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Zero-density estimates for L-functions

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1. Introduction and statement of results. The classical zero-density estimates of Ingham for the zeta-function, as well as their more recent analogues for L-functions, can be significantly improved in the region $\sigma > 3/4$ by using the Halász–Montgomery method ([4], [12], [13]), which yields sharp estimates, better than what one can deduce from known mean-value results, for the frequency of large values of Dirichlet polynomials

$$f(s, \chi) = \sum_{N+1}^{2N} a_n \chi(n) n^{-s}. \tag{1.1}$$

Recent developments of this method are based on the "reflection argument" of Huxley [6]–[8] which leads to the problem how often two different Dirichlet polynomials can be simultaneously large. In [11] we gave a simple variant of Huxley's method with an application to the zeta-zeros. In this paper we consider another variant which makes use of the mean fourth power estimates for L-functions on the critical line.

Let a Dirichlet polynomial of the type (1.1) with variable Dirichlet character χ and variable complex number s be given, and write

$$G = \sum_{N+1}^{2N} |a_n|^2.$$

Suppose we are given a set of pairs (s_r, χ_r), $r = 1, \ldots, R$, where the points $s_r = \sigma_r + it_r$ satisfy $\sigma_r \geqslant 0$, $|t_r - t_s| \leqslant T$, and for $r \neq s$ either $\chi_r \neq \chi_s$ or $|t_r - t_s| \geqslant 1$. Further, suppose that $|f(s_r, \chi_r)| \geqslant V > 0$ for all r. We shall estimate the number R in three cases: (i) all χ_r belong to the same modulus q; (ii) all χ_r are primitive characters of conductor at most Q; (iii) χ_r is the principal character $\chi_0 (\mathrm{mod}\ 1)$ for all r. Let R_1, R_2 and R_3 stand for the respective R-numbers. In the following, the constants implied by the symbols $\ll$ and $\gg$ will be absolute unless otherwise indicated by notation (e.g. $\ll_\varepsilon$). Our main result is the following

THEOREM. *Let ε be any fixed positive number and k any fixed positive integer. Then in the notation above (with $T \geqslant 2$), we have*

$$R_1 \ll_{\varepsilon,k} \big(GNV^{-2}+qT(G^4N^2V^{-8})^k+(qTG^2V^{-4})^k\big)(qT)^{\varepsilon}, \tag{1.2}$$

$$R_2 \ll_{\varepsilon,k} \big(GNV^{-2}+Q^2T(G^4N^2V^{-8})^k+(Q^2TG^2V^{-4})^k\big)(QT)^{\varepsilon}, \tag{1.3}$$

$$R_3 \ll_{\varepsilon,k} \big(GNV^{-2}+TG^{3-1/k}N^{1-1/k}V^{-6+2/k}+T(G^4N^2V^{-8})^k\big)T^{\varepsilon}. \tag{1.4}$$

This implies new zero-density estimates which are formulated in the corollary below. Let $N(\alpha, T, \chi)$ stand for the number of zeros of $L(s, \chi)$ in the rectangle $\alpha \leqslant \sigma \leqslant 1$, $|t| \leqslant T$, and write $N(\alpha, T) = N(\alpha, T, \chi_0)$. Let $\sum^*$ denote a sum over primitive characters.

COROLLARY. *Let ε and k be as in the theorem. Then for $3/4 \leqslant \sigma \leqslant 1$, $T \geqslant 2$, we have*

$$\sum_{\chi \bmod q} N(\alpha, T, \chi) \ll_{\varepsilon,k} (qT)^{A_1(\alpha)(1-\alpha)+\varepsilon}, \tag{1.5}$$

$$\sum_{q \leqslant Q} \sum_{\chi \bmod q}{}^{*} N(\alpha, T, \chi) \ll_{\varepsilon,k} (Q^2T)^{A_1(\alpha)(1-\alpha)+\varepsilon}, \tag{1.6}$$

$$\sum_{q \leqslant Q} \sum_{\chi \bmod q}{}^{*} N(\alpha, T, \chi) \ll_{\varepsilon,k} (Q^2T^2)^{A_2(\alpha)(1-\alpha)+\varepsilon}, \tag{1.7}$$

$$N(\alpha, T) \ll_{\varepsilon,k} T^{A_3(\alpha)(1-\alpha)+\varepsilon}, \tag{1.8}$$

where

$$A_1(\alpha) = \max\left(2, \frac{3}{(8k-3)\alpha+3-6k}, \frac{3k}{(4k-3)\alpha+3-2k}\right),$$

$$A_2(\alpha) = \max\left(2, \frac{5}{(16k-5)\alpha+5-12k}, \frac{5k}{(8k-5)\alpha+5-4k}\right),$$

$$A_3(\alpha) = \max\left(2, \frac{3}{(8k-3)\alpha+3-6k}, \frac{3k}{(3k-2)\alpha+2-k}\right).$$

In particular, choosing $k = 2$ in (1.5) and (1.6), $k = 4$ in (1.7) and $k = 3$ in (1.8), we have $A_1(\alpha) = 2$ for $\alpha \geqslant 21/26 = 0.8076\ldots$, $A_2(\alpha) = 2$ for $\alpha \geqslant 7/9$, $A_3(\alpha) = 2$ for $\alpha \geqslant 11/14 = 0.7857\ldots$

Remarks. 1) In the following discussion we omit factors like $(qT)^{\varepsilon}$, $(QT)^{\varepsilon}$, T^{ε}. If in (1.2) or (1.3) the number V exceeds respectively $G^{1/2}(qT)^{1/4}$ or $G^{1/2}(Q^2T)^{1/4}$, and $N \leqslant qT$ or $\leqslant Q^2T$, then the limiting case $k \to \infty$ gives the estimate GNV^{-2}, obtained by Montgomery in [12]. Recently Huxley [8] proved the estimate

$$R_1 \ll GNV^{-2}+qTG^3NV^{-6} \tag{1.9}$$

if $V \geqslant G^{1/2}N^{1/4}$ (and an analogous result for R_2). Neither of the estimates (1.2) and (1.9) contains the other; roughly speaking, (1.2) is sharper than

(1.9) if V is large. Accordingly (1.2) leads to new density estimates near $4/5$, while (1.9) is better near $3/4$.

2) As $k \to \infty$ in (1.4) and $V > G^{1/2}N^{1/4}$, we get the case $q = 1$ of Huxley's estimate (1.9) (this special case was treated already in [5]). A method of Ramachandra [15] gives

$$R_3 \ll GNV^{-2}+TG^{3-1/k}N^{1-1/k}V^{-6+2/k}+T(G^2NV^{-4})^k; \tag{1.10}$$

in [11] we proved the estimate

$$R_3 \ll GNV^{-2}+TG^{3-1/k}N^{1-1/k}V^{-6+2/k}+TG^{7-3/k}N^{3-1/k}V^{-14+6/k}. \tag{1.11}$$

Both (1.10) and (1.11) are contained in (1.4) if $V \geqslant G^{1/2}N^{1/4}$.

3) The results listed in the end of the corollary are new. The density hypothesis asserts that $A_i(\alpha) = 2$ for $1/2 \leqslant \alpha \leqslant 1$. Let α_0 be such that the density hypothesis for $\zeta(s)$ holds for $\alpha \geqslant \alpha_0$. The following results have been obtained: $\alpha_0 \leqslant 9/10$ (Montgomery [13]), $\alpha_0 \leqslant 5/6$ (Huxley [5]), $\alpha_0 \leqslant 21/26$ (Ramachandra [15]), $\alpha_0 \leqslant 0.8059\ldots$ (Forti and Viola [3]), $\alpha_0 \leqslant 4/5$ (Huxley [8]; weaker results in [6], [7]), $\alpha_0 \leqslant 43/54$ (Jutila [11]). In the cases (1.5)–(1.6) the corresponding density hypotheses were known for $\alpha \geqslant 5/6$ (Balasubramanian and Ramachandra [1], Huxley [6], Jutila [10]), and in the case (1.7) for $\alpha \geqslant 11/14$ (Huxley [9]).

In conclusion, I wish to express my gratitude to Prof. M. Huxley and K. Ramachandra for kindly informing me on their unpublished results.

2. Lemmas.

The proof of the theorem will be based, beside the Halász–Montgomery inequality (3.1), on three auxiliary lemmas. Lemma 1 is a straightforward generalization of the basic lemma of [11]; Lemma 2 is an elementary inequality which plays an important role in our argument since it enables us to make use of the mean value estimates for partial sums of L-series, stated in Lemma 3.

To formulate our first lemma, let h be a positive number to be specified below, and define

$$b(n) = e^{-(n/2N)^h} - e^{-(n/N)^h},$$

$$H(s, \chi) = \sum_{1}^{\infty} b(n)\chi(n)n^{-s}.$$

LEMMA 1. *Let χ be a character $(\bmod\, q)$, $\varepsilon > 0$, $T \geqslant 2$, $N \leqslant qT$, $h = \log^2 qT$, $s = \sigma+it$, $0 \leqslant \sigma \leqslant 1$, $|t| \leqslant T$; also suppose that $|t| \geqslant h^2$ if $\chi = \chi_0$. Let $q(|t|+h^3)(\pi N)^{-1} \leqslant M \leqslant (qT)^2$. Then we have*

$$H(s, \chi) \ll_{\varepsilon} N^{1/2}q^{\varepsilon} \int_{-h^2}^{h^2} \Big|\sum_{1}^{M} \bar{\chi}(n) n^{-1/2+i(t+\tau)}\Big| d\tau + 1; \tag{2.1}$$

the factor q^{ε} can be omitted if χ is primitive. (If $M < 1$, then the sum means zero.)

Proof. The proof is essentially the same as in [11] (in the case $\chi = \chi_0$). We sketch it for the sake of completeness.

We may suppose that qT is sufficiently large. Suppose first that χ is primitive. Starting from the identity [12]

$$H(s, \chi) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, w=2} L(s+w, \chi)\Gamma(1+wh^{-1})\left(\frac{(2N)^w - N^w}{w}\right) dw,$$

we move the integration to the line $\mathrm{Re}(s+w) = -1/2$ (at $w = 1-s$ we get a residue $\ll 1$ if $\chi = \chi_0$), use the functional equation for $L(s, \chi)$, split up the series of $L(1-s-w, \bar{\chi})$ into two parts: the partial sum of length M and the remainder, correspondingly getting two integrals I_1 and I_2. In I_1 and I_2 we move the integration respectively to the lines $\mathrm{Re}(s+w) = 1/2$ and $\mathrm{Re}\, w = -h/2$. Then I_1 gives the main term in (2.1) (without the factor q^ε, obviously the integral can be cut at $\mathrm{Im}\, w = \pm h^2$ with small error) and I_2 becomes an error term, in view of our choices for h and M.

If χ is imprimitive, then we first express $L(s+w, \chi)$ by $L(s+w, \chi^*)$ with χ^* primitive, and again $L(1-s-w, \bar{\chi}^*)$ by $L(1-s-w, \bar{\chi})$. This means inserting certain factors to the integrand, and this may give the factor q^ε in (2.1).

Lemma 2. *Let a_n, $n = 1, \ldots, N$, be complex numbers of absolute value at most A, let t_r be a real number and χ_r a Dirichlet character, for $r = 1, \ldots, R$. Then we have*

$$\sum_{r,s=1}^{R} \Big|\sum_{n=1}^{N} a_n \bar{\chi}_r(n)\chi_s(n) n^{-1/2+i(t_r-t_s)}\Big|^2 \leqslant A^2 \sum_{r,s=1}^{R} \Big|\sum_{n=1}^{N} \bar{\chi}_r(n)\chi_s(n) n^{-1/2+i(t_r-t_s)}\Big|^2.$$

Proof. In fact the assertion is the case $z_{rn} = \bar{\chi}_r(n) n^{-1/4+it_r}$ of the general inequality

$$\sum_{r,s=1}^{R} \Big|\sum_{n=1}^{N} a_n z_{rn}\bar{z}_{sn}\Big|^2 \leqslant A^2 \sum_{r,s=1}^{R} \Big|\sum_{n=1}^{N} z_{rn}\bar{z}_{sn}\Big|^2.$$

For the proof of this note that the left-hand side is

$$\sum_{m,n=1}^{N} a_m\bar{a}_n \Big|\sum_{r=1}^{R} z_{rm}\bar{z}_{rn}\Big|^2 \leqslant A^2 \sum_{m,n=1}^{N} \Big|\sum_{r}\Big|^2 = A^2 \sum_{r,s=1}^{R} \Big|\sum_{n=1}^{N} z_{rn}\bar{z}_{sn}\Big|^2.$$

Lemma 3. *Let for each $r = 1, \ldots, R$ a real number t_r with $|t_r| \leqslant T$ $(T \geqslant 2)$ and a character χ_r be given. Suppose that for $r \neq s$ either $\chi_r \neq \chi_s$ or $|t_r - t_s| \geqslant 1$. Let N be any positive integer.*

(i) *If all the χ_r belong to the same modulus q, then*

$$\sum_{r=1}^{R} \Big|\sum_{n=1}^{N} \chi_r(n) n^{-1/2+it_r}\Big|^2 \ll \big(N + (RTq)^{1/2}\big)\log^B(qT). \tag{2.2}$$

(ii) *If all the χ_r are primitive characters of conductor at most Q and χ is any character of modulus at most Q, then*

$$\sum_{r=1}^{R} \Big|\sum_{n=1}^{N} \chi_r(n)\chi(n) n^{-1/2+it_r}\Big|^2 \ll_\varepsilon \big(N + (RT)^{1/2}Q\big)(QT)^\varepsilon. \tag{2.3}$$

Proof. If χ is a character (mod q) and $|t| \leqslant T$, then for $N \geqslant qT^2$

$$\sum_{1}^{N} \chi(n) n^{-1/2+it} = \varepsilon(\chi)\frac{N^{1/2+it}}{1/2+it} + L(1/2-it, \chi) + O\big(\log(2q)\big),$$

where $\varepsilon(\chi) = \varphi(q)/q$ or 0 according to whether $\chi = \chi_0$ or $\chi \neq \chi_0$. Hence in this case the estimate (2.2) is an easy consequence of the discrete analogue of the estimate

$$\sum_{\chi \bmod q} \int_{-T}^{T} |L(1/2+it, \chi)|^4 dt \ll qT\log^B(qT) \tag{2.4}$$

([14], Th. 10.1 and 10.3; in these theorems the sum is actually over primitive characters only, but this restriction is easy to remove). In the case $N < qT^2$ we may apply (2.4) or its discrete analogue in the integral representation

$$\sum_{1}^{N} \chi(n) n^{-1/2+it}$$
$$= \varepsilon(\chi)\frac{N^{1/2+it}}{1/2+it} + \frac{1}{2\pi i}\int_{a-iqT}^{a+iqT} L(1/2-it+w, \chi) N^w w^{-1} dw + O(\log qT),$$

where $a = 1/\log(qT)$.

Similarly, for the proof of (2.3) we use the estimate

$$\sum_{q \leqslant Q} \sideset{}{^*}\sum_{\chi' \bmod q} \int_{-T}^{T} |L(1/2+it, \chi\chi')|^4 dt \ll_\varepsilon (Q^2T)^{1+\varepsilon} \tag{2.5}$$

or its discrete analogue. This can also be proved in the way explained in [14]. Alternatively, a nice and simple way for proving (2.4), (2.5) and similar estimates has recently been deviced by Ramachandra [16]; his method avoids the difficulties connected with the approximate functional equation for $L(s, \chi)$.

3. Proof of the theorem. Let us first consider the proof of the estimate (1.2). It suffices to prove it for systems satisfying

$$|t_r - t_s| \geqslant h^2 = \log^4(qT) \quad \text{for} \quad r \neq s,\ \chi_r = \chi_s.$$

Also we may suppose that

$$0 \leqslant \sigma_r \leqslant 1/2 \quad \text{and} \quad N \leqslant qT.$$

The Halász–Montgomery inequality [12] implies that

$$R^2V^2 \ll GRN + G\sum_{r\neq s} |H(s_r+\bar{s}_s,\, \chi_r\bar{\chi}_s)|. \tag{3.1}$$

By Lemma 1, we have

$$|H(s_r+\bar{s}_s,\, \chi_r\bar{\chi}_s)| \ll_\varepsilon N^{1/2}q^\varepsilon \int_{-h^2}^{h^2} \Big|\sum_1^M \bar{\chi}_r(n)\chi_s(n)n^{-1/2+i(t_r-t_s+\tau)}\Big|\,d\tau + 1,$$

where $q(T+h^3)/N \ll M \ll q(T+h^3)/N$. Substituting this into (3.1) and using Hölder's inequality, we get

$$R^2V^2 \ll_{\varepsilon,k} GRN + GN^{1/2}q^\varepsilon R^{2-1/k} \int_{-h^2}^{h^2} \Big\{\sum_{r\neq s}\Big|\Big(\sum_1^M\Big)^k\Big|^2\Big\}^{1/2k} d\tau + GR^2; \tag{3.2}$$

further, by Lemmas 2 and 3, this implies

$$R^2V^2 \ll_{\varepsilon,k} GRN + GN^{1/2}(qT)^\varepsilon R^{2-1/k}\big\{R\big((qT/N)^k + (RTq)^{1/2}\big)\big\}^{1/2k} + GR^2$$

(note that $\tau_k(n) \ll_{\varepsilon,k} n^\varepsilon$). The term GR^2 can be omitted here, so we get an inequality which implies (1.2). The proof of (1.3) is analogous.

Finally, for the proof of (1.4), we use (1.2) and the Huxley [5] subdivision of the points s_r. It means we apply (1.2) with $q = 1$, $T = T_0$ and multiply the resulting estimate by $1+T_0^{-1}T$ to get an estimate for R; an optimal choice of T_0 completes the proof.

4. Proof of the corollary. The deduction of a density estimate from a "large moduli" theorem for Dirichlet polynomials is well known, so we just sketch the proof of the corollary. Let us consider the proof of (1.5). We may restrict ourselves to zeros ϱ satisfying for certain a_n with $|a_n| \leqslant \tau(n)$ and for certain $U \in [(qT)^\varepsilon, (qT)^{1/2+2\varepsilon}]$

$$\Big|\sum_{U<n\leqslant 2U} a_n\chi(n)n^{-\varrho}\Big| \geqslant (\log qT)^{-1} \tag{4.1}$$

(see e.g. [3], Lemma 1.1). Raising this inequality to a suitable integral power, we get a "zero-detecting" polynomial with length between Z and $Z^{3/2}$, where Z is an arbitrarily chosen number $\geqslant (qT)^{(2+8\varepsilon)/3}$. Applying (1.2) and choosing Z optimally to minimize the estimate, we obtain (1.5). The estimates (1.6) and (1.8) are proved likewise. For the proof of (1.7), we raise (4.1) to such a power that the length of the resulting polynomial is between Z and $Z^{5/4}$ with $Z \geqslant (QT)^{(8+32\varepsilon)/5}$, and use (1.3).

5. Concluding remarks. The "long" partial sums of $L(1/2+it, \chi)$, occurring in the proof of the theorem, can be estimated directly by using known estimates for $|L(1/2+it, \chi)|$. For example, if

$$\zeta(1/2+it) \ll_\varepsilon (|t|+1)^{c+\varepsilon},$$

then this argument gives

$$R_3 \ll_{\varepsilon,k} (GNV^{-2} + TG^{3-1/k}N^{1-1/k}V^{-6+2/k} + TG^{1+k/c}N^{1+k/2c}V^{-2-2k/c})T^\varepsilon;$$

this is sharper than (1.4) for large V, and implies the truth of the density hypothesis for $\zeta(s)$ for $\sigma \geqslant \max\left(\frac{3k+c}{4k-2c}, \frac{5k-4}{6k-4}\right)$. With $k = 3$, $c = 1/6$, this gives a new proof for the bound 11/14.

It is also possible to estimate $L(1/2+it, \chi)$ by using zero-free regions, the functional equation and convexity as indicated by Bombieri in [2]; this argument leads to a new proof of the bound 21/26 in the corollary.

Appendix

In [11] we noted that Huxley's estimate (1.9) (and its Q^2T-analogue) can be proved in a simple way by using the present method. The proof can be sketched as follows.

Suppose we have the situation explained in the introduction and want to estimate R_1. In (3.1) there exists an index r_0 such that

$$R^2V^2 \ll GRN + GR\sum_{r\neq r_0} |H(\bar{s}_r + s_{r_0},\, \bar{\chi}_r\chi_{r_0})|.$$

Estimate here $H(\ldots)$ by Lemma 1, and multiply the term with index r by $V^{-1}|f(s_r, \chi_r)|$. As a result we get a sum of integrals having a Dirichlet polynomial (with variable coefficients) of length $\ll q(T+h^3)$ in the integrand. Carrying out the summation under the integral sign by appealing to a well-known mean-value estimate for Dirichlet polynomials, we get

$$R^2V^2 \ll_\varepsilon GRN + G^{3/2}R^{3/2}V^{-1}N^{1/2}(qT)^{1/2+\varepsilon}.$$

This implies (1.9). The proof of the Q^2T-analogue is similar.

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The algebraic independence of certain numbers to algebraic powers

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Dedicated to Professor Th. Schneider on the occasion of his 65th birthday

In 1949, A. O. Gelfond proved ([4], Theorem 1, pp. 132–133) that if a is an algebraic number ($a \neq 0$, $\log a \neq 0$) and β is a cubic irrational number, then the two numbers a^{β} and a^{β^2} are algebraically independent (over $\boldsymbol{Q}$). Shortly thereafter Gelfond and N. I. Feldman [5] gave a measure of algebraic independence of these two numbers. R. Wallisser has conjectured that, for β a cubic irrational, a^{β} and a^{β^2} are algebraically independent even when a is only well-approximated by algebraic numbers. In this paper, we establish Wallisser's conjecture when a is closely approximated by algebraic numbers of bounded degree. We wish to thank M. Mignotte for his helpful comments on an earlier draft of this paper.

THEOREM. *Let a be a complex number, $a \neq 0$, $\log a \neq 0$, and β a cubic irrational number. Let $f\colon \boldsymbol{N}\to\boldsymbol{R}$ with $f \nearrow \infty$ and let $d_0 \in \boldsymbol{N}$. Assume that for infinitely many $T \in \boldsymbol{N}$, there exist algebraic numbers a_T of degree $\leqslant d_0$ satisfying*

$$\operatorname{log\,height} a_T \leqslant T,$$
$$\log|a - a_T| < -e^{Tf(T)}.$$

Then the two numbers a^{β} and a^{β^2} are algebraically independent.

Remark 1. If a itself is algebraic, we let $a = a_T$ for $T \geqslant \operatorname{log\,height} a$.

Remark 2. If a is a complex number ($a \neq 0$, $\log a \neq 0$) and β a cubic irrational number, with a^{β}, a^{β^2} algebraically dependent, then for all $d_0 \in \boldsymbol{N}$ there exist two positive constants

$$C = C(a, \beta, d_0), \quad H = H(a, \beta, d_0)$$

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