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Polynomial identities which imply identities of Euler and Jacobi

by

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1. Introduction. We shall prove polynomial identities which imply the following identities, namely

$$\prod_{r=1}^{\infty}(1-x^r) = 1+\sum_{r=1}^{\infty}(-1)^r(x^{\frac{1}{2}(3r^2-r)}+x^{\frac{1}{2}(3r^2+r)}) \tag{1}$$

and

$$\prod_{r=1}^{\infty}(1-x^r)^3 = \sum_{r=0}^{\infty}(-1)^r(2r+1)x^{\frac{1}{2}(r^2+r)}. \tag{2}$$

The first of these is due to Euler [1] who found it while investigating $p(n)$, the number of unrestricted partitions of n; the second is due to C. G. J. Jacobi [3]. Both (1) and (2) can be derived from Jacobi's two-variable identity [2]

$$\prod_{r=1}^{\infty}\left(1+\frac{x^{2r-1}}{a}\right)(1+ax^{2r-1})(1-x^{2r}) = \sum_{r=-\infty}^{\infty}a^r x^{r^2}. \tag{3}$$

Both identities (1) and (2) gained great importance in the theory of partitions when S. Ramanujan [4] employed them to prove that

$$p(5n+4)\equiv 0 \bmod 5,$$
$$p(7n+6)\equiv 0 \bmod 7.$$

The object of this note is to prove (1) and (2) using functions of only one variable. Indeed, we prove the polynomial identities

$$\prod_{r=1}^{3n}(1-x^r) = (1-x^{3n+3})\dots(1-x^{6n})+ \\ +\sum_{r=1}^{n}(-1)^r(x^{\frac{1}{2}(3r^2-r)}+x^{\frac{1}{2}(3r^2+r)})(1-x^{3n-3r+3})\dots(1-x^{3n})\times \\ \times(1-x^{3n+3r+3})\dots(1-x^{6n}) \tag{4}$$

and

$$(5)\qquad \prod_{r=1}^{n}(1-x^r)^3 = \sum_{r=0}^{n}(-1)^r(2r+1)x^{\frac12(r^2+r)}(1-x^{n-r+1})\dots(1-x^n)\times$$
$$\times(1-x^{n+r+2})\dots(1-x^{2n+1}).$$

These identities, we will show, imply (1) and (2).

2. The polynomial identities. Before embarking on proofs of (4) and (5), we recall relevant properties of the generalized binomial coefficients, defined for $m \geqslant n \geqslant 0$ by

$$(6)\qquad \begin{bmatrix} m \\ n \end{bmatrix}_p = \prod_{r=1}^{m}(1-x^{pr}) \Big/ \prod_{r=1}^{n}(1-x^{pr})\prod_{r=1}^{m-n}(1-x^{pr}).$$

These generalized binomial coefficients satisfy

$$(7)\qquad \begin{bmatrix} m \\ n \end{bmatrix}_p = x^{pn}\begin{bmatrix} m-1 \\ n \end{bmatrix}_p + \begin{bmatrix} m-1 \\ n-1 \end{bmatrix}_p$$

and

$$(8)\qquad \begin{bmatrix} m \\ n \end{bmatrix}_p = \begin{bmatrix} m-1 \\ n \end{bmatrix}_p + x^{p(m-n)}\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}_p.$$

In order to prove (4), let

$$P_n = \begin{bmatrix} 2n \\ n \end{bmatrix}_3 + \sum_{r=1}^{n}(-1)^r\left(x^{\frac12(3r^2-r)}+x^{\frac12(3r^2+r)}\right)\begin{bmatrix} 2n \\ n-r \end{bmatrix}_3$$
$$= \sum_{r=-n}^{n}(-1)^r x^{\frac12(3r^2+r)}\begin{bmatrix} 2n \\ n+r \end{bmatrix}_3.$$

Then, by repeated application of (7) and (8), we have for $n > 0$,

$$P_n = \sum_{r=-n}^{n}(-1)^r x^{\frac12(3r^2+r)}\left\{x^{3n+3r}\begin{bmatrix} 2n-1 \\ n+r \end{bmatrix}_3 + \begin{bmatrix} 2n-1 \\ n+r-1 \end{bmatrix}_3\right\}$$
$$= x^{3n}\sum_{r=-n}^{n-1}(-1)^r x^{\frac12(3r^2+7r)}\begin{bmatrix} 2n-1 \\ n+r \end{bmatrix}_3 + \sum_{r=-n+1}^{n}(-1)^r x^{\frac12(3r^2+r)}\begin{bmatrix} 2n-1 \\ n+r-1 \end{bmatrix}_3$$
$$= x^{3n}\sum_{r=-n}^{n-1}(-1)^r x^{\frac12(3r^2+7r)}\left\{\begin{bmatrix} 2n-2 \\ n+r \end{bmatrix}_3 + x^{3n-3r-3}\begin{bmatrix} 2n-2 \\ n+r-1 \end{bmatrix}_3\right\}+$$
$$+\sum_{r=-n+1}^{n}(-1)^r x^{\frac12(3r^2+r)}\left\{\begin{bmatrix} 2n-2 \\ n+r-1 \end{bmatrix}_3 + x^{3n-3r}\begin{bmatrix} 2n-2 \\ n+r-2 \end{bmatrix}_3\right\}$$

$$= x^{3n}\sum_{r=-n}^{n-2}(-1)^r x^{\frac12(3r^2+7r)}\begin{bmatrix} 2n-2 \\ n+r \end{bmatrix}_3 +$$
$$+ x^{6n-3}\sum_{r=-n+1}^{n-1}(-1)^r x^{\frac12(3r^2+r)}\begin{bmatrix} 2n-2 \\ n+r-1 \end{bmatrix}_3 +$$
$$+ \sum_{r=-n+1}^{n-1}(-1)^r x^{\frac12(3r^2+r)}\begin{bmatrix} 2n-2 \\ n+r-1 \end{bmatrix}_3 +$$
$$+ x^{3n}\sum_{r=-n+2}^{n}(-1)^r x^{\frac12(3r^2-5r)}\begin{bmatrix} 2n-2 \\ n+r-2 \end{bmatrix}_3$$
$$= (1+x^{6n-3})P_{n-1} + x^{3n}\sum_{r=-n+1}^{n-1}(-1)^{r-1}x^{\frac12(3r^2+r-4)}\begin{bmatrix} 2n-2 \\ n+r-1 \end{bmatrix}_3 +$$
$$+ x^{3n}\sum_{r=-n+1}^{n-1}(-1)^{r+1}x^{\frac12(3r^2+r-2)}\begin{bmatrix} 2n-2 \\ n+r-1 \end{bmatrix}_3$$
$$= (1-x^{3n-2}-x^{3n-1}+x^{6n-3})P_{n-1}$$
$$= (1-x^{3n-2})(1-x^{3n-1})P_{n-1}.$$

Since $P_0 = 1$, it follows by induction that for $n \geqslant 0$,

$$P_n = \prod_{r=1}^{n}(1-x^{3r-2})(1-x^{3r-1}).$$

That is,

$$\prod_{r=1}^{n}(1-x^{3r-2})(1-x^{3r-1}) = \begin{bmatrix} 2n \\ n \end{bmatrix}_3 + \sum_{r=1}^{n}(-1)^r\left(x^{\frac12(3r^2-r)}+x^{\frac12(3r^2+r)}\right)\begin{bmatrix} 2n \\ n-r \end{bmatrix}_3.$$

If we now multiply by $\prod_{r=1}^{n}(1-x^{3r})$, we obtain (4).

To prove (5), let

$$Q_n = \sum_{r=0}^{n}(-1)^r(2r+1)x^{\frac12(r^2+r)}\begin{bmatrix} 2n+1 \\ n-r \end{bmatrix}_1.$$

Then for $n > 0$,

$$Q_n = \sum_{r=0}^{n}(-1)^r(2r+1)x^{\frac12(r^2+r)}\left\{x^{n-r}\begin{bmatrix} 2n \\ n-r \end{bmatrix}_1 + \begin{bmatrix} 2n \\ n-r-1 \end{bmatrix}_1\right\}$$
$$= x^{n}\sum_{r=0}^{n}(-1)^r(2r+1)x^{\frac12(r^2-r)}\begin{bmatrix} 2n \\ n-r \end{bmatrix}_1 +$$

$$+\sum_{r=0}^{n-1}(-1)^r(2r+1)x^{\frac12(r^2+r)}\begin{bmatrix}2n\\ n-r-1\end{bmatrix}_1$$

$$=x^n\sum_{r=0}^{n}(-1)^r(2r+1)x^{\frac12(r^2-r)}\left\{\begin{bmatrix}2n-1\\ n-r\end{bmatrix}_1+x^{n+r}\begin{bmatrix}2n-1\\ n-r-1\end{bmatrix}_1\right\}+$$

$$+\sum_{r=0}^{n-1}(-1)^r(2r+1)x^{\frac12(r^2+r)}\left\{\begin{bmatrix}2n-1\\ n-r-1\end{bmatrix}_1+x^{n+r+1}\begin{bmatrix}2n-1\\ n-r-2\end{bmatrix}_1\right\}$$

$$=x^n\sum_{r=0}^{n}(-1)^r(2r+1)x^{\frac12(r^2-r)}\begin{bmatrix}2n-1\\ n-r\end{bmatrix}_1+$$

$$+x^{2n}\sum_{r=0}^{n-1}(-1)^r(2r+1)x^{\frac12(r^2+r)}\begin{bmatrix}2n-1\\ n-r-1\end{bmatrix}_1+$$

$$+\sum_{r=0}^{n-1}(-1)^r(2r+1)x^{\frac12(r^2+r)}\begin{bmatrix}2n-1\\ n-r-1\end{bmatrix}_1+$$

$$+x^n\sum_{r=0}^{n-2}(-1)^r(2r+1)x^{\frac12(r^2+3r+2)}\begin{bmatrix}2n-1\\ n-r-2\end{bmatrix}_1$$

$$=(1+x^{2n})Q_{n-1}+$$

$$+x^n\left\{\begin{bmatrix}2n-1\\ n\end{bmatrix}_1-3\begin{bmatrix}2n-1\\ n-1\end{bmatrix}_1+\sum_{r=2}^{n}(-1)^r(2r+1)x^{\frac12(r^2-r)}\begin{bmatrix}2n-1\\ n-r\end{bmatrix}_1+\right.$$

$$\left.+\sum_{r=0}^{n-2}(-1)^r(2r+1)x^{\frac12(r^2+3r+2)}\begin{bmatrix}2n-1\\ n-r-2\end{bmatrix}_1\right\}$$

$$=(1+x^{2n})Q_{n-1}+$$

$$+x^n\left\{-2\begin{bmatrix}2n-1\\ n-1\end{bmatrix}_1+\sum_{r=1}^{n-1}(-1)^{r+1}(2r+3)x^{\frac12(r^2+r)}\begin{bmatrix}2n-1\\ n-r-1\end{bmatrix}_1+\right.$$

$$\left.+\sum_{r=1}^{n-1}(-1)^{r-1}(2r-1)x^{\frac12(r^2+r)}\begin{bmatrix}2n-1\\ n-r-1\end{bmatrix}_1\right\}$$

$$=(1+x^{2n})Q_{n-1}-2x^n\left\{\begin{bmatrix}2n-1\\ n-1\end{bmatrix}_1+\sum_{r=1}^{n-1}(-1)^r(2r+1)x^{\frac12(r^2+r)}\begin{bmatrix}2n-1\\ n-r-1\end{bmatrix}_1\right\}$$

$$=(1-2x^n+x^{2n})Q_{n-1}=(1-x^n)^2Q_{n-1}.$$

Since $Q_0=1$, it follows by induction that

$$Q_n=\prod_{r=1}^{n}(1-x^r)^2,$$

or,

$$\prod_{r=1}^{n}(1-x^r)^2=\sum_{r=0}^{n}(-1)^r(2r+1)x^{\frac12(r^2+r)}\begin{bmatrix}2n+1\\ n-r\end{bmatrix}_1.$$

(5) follows on multiplication by $\prod_{r=1}^{n}(1-x^r)$.

3. Identities of Euler and Jacobi. We now show that (4) implies (1), (5) implies (2). We have

$$\prod_{r=1}^{3n}(1-x^r)=(1-x^{3n+3})\dots(1-x^{6n})+$$

$$+\sum_{r=1}^{n}(-1)^r(x^{\frac12(3r^2-r)}+x^{\frac12(3r^2+r)})(1-x^{3n-3r+3})\dots(1-x^{3n})\times$$

$$\times(1-x^{3n+3r+3})\dots(1-x^{6n})$$

$$=1+\sum_{r=1}^{n}(-1)^r(x^{\frac12(3r^2-r)}+x^{\frac12(3r^2+r)})+\text{terms of degree}>3n,$$

since $\frac12(3r^2-r)+(3n-3r+3)=3n+\frac12(3r^2-7r+6)\geqslant 3n+1>3n$

$$=1+\sum_{\substack{r\geqslant1\\ \frac12(3r^2-r)\leqslant3n}}(-1)^rx^{\frac12(3r^2-r)}+\sum_{\substack{r\geqslant1\\ \frac12(3r^2+r)\leqslant3n}}(-1)^rx^{\frac12(3r^2+r)}+$$

$$+\text{terms of degree}>3n$$

since if $\frac12(3r^2-r)\leqslant 3n$, $r\leqslant n$, and if $\frac12(3r^2+r)\leqslant 3n$, $r\leqslant n$.

(1) follows on letting $n\to\infty$.

We also have

$$\prod_{r=1}^{n}(1-x^r)^3$$

$$=\sum_{r=0}^{n}(-1)^r(2r+1)x^{\frac12(r^2+r)}(1-x^{n-r+1})\dots(1-x^n)\times$$

$$\times(1-x^{n+r+2})\dots(1-x^{2n+1})$$

$$=\sum_{r=0}^{n}(-1)^r(2r+1)x^{\frac12(r^2+r)}+\text{terms of degree}>n,$$

since $\frac12(r^2+r)+(n-r+1)=n+\frac12(r^2-r+2)\geqslant n+1>n$,

$$=\sum_{\substack{r\geqslant0\\ \frac12(r^2+r)\leqslant n}}(-1)^r(2r+1)x^{\frac12(r^2+r)}+\text{terms of degree}>n,$$

since if $\frac12(r^2+r)\leqslant n$, $r\leqslant n$.

(2) follows on letting $n\to\infty$.

References

[1] L. Euler, *Opera omnia*, Series Prima, Vol. VIII, p. 334.
[2] C. G. J. Jacobi, *Gesammelte Werke*, Vol. 1, pp. 232–234.
[3] — ibid., pp. 236–237.
[4] S. Ramanujan, *Collected Papers*, pp. 210–212.

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Remarques sur les nombres de Pisot-Vijayaraghavan

par

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1. Introduction. Dans [3], Mendès-France donne la caractérisation suivante des nombres de Pisot–Vijayaraghavan:

THÉORÈME. *Soit θ un réel >1. Soit*

$$n = \sum_{r=0}^{+\infty} a_r(n) q^r$$

le développement de n en base q. Posons

$$f_{(\theta)}(n) = \sum_{r=0}^{+\infty} a_r(n)\theta^r.$$

Une condition nécessaire et suffisante pour que θ soit un nombre de Pisot est que la suite $\big(f_{(\theta)}(n)\big)_{n\in N}$ ne soit pas équirépartie modulo 1.

C'est ce résultat que nous généralisons. Nous avons besoin de quelques définitions et notations:

(1) $\|x\|$ désigne la distance du réel x à l'entier qui lui est le plus proche.

(2) $\varphi = (q_i)_{i\in N^*}$ est une suite d'entiers $\geqslant 2$. Posons $p_0 = 1$, et pour tout $i \in N^*$, $p_i = \prod_{j=1}^{i} q_j$.

Tout entier naturel n se développe de manière unique sous la forme:

$$n = \sum_{r=0}^{+\infty} a_r(n) p_r \quad \text{où} \quad \forall r\in N, a_r(n)\in\{0, \ldots, q_{r+1}-1\}.$$

Ce développement est appelé développement de n en base φ.

(3) Une application f de N dans C est dite φ-additive si, pour tout $i\in N^*$, on a:

$$f(ap_i+b) = f(ap_i)+f(b)$$

quel que soit le couple d'entiers (a, b) satisfaisant à:

$$a\in\{1, \ldots, q_{i+1}-1\} \quad \text{et} \quad b\in\{0, \ldots, p_i-1\}.$$