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Spinor genera under field extensions, I*

by

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Dedicated to A. E. Ross on his 70th birthday

How does the genus of a positive definite integral quadratic form over the rationals behave when lifted to a totally real algebraic number field? This question was raised by N. C. Ankeny over ten years ago, and was resurrected again at the more recent Quadratic Forms Conference (Baton Rouge, La.) in 1972. The genus is partitioned into patches of (proper) spinor genera. In the present paper we treat the closely related problem of how does the *spinor genus* behave upon inflation to an extension field. Our fields always have characteristic different from two. We shall approach this question from the geometric standpoint of quadratic lattices; indeed, we consider it in a more general scope in that we study the spinor genera associated with an arbitrary regular lattice, which needs not be free. We will show that (*modulo some restrictions on the lattice and on the behaviour of the dyadic primes*) *the proper spinor genera in the genus of the given lattice do not collapse when lifted to an odd degree field extension*. The oddness of the field degree is essential as examples below (see Appendix A) will show. More specifically, our main results are:

THEOREM I. *Let L be a regular quadratic lattice of rank $r(L) \geqslant 3$ and defined over a global field F with the property that at each localization with respect to a dyadic prime spot $\mathfrak{p}$ on F, the lattice $L_{\mathfrak{p}}$ is modular. Then, for any odd degree field extension E/F, Γ_L is injective.*

THEOREM II. *Let L be a regular quadratic lattice of arbitrary rank, and defined over an algebraic number field F. If E/F is an odd degree field extension such that 2 is unramified in E, then Γ_L is injective.*

The maps Γ_L are defined below (in § 1). Theorem II can be strengthened for binary lattices in that we need only assume 2 is unramified

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in the bottom field F. (See § 5C.) The proofs rely heavily on the exact knowledge of the spinor norms of integral rotations on $L_\mathfrak{p}$ at every discrete spot $\mathfrak{p}$. Earlier we had also exploited this information of $\theta(O^+(L_\mathfrak{p}))$ to sharpen a theorem of M. Kneser (see [2]), yielding the *best possible* upper bounds for the exponents of the reduced discriminant of an indefinite $\boldsymbol{Z}$-lattice to have class number 1. Finally, two appendices are offered. In the first appendix, we furnish counter-examples to the main theorems cited above when the oddness of the field degree is relaxed; the second appendix makes a remark on the definition of spinor genus.

An important special case where Theorem I immediately applies is when F is a function field and so has no dyadic spots. Another is the case when $F = \boldsymbol{Q}$, L an integral lattice (i.e. integral with respect to scale) with discriminant an odd integer. Note also that when L is indefinite, then the familiar theorem of Eichler–Kneser says the (proper) spinor genus coincides with the (proper) class, so that non-equivalent proper classes in a given genus remain non-equivalent when lifted to E. A *Hasse domain* is a Dedekind domain with quotient field a global field F and which can be obtained as the intersection of almost all valuation rings of F. Both Theorems I and II generalize to these Hasse domains.

1. Preliminaries. Let E/F be a finite relative extension of global fields, $\mathfrak{o}_E$, $\mathfrak{o}_F$ respectively the rings of integers in E and F, and L a lattice on a regular quadratic space V over F with rank $r(L)$. Set $\tilde{V} = V \otimes_F E$ and $\tilde{L} = L \otimes_{\mathfrak{o}_F} \mathfrak{o}_E$. The map f from the group J_V of split rotations of V (i.e. the adèle group of V) into the idèle group J_F of F given by $f(u)_\mathfrak{p} \in \theta(u_\mathfrak{p})$ where θ is the spinor norm function induces a homomorphism from J_V into the factor group $J_F/P_D J_F^L$ of J_F. Here P_D denotes the principal idèles with respect to the group $D = \theta(O^+(V))$ and $J_F^L = \{j = (j_\mathfrak{p}) \in J_F\colon j_\mathfrak{p} \in \theta(O^+(L_\mathfrak{p}))$ at each non-archimedean spot $\mathfrak{p}\}$. The kernel of this homomorphism is the subgroup $P_V J'_V J_L$, where P_V consists of the principal split rotations, J'_V all those split rotations which at every spot $\mathfrak{p}$ on F has the component lying in the spinorial kernel $O'(V_\mathfrak{p})$, and $J_L = \{u = (u_\mathfrak{p}) \in J_V\colon u_\mathfrak{p} L_\mathfrak{p} = L_\mathfrak{p}$ at every non-archimedean spot $\mathfrak{p}\}$. If Φ_L denotes the induced map from $J_V/P_V J'_V J_L$ to $J_F/P_D J_F^L$, then it is a familiar fact that Φ_L is an isomorphism whenever $r(L) \geqslant 3$. In general, Φ_L is at least injective. See [7]. The set of all proper spinor genera in the genus $\operatorname{gen}(L)$ of L corresponds bijectively to the cosets in $J_V/P_V J'_V J_L$ and thus can be endowed with a group structure. It should be mentioned here that while $J_V/P_V J'_V J_L$ is always a finite elementary 2-abelian group, the group $J_F/P_D J_F^L$ can have *infinite* order when $r(L) = 2$ (e.g., $F = \boldsymbol{Q}$, $L \cong \langle 1, 1\rangle$). Next, define $g\colon J_F \to J_E$ by $(g(j_\mathfrak{p}))_\mathfrak{P} = j_\mathfrak{p}$ and $h\colon J_V \to J_{\tilde{V}}$ by $(h(u_\mathfrak{p}))_\mathfrak{P} = u_\mathfrak{p} \otimes_{F_\mathfrak{p}} E_\mathfrak{P}$ whenever $\mathfrak{P}|\mathfrak{p}$ in both instances. Then g and h induce the vertical homomorphisms Ψ_L and Γ_L respectively in the commutative diagram

$$\begin{array}{ccc} J_V/P_V J'_V J_L & \overset{\Phi_L}{\rightarrowtail} & J_F/P_D J_F^L \\ \downarrow{\scriptstyle \Gamma_L} & & \downarrow{\scriptstyle \Psi_L} \\ J_{\tilde{V}}/P_{\tilde{V}} J'_{\tilde{V}} J_{\tilde{L}} & \overset{\Phi_{\tilde{L}}}{\rightarrowtail} & J_E/P_{\tilde{D}} J_E^{\tilde{L}} \end{array}$$

so that Γ_L is injective if (and only if when $r(L) \geqslant 3$) Ψ_L is injective. In our proofs below we shall show that Ψ_L is injective.

Unexplained notations and terminologies are from [7].

2. Modular case. In this section we assume L is $\mathfrak{A}$-modular, $\mathfrak{A}$ a fractional ideal. Let $\mathfrak{p}$ be a discrete spot on F, E/F a finite extension, and $\mathfrak{P}_1, \ldots, \mathfrak{P}_g$ all the spots on E lying above $\mathfrak{p}$. For any $c \in E^*$, we have

$$|N_{E/F}(c)|_\mathfrak{p} = \prod_{i=1}^{g} |N_{\mathfrak{P}_i|\mathfrak{p}}(c)|_\mathfrak{p} = \prod_{i=1}^{g} |c|_{\mathfrak{P}_i}.$$

And since

$$|N_{E/F}(c)|_\mathfrak{p} = (1/N\mathfrak{p})^{\operatorname{ord}_\mathfrak{p} N_{E/F}(c)} \quad \text{and} \quad |c|_{\mathfrak{P}_i} = (1/N\mathfrak{P}_i)^{\operatorname{ord}_{\mathfrak{P}_i}(c)},$$

we have:

LEMMA 2.1.

$$\operatorname{ord}_\mathfrak{p} N_{E/F}(c) = \sum_{i=1}^{g} f(\mathfrak{P}_i|\mathfrak{p}) \operatorname{ord}_{\mathfrak{P}_i}(c).$$

LEMMA 2.2. *Let $D = \theta(O^+(V))$, $\tilde{D} = \theta(O^+(\tilde{V}))$ and $c \in \tilde{D}$. If $\dim_F V \geqslant 3$, then $N_{E/F}(c) \in D$ whenever $[E:F]$ is odd. If $\dim_F V \leqslant 2$, then we always have $N_{E/F}(c) \in D$.*

Proof. When $\dim_F V \geqslant 3$, 101:8, [7], characterizes the set D (and similarly for $\tilde{D}$) as the set of all the elements of F^* which are positive at all the real spots $\mathfrak{p}$ on F for which $V_\mathfrak{p}$ is anisotropic. For every such real spot $\mathfrak{p}$ on F, there is an odd number of such real spots on E since $[E:F]$ is odd. Of course, every such real spot $\mathfrak{P}$ on E comes from such a real spot $\mathfrak{p}$ on F.

When $\dim_F V = 2$, a stronger assertion is possible here because (via scaling) we may suppose the quadratic space V (hence also $\tilde{V}$) is a Pfister space; i.e. $V \cong [1, d]$ where $d =$ the discriminant of V. Hence, the conclusion follows immediately from either the Norm Principle of Scharlau or of Knebusch (Theorems 4.3, 5.1, Chap. VII, [6] resp.). This proves Lemma 2.2.

THEOREM 2.3. *Let L be a modular lattice on a regular quadratic space defined over a global field F, and having rank $r(L) \geqslant 3$. Then for any odd degree field extension E/F, the map Γ_L is injective.*

Proof. We will show Ψ_L is injective instead. If $\bar{j}\in \mathrm{Ker}(\Psi_L)$, $j\in J_F$, then there exists $c\in\tilde{D}$ for which

$$c\cdot j_{\mathfrak p}\in\theta\big(O^+(\tilde{L}_{\mathfrak P})\big)\quad \text{for every } \mathfrak P|\mathfrak p.$$

It is therefore enough to show that $N_{E/F}(c)\cdot j\in J_F^L$ by Lemma 2.2.

By 92 : 5, [7] and Proposition A, [4], we see that the spinor norm groups almost always satisfy

$$\theta\big(O^+(L_{\mathfrak p})\big)=\mathfrak U_{\mathfrak p}F_{\mathfrak p}^{*2}.$$

In the exceptional cases, $\mathfrak p$ must be dyadic and then we get $F_{\mathfrak p}^*$. Moreover, if $\mathfrak p$ is not exceptional for L, then every $\mathfrak P$ on E lying above $\mathfrak p$ cannot also be exceptional for $\tilde{L}$. Hence, we need only to show that $c\cdot j_{\mathfrak p}\in\mathfrak U_{\mathfrak P}E_{\mathfrak P}^{*2}$ for all $\mathfrak P|\mathfrak p$ implies $N_{E/F}(c)\cdot j_{\mathfrak p}$ lies in $\mathfrak U_{\mathfrak p}F_{\mathfrak p}^{*2}$. By Lemma 2.1,

$$\begin{aligned}\mathrm{ord}_{\mathfrak p}N_{E/F}(c)&=\sum_{i=1}^{g}f(\mathfrak P_i|\mathfrak p)\cdot\mathrm{ord}_{\mathfrak P_i}(c)\equiv\sum_{i=1}^{g}f(\mathfrak P_i|\mathfrak p)\cdot\mathrm{ord}_{\mathfrak P_i}(j_{\mathfrak p})\pmod 2\\&=\sum_{i=1}^{g}f(\mathfrak P_i|\mathfrak p)\cdot e(\mathfrak P_i|\mathfrak p)\cdot\mathrm{ord}_{\mathfrak p}(j_{\mathfrak p})=[E:F]\cdot\mathrm{ord}_{\mathfrak p}(j_{\mathfrak p})\equiv\mathrm{ord}_{\mathfrak p}(j_{\mathfrak p})\pmod 2.\end{aligned}$$

This completes the proof of the theorem.

3. Non-dyadic results. Consider a fixed non-dyadic prime spot $\mathfrak p$ on F, and $\mathfrak P_1,\dots,\mathfrak P_g$ all the distinct spots on E above $\mathfrak p$. Here E/F may be any finite extension. Again, scaling will not affect the local spinor norm groups so that we may suppose $L_{\mathfrak p}$ represents 1. By local theory, $L_{\mathfrak p}$ has a Jordan decomposition into modular components $L_{\mathfrak p}=L_1\perp\dots\perp L_t$.

PROPOSITION 3.1. *Let $c\in E^*$ such that at each $\mathfrak P_i$ we have $c\in\theta\big(O^+(\tilde{L}_{\mathfrak P_i})\big)$. Then, $N_{E/F}(c)\in\theta\big(O^+(L_{\mathfrak p})\big)$.*

Proof. Suppose that $L_{\mathfrak p}$ has a Jordan component of dimension $\geqslant 2$. Then $\theta\big(O^+(L_{\mathfrak p})\big)\supseteq\mathfrak U_{\mathfrak p}F_{\mathfrak p}^{*2}$. Should there exists an integral symmetry on $L_{\mathfrak p}$ having an odd order for its spinor norm, then $\theta\big(O^+(L_{\mathfrak p})\big)=F_{\mathfrak p}^*$ and there is nothing to prove. Thus, we assume here $\theta\big(O^+(L_{\mathfrak p})\big)=\mathfrak U_{\mathfrak p}F_{\mathfrak p}^{*2}$, which implies all the Jordan components L_j has even scale. But then so do all the Jordan components of $\tilde{L}_{\mathfrak P_i}$, and we obtain $\theta\big(O^+(\tilde{L}_{\mathfrak P_i})\big)=\mathfrak U_{\mathfrak P_i}E_{\mathfrak P_i}^{*2}$, $i=1,\dots,g$. Lemma 2.1 finishes this case.

Next, we consider the case where all the Jordan components of $L_{\mathfrak p}$ are 1-dimensional; say,

$$L_{\mathfrak p}\cong\langle 1\rangle\perp\langle\pi^{r_2}u_2\rangle\perp\dots\perp\langle\pi^{r_n}u_n\rangle$$

where $0=r_1<r_2<\dots<r_n$ are integers and the u_j's are units. The key to this case is Kneser's computation for the spinor norm group. (See Satz 3, [5].) His theorem states that if

$$\mathfrak M_j=\{\pi^{r_j}u_jc^2\colon\ c\in\mathfrak U_{\mathfrak p}\},\quad j=1,\dots,n,$$

and $\mathfrak M(L_{\mathfrak p})$ denotes the set of all elements of $F_{\mathfrak p}^*$ which can be expressed as a product of even number of factors from $\bigcup_{j=1}^{n}\mathfrak M_j$. Then

$$\theta\big(O^+(L_{\mathfrak p})\big)=\mathfrak M(L_{\mathfrak p})\cdot F_{\mathfrak p}^{*2}.$$

Clearly, the Jordan decomposition of $L_{\mathfrak p}$ also gives a Jordan decomposition for $\tilde{L}_{\mathfrak P_i}$ for every i. Thus, putting $\tilde{\mathfrak M}_j$ as $\mathfrak M_j$ above except having c ranging over $\mathfrak U_{\mathfrak P_i}$ and defining $\tilde{\mathfrak M}(\tilde{L}_{\mathfrak P_i})$ in a similar fashion, we have:

$$\theta\big(O^+(\tilde{L}_{\mathfrak P_i})\big)=\tilde{\mathfrak M}(\tilde{L}_{\mathfrak P_i})\cdot E_{\mathfrak P_i}^{*2}\quad\text{for every } i.$$

If

$$c=\Big(\prod_{m=1}^{2s_i}\pi^{r_{j_m}}u_{j_m}\Big)\eta^2\quad\text{where}\quad \eta\in E_{\mathfrak P_i}^*,$$

then

$$N_{E/F}(c)=\prod_{i=1}^{g}\prod_{m=1}^{2s_i}(\pi^{r_{j_m}}u_{j_m})^{k_i}\gamma^2$$

where $\gamma\in F_{\mathfrak p}^*$ and the exponents k_i is either 1 or 0 according to the local degree $n(\mathfrak P_i|\mathfrak p)$ is odd or even respectively. Hence, $N_{E/F}(c)$ will always lie in $\theta\big(O^+(L_{\mathfrak p})\big)$, and this completes the proof.

4. Dyadic results. This section is central to the proofs of our main results. We assume throughout this section that $\mathfrak p$ is an unramified dyadic spot on F (i.e. the element 2 is a prime in $F_{\mathfrak p}$). E/F may still be any finite extension. Our aim is to prove:

PROPOSITION 4.1. *Let $c\in E^*$ such that $c\in\theta\big(O^+(\tilde{L}_{\mathfrak P})\big)$, $\forall\mathfrak P|\mathfrak p$. Assume further that $e(\mathfrak P|\mathfrak p)=1$ for every $\mathfrak P|\mathfrak p$. Then $N_{E/F}(c)\in\theta\big(O^+(L_{\mathfrak p})\big)$.*

To accomplish this, heavy reliance is placed on the exact knowledge of local spinor norm groups as was for the non-dyadic case in § 3. The computations for the spinor norm groups needed here can be found in [2]. Our proof is divided into various cases depending upon the form of a Jordan splitting for $L_{\mathfrak p}$. By scaling, we may suppose $L_{\mathfrak p}$ has scale $s(L_{\mathfrak p})=\mathfrak o_{F_{\mathfrak p}}$.

Case I. The Jordan components of $L_{\mathfrak p}$ are all 1-dimensional; say,

$$L_{\mathfrak p}\cong\langle 1\rangle\perp\langle 2^{r_2}u_2\rangle\perp\dots\perp\langle 2^{r_n}u_n\rangle,$$

where $0 = r_1 < r_2 < \ldots < r_n$ are integers and the u_j's are units. It is convenient to introduce the following notations: for $j = 1, \ldots, n-1$ set

$$K_{j,j+1} = \langle 2^{r_j} u_j \rangle \perp \langle 2^{r_{j+1}} u_{j+1} \rangle,$$

$$D_j = 2^{r_{j+1} - r_j} u_{j+1} u_j^{-1},$$

$$r(K_{j,j+1}) = r_{j+1} - r_j,$$

$$I_{j,j+1} = [F_\mathfrak{p}^*/F_\mathfrak{p}^{*2} : \theta(O^+(K_{j,j+1}))].$$

Denote by $\boldsymbol{P}(M)$ to be the set of all primitive vectors in the lattice M which give rise to integral symmetry on M. Since the rank of $K_{j,j+1}$ is always 2, there will be no confusion for the notation $r(K_{j,j+1})$.

The next two results can be found in Theorems 2.2, 2.7, [2] resp.:

LEMMA 4.2. *In the setting here for $L_\mathfrak{p}$, assume there is at least one j for which $r(K_{j,j+1}) = 1$ or 3. Then if $r_s - r_t = 2$ or 4 for any $s, t = 1, \ldots, n$, we have $\theta(O^+(L_\mathfrak{p})) = F_\mathfrak{p}^*$.*

LEMMA 4.3. *In the setting here for $L_\mathfrak{p}$, assume that $L_\mathfrak{p}$ does not satisfy the hypotheses of Lemma 4.2 above. Then we have:*

$$\theta(O^+(L_\mathfrak{p})) = \Big\{\prod_{i=1}^{\text{even}} Q(v_i):\ v_i \in \boldsymbol{P}(K_{j_i,j_i+1}),\ 1 \leqslant j_i \leqslant n-1\Big\}.$$

We also need the binary computations (see 1.9, [2]).

LEMMA 4.4. *Suppose $L_\mathfrak{p} = \langle 1 \rangle \perp \langle 2^r u \rangle$, $r \geqslant 1$, $u \in \mathfrak{U}_\mathfrak{p}$. Then*

$$\theta(O^+(L_\mathfrak{p})) = \begin{cases} \{c \in F_\mathfrak{p}^*:\ (c, -2u)_\mathfrak{p} = +1\} & \text{if} \quad r = 1 \text{ or } 3,\\ \{c \in \mathfrak{U}_\mathfrak{p} F_\mathfrak{p}^{*2}:\ (c, -u)_\mathfrak{p} = +1\} & \text{if} \quad r = 2,\\ F_\mathfrak{p}^{*2} \cup uF_\mathfrak{p}^{*2} \cup \Delta F_\mathfrak{p}^{*2} \cup u\Delta F_\mathfrak{p}^{*2} & \text{if} \quad r = 4,\\ F_\mathfrak{p}^{*2} \cup 2^r u F_\mathfrak{p}^{*2} & \text{if} \quad r \geqslant 5. \end{cases}$$

Should $\theta(O^+(L_\mathfrak{p})) = F_\mathfrak{p}^*$, Proposition 4.1 follows trivially. So, we shall suppose below that this is *not* the situation. Three subcases present themselves:

(i) There exists an index j for which $r(K_{j,j+1}) = 1$ or 3.
(ii) There exists an index j for which $r(K_{j,j+1}) = 2$ or 4.
(iii) $r(K_{j,j+1}) \geqslant 5$ for all $j = 1, \ldots, n-1$.

Proof of Subcase (i). From Lemma 4.4, $I_{j,j+1} = 2$. Hence,

$$\theta(O^+(L_\mathfrak{p})) = \theta(O^+(K_{j,j+1})) = \{c \in F_\mathfrak{p}^*:\ (c, -D_j)_\mathfrak{p} = +1\}.$$

It suffices to show that

$$(N_{\mathfrak{P}|\mathfrak{p}}(c), -D_j)_\mathfrak{p} = +1 \quad \text{for each } \mathfrak{P}|\mathfrak{p}.$$

By Lemma 4.3, c can be expressed as

$$c = \prod_{m=1}^{\text{even}} c_m, \quad c_m \in E_\mathfrak{P}^*$$

where

$$c_m \in Q(\boldsymbol{P}(\tilde{K}_{s_m, s_m+1})) \quad \text{for each } m.$$

This is because the Jordan splitting for $L_\mathfrak{p}$ provides also a Jordan splitting for $\tilde{L}_\mathfrak{P}$, and since $\mathfrak{P}|\mathfrak{p}$ is unramified, Lemma 4.3 is applicable to $\tilde{L}_\mathfrak{P}$ as well. Thus, it is enough to show that for each $\mathfrak{P}|\mathfrak{p}$,

$$(N_{\mathfrak{P}|\mathfrak{p}}(c_m), -D_j)_\mathfrak{p} = +1.$$

Suppose $r(\tilde{K}_{s_m,s_m+1}) = 1$ or 3. Then, Lemma 4.5 below implies that $(c_m, -D_j)_\mathfrak{P} = +1$. Hence, $(N_{\mathfrak{P}|\mathfrak{p}}(c_m), -D_j)_\mathfrak{p} = +1$. On the other hand, should $r(\tilde{K}_{s_m,s_m+1})$ be even, and hence $\geqslant 6$ by Lemma 4.2, then one sees immediately from Lemma 4.4 that we also have $(N_{\mathfrak{P}|\mathfrak{p}}(c_m), -D_j)_\mathfrak{p} = +1$. Therefore, Subcase (i) will be finished once we prove:

LEMMA 4.5. *In the setting at hand for $L_\mathfrak{p}$ with index j satisfying $r(\tilde{K}_{j,j+1}) = 1$ or 3, if t is any index also with $r(\tilde{K}_{t,t+1}) = 1$ or 3, then for each $\mathfrak{P}|\mathfrak{p}$, we have $b \in Q(\boldsymbol{P}(\tilde{K}_{t,t+1}))$ implies $(b, -D_j)_\mathfrak{P} = +1$.*

Proof. We use $\langle, \ldots, \rangle$ to denote lattices and $[, \ldots,]$ for spaces. We make two assertions:

(A) if the binary $F_\mathfrak{p}$-space $[1, D_t]$ represents $2^{r_t} u_t$, then $b \in Q(\boldsymbol{P}(\tilde{K}_{t,t+1}))$ implies

$$(b, -D_t)_\mathfrak{P} = +1 \quad \text{for all } \mathfrak{P}|\mathfrak{p};$$

(B) if it does not represent $2^{r_t} u_t$, then $b \in Q(\boldsymbol{P}(\tilde{K}_{t,t+1}))$ implies

$$(b, -D_t)_\mathfrak{P} = (-1)^{n(\mathfrak{P}|\mathfrak{p})} \quad \text{for all } \mathfrak{P}|\mathfrak{p}.$$

We shall just do case (A). Here, we have $(2^{r_t} u_t, -D_t)_\mathfrak{p} = 1$, and an isometry between $E_\mathfrak{P}$-spaces $[2^{r_t} u_t, 2^{r_{t+1}} u_{t+1}] \cong [b, bD_t]$. When r_t is even, $(b, -D_t)_\mathfrak{P} = (u_t, 2u_{t+1})_\mathfrak{P}$ and $(u_t, 2u_{t+1})_\mathfrak{p} = 1$, regardless of the parity of the local degree. When r_t is odd, $(b, -D_t)_\mathfrak{P} = (2u_t, u_{t+1})_\mathfrak{P}$, which is always 1. Computations for Hilbert symbols are aided by Corollary 1, [1]. A similar (slightly longer) calculation applies to case (B).

We have, therefore, in case (A) $1 = (b, -D_t)_\mathfrak{P} = (N_{\mathfrak{P}|\mathfrak{p}}(b), -D_t)_\mathfrak{p}$. Since $I_{j,j+1} = 2$, the latter must be $(N_{\mathfrak{P}|\mathfrak{p}}(b), -D_j)_\mathfrak{p}$, which equals $(b, -D_j)_\mathfrak{P}$. Fortunately, case (B) really cannot occur in the present setting for $L_\mathfrak{p}$. Indeed, if $2^{r_t} u_t$ is not represented by the binary $F_\mathfrak{p}$-space $[1, D_t]$, then $2^{r_t} u_t$ does not belong to $\theta(O^+(K_{t,t+1}))$, which must also equal to $\theta(O^+(L_\mathfrak{p}))$ since $I_{t,t+1} = 2$ by Lemma 4.4. As $L_\mathfrak{p}$ represents 1, there is a rotation (a product of two *integral* symmetries) with spinor norm $2^{r_t}u_t$. Thus, a contradiction (!) is achieved and finishes Lemma 4.5

Proof of Subcase (ii). We break into two parts still: (a) when $r(K_{j,j+1}) = 2$, and (b) when $r(K_{j,j+1}) = 4$.

(a) Here $I_{j,j+1}$ may have the value either 2 or 4, depending upon whether $-D_j$ is one of the two units 1 and Δ (modulo squares) or not. If $I_{j,j+1} = 2$, then surely r_s must be even for all s. Lemma 4.4 then gives $\theta(O^+(\tilde{L}_{\mathfrak{P}})) = \mathfrak{U}_{\mathfrak{P}}E_{\mathfrak{P}}^{*2}$ for all $\mathfrak{P}|\mathfrak{p}$. Clearly, $N_{E/F}(c)\in\mathfrak{U}_{\mathfrak{p}}E_{\mathfrak{p}}^{*2} = \theta(O^+(L_{\mathfrak{p}}))$. So, let $I_{j,j+1} = 4$. Moreover, we may suppose that $I_{s,s+1} \neq 2$ for any s. It follows from Lemmas 4.3 and 4.4 that $[F_{\mathfrak{p}}^*/F_{\mathfrak{p}}^{*2}: \theta(O^+(L_{\mathfrak{p}}))] = I_{j,j+1} = 4$ if and only if

(i) all r_s are even,

(ii) $(b, -D_j)_{\mathfrak{p}} = (b, -D_s)_{\mathfrak{p}}$ whenever $b\in\mathfrak{U}_{\mathfrak{p}}$ and $r(K_{s,s+1}) = 2$,

(iii) $(D_s, -D_j)_{\mathfrak{p}} = 1$ whenever $r(K_{s,s+1}) = 4$.

In this situation, $\theta(O^+(L_{\mathfrak{p}})) = \{c\in\mathfrak{U}_{\mathfrak{p}}F_{\mathfrak{p}}^{*2}: (c, -D_j)_{\mathfrak{p}} = +1\}$. As all the r_s are even, $N_{E/F}(c)\in\mathfrak{U}_{\mathfrak{p}}F_{\mathfrak{p}}^{*2}$. Using a similar calculation as in Subcase (i), it can be shown that $(N_{\mathfrak{P}|\mathfrak{p}}(c_m), -D_j)_{\mathfrak{p}} = 1$, for each m and $\mathfrak{P}|\mathfrak{p}$ (using the same notation as before).

Whereas if either condition (ii) or (iii) is violated, then $\theta(O^+(L_{\mathfrak{p}}))$ can only be $\mathfrak{U}_{\mathfrak{p}}F_{\mathfrak{p}}^{*2}$ (since it is excluded from being equal to $F_{\mathfrak{p}}^*$), and so the conclusion follows. Finally, suppose just condition (i) is not satisfied, consider a binary sublattice K_{s_m,s_m+1} with $r(K_{s_m,s_m+1}) = 2$ (note by Subcase (i), it cannot be 1 nor 3). Let

$$c_m\in Q(\boldsymbol{P}(\tilde{K}_{s_m,s_m+1})) = 2^{r_{s_m}}\cdot u_{s_m}\cdot Q(\boldsymbol{P}(\langle \overline{1, D_{s_m}}\rangle)), \quad \text{and} \quad \mathfrak{P}|\mathfrak{p}.$$

Then, we have

$$N_{\mathfrak{P}|\mathfrak{p}}(c_m) = N_{\mathfrak{P}|\mathfrak{p}}(2^{r_{s_m}}\cdot u_{s_m})\cdot N_{\mathfrak{P}|\mathfrak{p}}(a\cdot t^2),$$

where $a\in\mathfrak{U}_{\mathfrak{P}}$, $t\in E_{\mathfrak{P}}^*$. Since $(a, -D_{s_m})_{\mathfrak{P}} = +1$, using condition (ii), we see that

$$(N_{\mathfrak{P}|\mathfrak{p}}(a), -D_{s_m})_{\mathfrak{p}} = (N_{\mathfrak{P}|\mathfrak{p}}(a), -D_j)_{\mathfrak{p}} = 1.$$

This gives $(N_{\mathfrak{P}|\mathfrak{p}}(c_m), -D_j)_{\mathfrak{p}} = 1$ when the local degree $n(\mathfrak{P}|\mathfrak{p})$ is even. On the other hand, if $n(\mathfrak{P}|\mathfrak{p})$ is odd, we still have that $2u_{s_m}\in\theta(O^+(L_{\mathfrak{p}}))$ — here we assume that s_m was chosen at the outset to have r_{s_m} being odd. Thus, we always have: $(N_{\mathfrak{P}|\mathfrak{p}}(c_m), -D_j)_{\mathfrak{p}} = 1$, and the desired conclusion follows. Similar considerations, using Lemma 4.4 always, handle the remaining cases of $r(K_{s_m,s_m+1}) = 4$, and $\geqslant 5$. This completes the proof for part (a).

(b) This part is handled in an analogous fashion.

Subcase (iii) is the easiest of all three subcases, and we omit its proof. We are now completely finished with Case I.

Case II. There is at least one Jordan component of $L_{\mathfrak{p}}$ having rank three or greater. Recall from [2] that a lattice N over $\mathfrak{o}_{F_{\mathfrak{p}}}$ is said to have *even order* if $Q(\boldsymbol{P}(N)) \subseteq \mathfrak{U}_{\mathfrak{p}}F_{\mathfrak{p}}^{*2}$, and has *odd order* if $Q(\boldsymbol{P}(N)) \subseteq 2\mathfrak{U}_{\mathfrak{p}}F_{\mathfrak{p}}^{*2}$. Since we are excluding the case where $\theta(O^+(L_{\mathfrak{p}}))$ can be $F_{\mathfrak{p}}^*$, it follows from Theorem 3.8, [2] that if

$$L_{\mathfrak{p}} = L_1 \perp \ldots \perp L_t$$

is a Jordan decomposition for $L_{\mathfrak{p}}$ (remember $L_{\mathfrak{p}}$ is not modular), then L^j must have odd order all $j = 1, \ldots, t$. Furthermore, we have $\theta(O^+(L_{\mathfrak{p}})) = \mathfrak{U}_{\mathfrak{p}}F_{\mathfrak{p}}^{*2}$. A unary Jordan component $\langle 2^r u\rangle$ for $L_{\mathfrak{p}}$ has odd order if and only if r is odd, and hence lifts to an odd order unary component above for $\tilde{L}_{\mathfrak{P}}$. A binary component 2^rM, where M is unimodular, has odd order when: (i) r is even and $M \cong A(0, 0)$ or $A(2, 2\varrho)$, (ii) r is odd and $M = A(1, 0)$ or $A(1, 4\varrho)$. $A(0, 0)$ remains unchanged upstairs. $A(2, 2\varrho)$ remains also unchanged unless $\Delta_{\mathfrak{p}}$ becomes a square in $E_{\mathfrak{P}}$ in which case it becomes $A(0, 0)$. Similarly, $A(1, 0)$ remains unchanged, and $A(1, 4\varrho)$ changes to $A(1, 0)$ if and only if $\Delta_{\mathfrak{p}}\in E_{\mathfrak{P}}^{*2}$. Thus, a binary component of odd order remains odd order upstairs. Finally, if a component L_j with rank three or greater has odd order, then L_j must be totally improper and so has even rank. Write $L_j = 2^rM$, where M is unimodular. It is not difficult to see that r must be even and M is either hyperbolic or else isometric to

$$A(0, 0) \perp \ldots \perp A(0, 0) \perp A(2, 2\varrho).$$

In any case, L_j clearly lifts to a component of odd order as well. Therefore, $\theta(O^+(\tilde{L}_{\mathfrak{P}})) = \mathfrak{U}_{\mathfrak{P}}E_{\mathfrak{P}}^{*2}$ for all $\mathfrak{P}|\mathfrak{p}$. Hence, Proposition 4.1 is valid for this Case II.

Case III. Every Jordan component has rank less or equal to two, but on the other hand, there is at least one binary component. In this situation, $\theta(O^+(L_{\mathfrak{p}}))$ is fully described by Theorem 3.14, [2]. Once again since we are excluding the spinor norm group from being $F_{\mathfrak{p}}^*$, only the following possibilities can occur: writing

$$L_{\mathfrak{p}} = L_1 \perp 2^{r_2}L_2 \perp \ldots \perp 2^{r_t}L_t$$

for a Jordan splitting of $L_{\mathfrak{p}}$ with L_j's unimodular, then

(A) all Jordan components have odd order;

(B) all Jordan components have even order;

or

(C) all binary components, say, $2^{r_j}L_j$ have the form where $L_j = A(a_j, 2b_j)$, $a_j, b_j\in\mathfrak{U}_{\mathfrak{p}}$, and moreover we must have:

(Ci) the associated spaces of all these binary components are isometric, (Cii) for any unary component, say, $2^{r_k}L_k$, where $L_k \cong \langle u_k\rangle$, $u_k \in \mathfrak{U}_\mathfrak{p}$, the Hilbert symbol $\big(2^{r_k}u_k, -\det\big(A(a_j, 2b_j)\big)\big)_\mathfrak{p} = +1$,

(Ciii) $r_{s+1}-r_s \geqslant 4$ for all $s = 1, \ldots, t-1$.

In the exceptional cases (A) and (B), $\theta\big(O^+(L_\mathfrak{p})\big) = \mathfrak{U}_\mathfrak{p}F_\mathfrak{p}^{*2}$, while in exceptional case (C),

$$\theta\big(O^+(L_\mathfrak{p})\big) = \theta\big(O^+(L_j)\big) = \big\{c \in F_\mathfrak{p}^*:\ \big(c, -\det(L_j)\big)_\mathfrak{p} = +1\big\}.$$

By reasoning in a similar fashion as given in Case II we can handle exceptional cases (A) and (B). So, let us suppose then that we are in the exceptional case (C). Clearly, each binary component $2^{r_j}A(a_j, 2b_j)$ lifts upstairs to the same since $\mathfrak{P}|\mathfrak{p}$ is unramified. Also, conditions (Ci), (Cii), and (Ciii) are all preserved upstairs as well. Hence, we deduce that

$$\theta\big(O^+(\tilde{L}_\mathfrak{P})\big) = \theta\big(O^+(\tilde{L}_j)\big) = \big\{b \in E_\mathfrak{P}^*:\ \big(b, -\det(L_j)\big)_\mathfrak{P} = +1\big\},$$

for every $\mathfrak{P}|\mathfrak{p}$. Thus, if $c \in \theta\big(O^+(\tilde{L}_\mathfrak{P})\big)$, $\big(N_{\mathfrak{P}|\mathfrak{p}}(c), -\det(L_j)\big)_\mathfrak{p} = +1$ so that $N_{\mathfrak{P}|\mathfrak{p}}(c)$ belongs to $\theta\big(O^+(L_\mathfrak{p})\big)$, $\forall\, \mathfrak{P}|\mathfrak{p}$. Hence, $N_{E/F}(c)$ does as well. This finishes Case III, and therefore, completes the proof of Proposition 4.1.

5. General cases

5A. Proof of Theorem I. Let $\bar{j} \in J_F/P_DJ_F^L$ such that $\Psi_L(\bar{j}) = \bar{1}$. Put

$$\boldsymbol{T} = \{\text{spot } \mathfrak{p} \text{ on } F\text{: either (i) } \mathfrak{p} \text{ is dyadic, or else (ii) } \mathfrak{p} \text{ is non-dyadic and } \theta\big(O^+(L_\mathfrak{p})\big) \text{ non} \supseteq \mathfrak{U}_\mathfrak{p}F_\mathfrak{p}^{*2}\},$$

and

$$\boldsymbol{R} = \{\text{spot } \mathfrak{p} \text{ on } F\text{: } \mathfrak{p} \text{ is real archimedean and } V_\mathfrak{p} \text{ is anisotropic}\}.$$

Then, $\boldsymbol{T} \cup \boldsymbol{R}$ is a finite set. By the weak approximation theorem (11 : 8, [7]), there is an element $d \in F^*$ satisfying:

(a) d is positive at each $\mathfrak{p} \in \boldsymbol{R}$,

(b) $d \cdot j_\mathfrak{p} \in F_\mathfrak{p}^{*2}$, $\forall\, \mathfrak{p} \in \boldsymbol{T}$.

Since $r(L) \geqslant 3$, 101 : 8, [7] gives $d \in D = \theta\big(O^+(V)\big)$. But, $\overline{(d)j} = \bar{j}$. Therefore, without loss of generality, we may suppose at the outset our idèle j satisfies

$$j_\mathfrak{p} \in F_\mathfrak{p}^{*2} \qquad \forall\, \mathfrak{p} \in \boldsymbol{T}.$$

Now, $\bar{j} \in \operatorname{Ker}(\Psi_L)$ implies that there exists $c \in \tilde{D}$ such that

$$c \cdot j_\mathfrak{p} \in \theta\big(O^+(\tilde{L}_\mathfrak{P})\big), \qquad \mathfrak{P}|\mathfrak{p}.$$

If $\mathfrak{p}$ is a discrete spot on F and not belonging to $\boldsymbol{T}$, then $\theta\big(O^+(L_\mathfrak{p})\big) = \mathfrak{U}_\mathfrak{p}F_\mathfrak{p}^{*2}$ or $F_\mathfrak{p}^*$. We want to show $N_{E/F}(c) \cdot j_\mathfrak{p} \in \theta\big(O^+(L_\mathfrak{p})\big)$, so we may assume it is in the $\mathfrak{U}_\mathfrak{p}F_\mathfrak{p}^{*2}$ case. From non-dyadic considerations, we have $\theta\big(O^+(\tilde{L}_\mathfrak{P})\big) \subseteq \mathfrak{U}_\mathfrak{P}E_\mathfrak{P}^{*2}$ for $\mathfrak{P}|\mathfrak{p}$. As in the proof of Theorem 2.3 above,

$$\operatorname{ord}_\mathfrak{p} N_{E/F}(c) \equiv [E:F] \cdot \operatorname{ord}_\mathfrak{p}(j_\mathfrak{p}) \pmod 2.$$

Since E/F is an odd extension, we have $N_{E/F}(c) \cdot j_\mathfrak{p} \in \theta\big(O^+(L_\mathfrak{p})\big)$.

Now let $\mathfrak{p} \in \boldsymbol{T}$. If $\mathfrak{p}$ is dyadic, use the argument in the proof of Theorem 2.3. When $\mathfrak{p}$ is non-dyadic, $j_\mathfrak{p}$ is a square in $F_\mathfrak{p}^*$. Apply Proposition 3.1 to c. Therefore, at every discrete spot $\mathfrak{p}$ on F, we always have $N_{E/F}(c) \cdot j_\mathfrak{p} \in \theta\big(O^+(L_\mathfrak{p})\big)$. This means $N_{E/F}(c) \cdot j \in J_F^L$. Since $c \in \tilde{D}$, Lemma 2.2 yields $j \in P_DJ_F^L$. This proves the injectivity of Ψ_L.

5B. Proof of Theorem II. The reasoning is entirely analogous. We want to show that $N_{E/F}(c) \cdot j \in J_F^L$. The proofs for the non-dyadic spots remain intact. At the dyadic spots, we need to invoke Proposition 4.1. When the rank $r(L)$ is less than three, apply Theorem 5.1 given below.

5C. A stronger Theorem II for binary lattices. We wish to prove the following result:

THEOREM 5.1. *Let L be a regular binary quadratic lattice and defined over an algebraic number field F in which 2 is unramified. For any odd degree field extension E/F, Ψ_L is injective.*

Proof. The proof is not quite the same because the set $D = \theta\big(O^+(V)\big)$ can no longer be characterized by 101 : 8, [7]. As observed in the introduction, even though $\Phi_L\colon J_V/P_VJ_V'J_L \to J_F/P_DJ_F^L$ may not be an isomorphism, it is still a monomorphism (the same goes for $\Phi_{\tilde{L}}$) so that still Γ_L is injective if Ψ_L is injective. We will again show this latter map is a monomorphism.

The non-dyadic arguments are essentially the same as in the proofs of Theorem 2.3 and Proposition 3.1. At the dyadic primes, also the proof of Theorem 2.3 will take care of the case when $L_\mathfrak{p}$ is modular. So, we suppose that we are in the situation where $\mathfrak{p}$ is dyadic and $L_\mathfrak{p}$ is non-modular. We have then

$$L_\mathfrak{p} \cong 2^t\langle u_1, 2^r u_2\rangle, \qquad r \geqslant 1,\ u_1, u_2 \in \mathfrak{U}_\mathfrak{p}.$$

If $r = 1$ or 3, Lemma 5.2 below will finish the case. If $r = 2$, Lemma 5.2 together with the fact that $b \in \mathfrak{U}_\mathfrak{P}E_\mathfrak{P}^{*2}$ for every $\mathfrak{P}|\mathfrak{p}$ will imply $N_{E/F}(b) \in \mathfrak{U}_\mathfrak{p}F_\mathfrak{p}^{*2}$. Finally, if $r \geqslant 4$, use Lemma 5.3 below and the Local Square Theorem (63 : 1, [7]). Therefore, we shall be through after proving the two lemmas below.

LEMMA 5.2. *Let E/F be an extension of odd degree, and $\mathfrak{p}$ any discrete prime spot on F. If $a \in E^*$, $b \in F_\mathfrak{p}^*$, and $d \in F^*$, then $(ab, -d)_\mathfrak{P} = 1$ for all $\mathfrak{P}|\mathfrak{p}$ implies $\big(N_{E/F}(a)b, -d\big)_\mathfrak{p} = 1$.*

Proof. For every $\mathfrak{P}|\mathfrak{p}$, $(a, -d)_\mathfrak{P} = (b, -d)_\mathfrak{P}$. If $n(\mathfrak{P}|\mathfrak{p})$ is odd, $\big(N_{\mathfrak{P}|\mathfrak{p}}(a), -d\big)_\mathfrak{p} = (a, -d)_\mathfrak{P} = (b, -d)_\mathfrak{p}$. If $n(\mathfrak{P}|\mathfrak{p})$ is even, then as $(b, -d)_\mathfrak{P}$

$= 1$, we see $(N_{\mathfrak{P}|\mathfrak{p}}(a), -d)_{\mathfrak{p}} = 1$. Thus,

$$(N_{E/F}(a), -d)_{\mathfrak{p}} = \prod_{\substack{n(\mathfrak{P}|\mathfrak{p})\\ \text{odd}}} (N_{\mathfrak{P}|\mathfrak{p}}(a), -d)_{\mathfrak{p}} \prod_{\substack{n(\mathfrak{P}|\mathfrak{p})\\ \text{even}}} (N_{\mathfrak{P}|\mathfrak{p}}(a), -d)_{\mathfrak{p}} = (b, -d)_{\mathfrak{p}}$$

since $[E:F] = \sum_{\mathfrak{P}|\mathfrak{p}} n(\mathfrak{P}|\mathfrak{p})$ is odd. This proves the lemma.

LEMMA 5.3. *Let* $\mathfrak{p}$ *be a dyadic spot and* $L_{\mathfrak{p}} \cong \langle 1, 2^r u\rangle$, $r \geqslant 3$, $u \in \mathfrak{U}_{\mathfrak{p}}$, *adapted to the orthogonal basis* $\{x, y\}$. *Let* E/F *be any finite extension with* $\mathfrak{P}$ *a spot lying above* $\mathfrak{p}$, $v \in P(\tilde{L}_{\mathfrak{P}})$ *with* $v = Ax+By$, $A, B \in \mathfrak{o}_{E_{\mathfrak{P}}}$. *Then, one of the following must occur:* (i) $A \in \mathfrak{U}_{\mathfrak{P}}$, (ii) $B \in \mathfrak{U}_{\mathfrak{P}}$ *with* $\operatorname{ord}_{\mathfrak{P}}(A) \leqslant e(\mathfrak{P}|\mathfrak{p})$, *or* (iii) $B \in \mathfrak{U}_{\mathfrak{P}}$ *with* $\operatorname{ord}_{\mathfrak{P}}(A) \geqslant (r-1)\cdot e(\mathfrak{P}|\mathfrak{p})$.

Proof. Since $v \in \boldsymbol{P}(\tilde{L}_{\mathfrak{P}})$, if $A \notin \mathfrak{U}_{\mathfrak{P}}$, then B must be a unit. Also, the symmetry $S_v \in O(\tilde{L}_{\mathfrak{P}})$, which gives $2A\cdot \mathfrak{o}_{E_{\mathfrak{P}}} = 2B(v, x)\cdot \mathfrak{o}_{E_{\mathfrak{P}}} \subseteq Q(v)\cdot \mathfrak{o}_{E_{\mathfrak{P}}}$. Hence,

$$\operatorname{ord}_{\mathfrak{P}}(2A) \geqslant \begin{cases} \operatorname{ord}_{\mathfrak{P}}(A^2) & \text{if} \quad \operatorname{ord}_{\mathfrak{P}}(A^2) < r\cdot e(\mathfrak{P}|\mathfrak{p}),\\ r\cdot \operatorname{ord}_{\mathfrak{P}}(2) & \text{if} \quad \operatorname{ord}_{\mathfrak{P}}(A^2) \geqslant r\cdot e(\mathfrak{P}|\mathfrak{p}).\end{cases}$$

These conditions force the conclusions.

Remark 5.4. It is now quite apparent that if we are in the function field case, then there would be no need to impose the restriction on the rank either. Thus, we have the following proposition: *Let* L *be a regular quadratic lattice of arbitrary rank and defined over a (global) function field* F. *For any odd degree field extension* E/F, Γ_L *is injective.*

Appendix A. In this appendix we will present examples which show that the oddness of the field extension degree in both Theorems I and II is necessary.

A1. Let $q_0, q_1, \ldots, q_m$ be distinct odd primes each congruent to 5 (mod 8), so that $(2/q_i) = -1$ for every i. Denote by Δ_k a fixed non-square unit in $\boldsymbol{Q}_{q_k}$, $k = 0, 1, \ldots, m$. Putting

$$L \cong \langle 1, (q_0\ldots q_m)^2, (q_0\ldots q_m)^4, \ldots, (q_0\ldots q_m)^{2d}\rangle$$

to be a lattice over $\boldsymbol{Z}$ of rank $d+1 \geqslant 3$. Then, it can be shown that $J_{\boldsymbol{Q}}/P_{\boldsymbol{Q}}J_{\boldsymbol{Q}}^L$ is a $\boldsymbol{Z}_2$-space of dimension m. Similarly, the indefinite lattice $L^- \cong \langle -1, (q_0\ldots q_m)^2, \ldots, (q_0\ldots q_m)^{2d}\rangle$ also gives $g^+(L^-) = 2^m$. Consider now $E = \boldsymbol{Q}(\sqrt{q_1})$. Let ξ_i denote the idèle which is 1 at all components $(\xi_i)_p$ for $p \neq q_i$ and which is Δ_i at $p = q_i$, $i = 1, \ldots, m$. Then, $\{\xi_1, \ldots, \xi_m\}$ is a basis for $J_{\boldsymbol{Q}}/P_{\boldsymbol{Q}}J_{\boldsymbol{Q}}^L$. (Note: Since $(-1/q_i) = 1$ at every i, $P_D J_{\boldsymbol{Q}}^L = P_{\boldsymbol{Q}}J_{\boldsymbol{Q}}^L$.) But, the cosets in $J_{\boldsymbol{Q}}/P_D J_{\boldsymbol{Q}}^L$ represented by these basis vectors ξ_i, $i \neq 1$, all collapse in E. To see this, one notes that the q_i-th component of the idèle ξ_i may be taken as q_1 since $(q_1/q_i) = -1$. But then, q_1 is a square in E. It is now quite apparent that a more patient analysis will reveal a method whereby any desired amount of collapsing takes place. Similarly, since -1 is a square at all the q's, considering $E^- = \boldsymbol{Q}(\sqrt{-q_1})$ will yield similar behaviour, as is the case with the lattice L^- with respect to either E or E^-.

A2. We now show that Theorem I will also fail under relative quadratic extensions. Let $F = \boldsymbol{Q}(\sqrt{-5})$ and $E = \boldsymbol{Q}(\sqrt{-5}, \sqrt{-1})$ its Hilbert class field. Consider the unimodular lattice $L \cong \langle 1, 1, \ldots, 1\rangle$ associated to the quadratic form representing sums of three or more squares. Clearly, $D = F^*$ here. Using the calculations for the local spinor norm groups, it is not difficult to see that $P_D J_F^L = P_F J_F^{\text{even}}$. But, $J_F/P_F J_F^{\text{even}}$ is isomorphic to the factor group $\mathfrak{C}_F/\mathfrak{C}_F^2$, where $\mathfrak{C}_F$ denotes the ideal class group of F. Since the ideal class number for F is 2, and for E is 1, one sees that $g^+(L) = 2$, and $g^+(\tilde{L}) = 1$ (since $P_{\tilde{D}} J_E^{\tilde{L}}$ also equals $P_E J_E^{\text{even}}$). Hence, Γ_L must collapse. Of course, this same example also serves to show Theorem 2.3 fails for even degree extensions.

Appendix B. *A note on the definition of spinor genus.* The purpose here is to make the observation that the concept of a spinor genus can be introduced without mentioning spinors at all, and instead, adhering only to the much more elementary concept of commutators (so that even a naive audience finds it accessible). The introduction of spinors usually necessitates resorting to more algebraically preparatory machinery; namely, Clifford algebra, opposite algebra, anti-isomorphism, etc., in order to justify that the spinor norm is indeed an invariant. See [7]. Even when Clifford algebra is not used, the procedure is still quite complicated (see [8]). We want to emphasize that our "new" definition has almost no value from a *practical* point of view, because spinorial rotations are quite accessible to computations, and indeed, only through its use can one really do something significant about this spinor genus. Our point, however, is a philosophical one! For some reason the notion of a spinor genus has acquired a certain "mystique" (probably because quite a few truly deep results were established only after its introduction). Thus, we aim to point out in this appendix that employing just the group commutators, the same creature is obtained, so that perhaps *commutator genus* is an equally apt name. Let us call the commutator genus $\operatorname{Cgn}(L)$ of a lattice L the set of all lattices K on $V = FL$ such that there is a $\sigma \in O(V)$ and at each spot $\mathfrak{p} \in \boldsymbol{S}$, there is a commutatorial rotation $\gamma_{\mathfrak{p}} \in \Omega(V_{\mathfrak{p}})$ satisfying $K_{\mathfrak{p}} = \sigma\gamma_{\mathfrak{p}}L_{\mathfrak{p}}$ for all $\mathfrak{p} \in \boldsymbol{S}$. Similarly, one introduces $\operatorname{Cgn}^+(L)$. Clearly, we have: $\operatorname{Cls}(L) \subseteq \operatorname{Cgn}(L) \subseteq \operatorname{Spn}(L) \subseteq \operatorname{Gen}(L)$, and $\operatorname{Cls}^+(L) \subseteq \operatorname{Cgn}^+(L) \subseteq \operatorname{Spn}^+(L) \subseteq \operatorname{Gen}^+(L) = \operatorname{Gen}(L)$. We claim the following equalities prevail:

$$\operatorname{Cgn}(L) = \operatorname{Spn}(L), \quad \operatorname{Cgn}^+(L) = \operatorname{Spn}^+(L).$$

To achieve these, we employ the usual adèlized language of split rotations. It is quite clear that it suffices to prove the following statement:

Given any $\Sigma \in J'_V$, there is a global rotation $\sigma \in O'(V)$ with spinor norm 1 such that $\sigma\Sigma \in J^c_V$.

Here

$$J^c_V = \{\Sigma \in J_V\colon \Sigma_\mathfrak{p} \in \Omega(V_\mathfrak{p}) \text{ for all (finite \& infinite) spots } \mathfrak{p}\}.$$

For any spot $\mathfrak{p}$, $O'(V_\mathfrak{p})$ always equals to $\Omega(V_\mathfrak{p})$ except when $\mathfrak{p}$ is discrete, non-dyadic and $V_\mathfrak{p}$ is the unique four dimensional anisotropic space. See 61C, 95 : 1, [7]. Thus, we may suppose $\dim(V) = 4$. Let $\boldsymbol{T} = \{\mathfrak{p}\colon \Sigma_\mathfrak{p} \notin \Omega(V_\mathfrak{p})\}$ and let $\boldsymbol{X}$ be the remaining real and non-dyadic spots at which V is anisotropic. We may suppose $\boldsymbol{T}$ is not empty. By scaling, we may further assume V represents 1. Fix a unit $\Delta_\mathfrak{p}$ of quadratic defect $4\mathfrak{o}_{F_\mathfrak{p}}$ and a prime element $\pi_\mathfrak{p}$ at each $\mathfrak{p} \in \boldsymbol{T}$. The Weak Approximation Theorem gives us two elements α, β in the global field F such that: α is close to $\Delta_\mathfrak{p}$ at $\mathfrak{p} \in \boldsymbol{T}$, and close to 1 at $\mathfrak{p} \in \boldsymbol{X}$; β is close to $\pi_\mathfrak{p}$ at $\mathfrak{p} \in \boldsymbol{T}$, and close to 1 at $\mathfrak{p} \in \boldsymbol{X}$. Then, $\alpha\beta$ is close to $\Delta_\mathfrak{p}\pi_\mathfrak{p}$ at $\mathfrak{p} \in \boldsymbol{T}$ and to 1 at $\mathfrak{p} \in \boldsymbol{X}$. Locally, $V_\mathfrak{p}$ (being 4-dimensional) is universal for every non-real spot $\mathfrak{p}$ and of course also at all the real spots with $V_\mathfrak{p}$ isotropic. On the other hand, $V_\mathfrak{p}$ is positive definite for the remaining real spots. Thus, the Local-Global Representation Theorem gives us vectors $u, v, w \in V$ such that $Q(u) = \alpha$, $Q(v) = \beta$, and $Q(w) = \alpha\beta$. Choose $x \in V$ with $Q(x) = 1$, and put $\sigma = S_x S_u S_v S_w$. One sees that σ lies in $O'(V)$ and $\sigma\Sigma_\mathfrak{p} \in \Omega(V_\mathfrak{p})$ at every $\mathfrak{p}$, see 95 : 1a, [7]. Hence, $\sigma\Sigma \in J^c_V$.

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Asymptotic expansions of finite theta series

by

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Dedicated to Th. Schneider on his 65th birthday

1. Introduction. We are interested in the approximate evaluation of sums like

$$S_N(x) = \sum_{n=1}^{N} e^{\pi i n^2 x} \quad (x \text{ real}).$$

For instance we shall prove (see Remark 2):

THEOREM 1.

$$(1)\qquad S_N(x) = a(p, q)\int_0^N e^{\pi i t^2\xi/q}\,dt + O(\sqrt{q})$$

for

$$(2)\qquad x = \frac{p}{q} + \frac{\xi}{q}, \quad |\xi| \leqslant \frac{1}{4N}, \quad 0 < q \leqslant 4N, \quad (p, q) = 1.$$

Here $a(p, q)$ is an arithmetical function of p and q, whose modulus is zero or $1/\sqrt{q}$ according as pq is odd or even. The exact order of magnitude of the integral in (1) is known (see (9)) to be

$$\asymp \frac{N\sqrt{q}}{\sqrt{q} + N\sqrt{|\xi|}}.$$

Hence the main term in (1) is zero for odd pq and dominates $\sqrt{q}$ for even pq in the permitted range. (The symbols $O(\ )$ and $\asymp$ are explained at the beginning of the next section.)

By Dirichlet's box principle one can find for any given pair x, N a triple p, q, ξ satisfying (2). Therefore (1) is applicable to every real x and evaluates $S_N(x)$ up to an error $O(\sqrt{N})$ at most. (Observe that $\sqrt{N}$ is the exact order of the L_2-norm of $S_N(x)$.)

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