}{x^{1-\tau}}\right\} \\ &= \left(c_\tau - \frac{1}{\tau}\right)\sum_{d\leqslant z}\frac{\chi(d)\log d}{d^{1-\tau}} + \left(c'_\tau + \frac{1}{\tau^2}\right)\sum_{d\leqslant z}\frac{\chi(d)}{d^{1-\tau}} - \\ &\quad - \frac{x^\tau}{\tau}\left(\frac{1}{\tau} - \log x\right)\sum_{d\leqslant z}\frac{\chi(d)}{d} + \vartheta\sum_{d\leqslant z}\frac{\log d + \log(x/d)}{x^{1-\tau}}.\end{aligned} \tag{3.5}$$

On the other hand using the Pólya–Vinogradov inequality

$$\Big|\sum_{d=a}^{b}\chi(d)\Big| \leqslant 2\sqrt{D}\log D = A,$$

Abel's inequality, and considering that the functions

$$\frac{\log d}{d^{1-\tau}}, \quad \frac{1}{d^{1-\tau}}, \quad \frac{1}{d}, \quad \sum_{m\leqslant x/d}\frac{\log md}{(md)^{1-\tau}}$$

are monotonically decreasing in d, we get the following inequalities:

$$\left|\left(\frac{1}{\tau} - c_\tau\right)\sum_{d>z}\frac{\chi(d)\log d}{d^{1-\tau}}\right| \leqslant \frac{1}{\tau}\cdot\frac{A\log z}{z^{1-\tau}}, \tag{3.6}$$

$$\left|\left(c'_\tau + \frac{1}{\tau^2}\right)\sum_{d>z}\frac{\chi(d)}{d^{1-\tau}}\right| \leqslant \frac{2}{\tau^2}\cdot\frac{A}{z^{1-\tau}}, \tag{3.7}$$

$$\left|\frac{x^\tau}{\tau}\left(\frac{1}{\tau} - \log x\right)\sum_{d>z}\frac{\chi(d)}{d}\right| \leqslant \frac{x^\tau\log x}{\tau^2}\cdot\frac{A}{z}, \tag{3.8}$$

$$\left|\sum_{z<d\leqslant x}\chi(d)\sum_{m\leqslant x/d}\frac{\log md}{(md)^{1-\tau}}\right| \leqslant \frac{A}{z^{1-\tau}}\cdot\log x\cdot\frac{1}{\tau}\cdot\frac{x^\tau}{z^\tau}. \tag{3.9}$$

The formulae (3.4)–(3.9) now imply the relation

$$\sum_{n\leqslant x}\frac{g(n)\log n}{n^{1-\tau}} = \left(\frac{1}{\tau} - c_\tau\right)L'(1-\tau) + \left(c'_\tau + \frac{1}{\tau^2}\right)L(1-\tau) - \frac{x^\tau}{\tau}\left(\frac{1}{\tau} - \log x\right)L(1) + \frac{\vartheta z\log x\cdot x^\tau}{x} + \frac{5\vartheta A\log x\cdot x^\tau}{\tau^2 z}. \tag{3.10}$$

If we now choose $z = \sqrt{Ax}$ $(\leqslant x)$ we get Lemma 3.

On the other hand we have

$$g(n) = \prod_{p^\alpha \| n} \left(1+\chi(p)+\ldots+\chi^\alpha(p)\right) \geqslant 0 \quad \text{and} \quad g(m^2) \geqslant 1.$$

Thus for $x \geqslant 4$

$$\sum_{n\leqslant x} \frac{g(n)\log n}{n^{1-\tau}} \geqslant \frac{\log 4}{4}. \tag{3.11}$$

Let us assume, that contrary to Theorem 2 $L(s)$ has two zeros or a non-simple zero in the interval

$$\left[1-\frac{2-3\varepsilon}{\log D}, 1\right] \subset \left[1-\frac{1-\varepsilon}{\log A}, 1\right] \tag{3.12}$$

(if $D > D_0(\varepsilon)$ effective constant).

Then there exists a τ with $0 < \tau \leqslant (1-\varepsilon)/\log A$ $(< \frac{1}{3})$ for which $L(1-\tau) \leqslant 0$ and $L'(1-\tau) = 0$. Applying now Lemma 3 for this τ, and for $x = A/\tau^8$ as $A \geqslant \sqrt{D} \to \infty$ and so $\tau \to 0$ we have

$$\frac{1}{\tau} - \log x = \frac{1}{\tau} - 8\log\frac{1}{\tau} - \log A \geqslant \frac{\varepsilon}{\tau} - 8\log\frac{1}{\tau} > 0 \tag{3.13}$$

since $1/\tau \geqslant \log A/(1-\varepsilon) > (1+\varepsilon)\log A$ and so the right side of (3.1) is in consequence of $L(1) \geqslant 0$ and $|c_\tau'| < 3$,

$$\left(\frac{1}{\tau^2}+c_\tau'\right)L(1-\tau) - \frac{x^\tau}{\tau}\left(\frac{1}{\tau}-\log x\right)L(1) + \frac{6\vartheta x^\tau \log x\sqrt{A}}{\tau^2 x} \tag{3.14}$$

$$\leqslant 0+0+\frac{6e^{\tau\log x}\cdot\frac{1}{\tau}\sqrt{A}}{\tau^2\frac{\sqrt{A}}{\tau^4}} < 6e\tau < \frac{\log 4}{4}$$

which contradicts to (3.11) and so proves Theorem 2.

We note that as one can see from this proof Theorem 2 is valid for an arbitrary real valued completely multiplicative number theoretical function θ, for which $|\theta(n)| \leqslant 1$ and $\left|\sum_{d=a}^{b}\theta(d)\right| \leqslant A$. The corresponding interval in (1.6) is then $\left[1-\frac{1+o(1)}{\log A}, 1\right]$.

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Received on 8. 9. 1975 (762)