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The Fourier expansion of Epstein's zeta function over an algebraic number field and its consequences for algebraic number theory

by

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0. Introduction. In [22] we obtained a formula relating the Dedekind zeta function of an algebraic number field K, $\zeta_K(s)$, and $\zeta_K(s-1)$ via a series of modified Bessel functions of the second kind, provided that the number field K was:

1. totally real,
2. of class number one.

Here we eliminate both of these restrictive hypotheses. And as a consequence we derive a formula for the product of the class number and the regulator of K in terms of $\zeta_K(2)$ and a series of modified Bessel functions of the second kind (Theorem 3). The result is stated separately for quadratic fields in formulas (2.2) and (2.3).

Let us now give an outline of the results with some relevant history. In Section 1 we consider the Epstein zeta function associated with an ideal of the ring of integers of K. These are Eisenstein series for the general linear group over K. We develop the theory at the infinite primes of K rather than adelically. The subject of Eisenstein series for Lie groups over number fields has a long history. For $K = \boldsymbol{Q}$, such functions are studied in Maass [10], and Terras [18]–[21], the Lie groups involved being the general linear and the symplectic groups. Here we consider the Fourier expansions for Eisenstein series. The simplest of these is the Selberg–Chowla formula [13], which is the Fourier expansion of the Eisenstein series for GL(2) over $\boldsymbol{Q}$ (the usual Epstein zeta function). Hecke obtains Fourier expansions for complex-analytic Eisenstein series for Sp(1) = SL(2) over real quadratic fields ([5], pp. 345 and 385). Siegel ([15], pp. 291 ff.) is a good reference for complex analytic Eisenstein series for Sp(1) = SL(2) over totally real fields (i.e., the Hilbert modular group). Kubota ([6], [7]) obtains Fourier expansions of "non-analytic"

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(meaning non-complex analytic) Eisenstein series for SL(2) over totally imaginary number fields (e.g., the Picard modular group). Asai [1] extends the latter results to all number fields K of class number one. Tamagawa [17] obtains the functional equation but not the Fourier expansion for the Epstein zeta function over totally real fields. Ramanathan [12] considers the more general situation of Epstein zeta functions for indefinite quadratic forms over number fields, again obtaining the functional equation, but not a Fourier expansion. The functional equations of Eisenstein series for $\mathrm{GL}(n)$ over algebraic number fields were obtained by Langlands in [9]. The collection [3] provides a general framework for the theory. One should look, in particular, at the articles of Borel.

The main result of Section 1 (Theorem 1) of the present paper is the Fourier expansion of the Epstein zeta function associated with ideals of the algebraic number field K. Here the constant term involves the part of the Dedekind zeta function coming from an ideal class of K. The other terms include products of modified Bessel functions of the second kind. We then sum up a finite number of Epstein zeta functions corresponding to the elements of the ideal class group of K to obtain a function with $\zeta_K(s)$ and $\zeta_K(s-1)$ in the constant term of its Fourier expansion (Corollary to Theorem 1).

In Section 2 we use the invariance properties of the Epstein zeta function over K (i.e., the fact that is unchanged under transformation by matrices A such that A and A^{-1} have entries which are integers of K) and the Fourier expansion to derive some relations between $\zeta_K(s)$ and $\zeta_K(s-1)$. For $K = \boldsymbol{Q}$, such results have a long history of discovery and rediscovery. The paper of Berndt [2] contains many references. We then mention two applications of the result of Theorem 2. The first is to the study of the Dedekind zeta function in the critical strip. For example, a simple formula is found relating $\zeta_K(\frac{1}{2})$ and $\zeta_K(\frac{3}{2})$ for an imaginary quadratic field (formula (2.1)). The formula involves a sum of exponentials over the inverse different of the field K. One could also apply Theorem 2 to the study of ζ_K at odd integer argument. In this connection, one should compare the results with those of Grosswald [4], which involve Meyer functions rather than the modified Bessel functions of the second kind.

The final result of Section 2 is a formula for the product of the class number and the regulator of K (Theorem 3). One would hope to be able to use the result and the rapid decrease of the modified Bessel functions of the second kind to say something about the Brauer–Siegel theorem ([8], pp. 321 ff.).

Finally we note that it is possible to obtain similar results for Epstein zeta functions with characters. They would generalize functions used by Stark in the class number one problem [16].

1. Epstein zeta functions over number fields and their Fourier expansions. Let K be an algebraic number field of degree m over $\boldsymbol{Q}$. Suppose K has r_1 real embeddings (over $\boldsymbol{Q}$) denoted $x \mapsto x^{(j)}$, $j = 1, \ldots, r_1$, and r_2 pairs of conjugate complex embeddings $x \mapsto x^{(j)}$, $x \mapsto \overline{x^{(j)}} = x^{(j+r_2)}$, $j = r_1+1, \ldots, r_1+r_2$. Then $K \otimes_{\boldsymbol{Q}} \boldsymbol{R} \cong \boldsymbol{R}^{r_1} \oplus \boldsymbol{C}^{r_2}$ (algebra direct sum). Let O_K be the ring of integers of K, U_K be the units of O_K, $\mathfrak{d}_K$ be the different of K, d_K be the absolute value of the discriminant of K. If $\mathfrak{a}$ is an ideal of O_K, let $N\mathfrak{a}$ = Norm of $\mathfrak{a} = [O_K : \mathfrak{a}]$.

If $K = \boldsymbol{Q}$, the Epstein zeta function has a complex argument s and an $n \times n$ positive definite symmetric (real) matrix argument P. The latter should be thought of as lying in the symmetric space $\mathrm{GL}(n, \boldsymbol{R})/O(n)$ under the identification $gO(n) \mapsto g\,{}^t g$ for g in $\mathrm{GL}(n, \boldsymbol{R})$. Here ${}^t g$ = transpose of g. Note that, if Γ is the group of all $n \times n$ matrices with integer entries and determinant ± 1, then A in Γ acts on P via $P \mapsto P[A] = {}^t A P A$. And the Epstein zeta function for $\boldsymbol{Q}$

$$Z(P, s) = \tfrac{1}{2} \sum_{0 \neq g \in \boldsymbol{Z}^n} P[g]^{-s}, \quad \text{for} \quad \operatorname{Re} s > n/2, \tag{1.1}$$

is invariant under $P \mapsto P[A]$, for $A \in \Gamma$.

To lift the definition (1.1) to the number field K one needs the idea of a positive n-form over K. This is really an element of the infinite part of the adelized general linear group over K modulo the maximal compact subgroup (cf. [3], pp. 113 ff.). The reader can refer to Ramanathan [11] or Weil [24] for a complete treatment of the reduction theory of positive forms over division algebras. Other references are Siegel [15], Vol. I, pp. 459 ff.; Vol. II, pp. 390 ff., and Weyl [25]. The general theory is described in Borel [3], pp. 20 ff. For our purposes a positive n-form P over K is defined to be a vector $P = (P^{(1)}, \ldots, P^{(r_1+r_2)})$ whose first r_1 components are $n \times n$ positive definite symmetric (real) matrices and whose last r_2 components are $n \times n$ positive definite hermitian (complex) matrices. Let $\mathscr{P}_n^K$ denote the space of all positive n-forms P over K. Define Γ_n^K to be the group of all $n \times n$ matrices A such that both A and A^{-1} have entries in O_K, the ring of integers of K. It is clearly equivalent to require that A have entries in O_K and $\det A$ lie in U_K, the group of units of O_K. One then shows that Γ_n^K acts discontinuously on $\mathscr{P}_n^K$ by $P \mapsto P\{A\}$, where

$$(P\{A\})^{(j)} = P^{(j)}\{A^{(j)}\} = {}^t\overline{A^{(j)}} P^{(j)} A^{(j)}, \quad j = 1, \ldots, r_1+r_2.$$

Here $A \mapsto A^{(j)}$ means replace every entry of A in Γ_n^K by its jth conjugate, and $A \mapsto \overline{A^{(j)}}$ means replace every entry of A in Γ_n^K by the complex conjugate of its jth conjugate. Then $\operatorname{Tr}(P\{x\})$ gives a positive quadratic form in mn variables if $m = [K : \boldsymbol{Q}]$ and x is a column vector in $(K \otimes_{\boldsymbol{Q}} \boldsymbol{R})^n$. Here Tr denotes the trace from $K \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ to $\boldsymbol{R}$. It is this trace form which is used classically to obtain the fundamental domain for $\mathcal{P}_n^K$ under Γ_n^K in a way similar to the method (for $K = \boldsymbol{Q}$) used by Minkowski. We shall use N to denote the norm on the algebra $K \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ over $\boldsymbol{R}$ or the reduced norm on the algebra on $n \times n$ matrices over $K \otimes_{\boldsymbol{Q}} \boldsymbol{R}$.

As usual in this theory one uses the Jacobi decomposition ([15], Vol. II, p. 402) or Babylonian reduction ([24], p. 7) of a positive n-form P over K. This is related to the Iwasawa decomposition. We shall use the Jacobi decomposition of P in the following form. Let k be an integer $1 \leqslant k \leqslant n-1$ and write

$$P^{(j)} = \begin{pmatrix} P_1^{(j)} & P_{12}^{(j)} \\ {}^t\overline{P_{12}^{(j)}} & P_2^{(j)} \end{pmatrix},$$

where $P_2^{(j)}$ is a k-form over K, $j = 1, \ldots, r_1+r_2$. Then, using vector notation,

$$P = \begin{pmatrix} T & 0 \\ 0 & P_2 \end{pmatrix} \left\{ \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix} \right\}, \tag{1.2}$$

where I is the identity matrix, $T = P_1 - P_2\{Q\}$, $Q = P^{-1}\,{}^t\overline{P_{12}}$.

We are now ready to define the Epstein zeta function associated with $P \in \mathcal{P}_n^K$, $s \in C$ with $\operatorname{Re} s > n/2$, and an ideal $\mathfrak{a}$ of K, by

$$Z_n^{\mathfrak{a}}(P, s) = \sum_{0 \neq g \in \mathfrak{a}^n/U_K} \prod_{j=1}^{r_1+r_2} (P^{(j)}\{g^{(j)}\})^{-e_j s}, \tag{1.3}$$

where

$$e_j = \begin{cases} 1, & j = 1, \ldots, r_1, \\ 2, & j = r_1+1, \ldots, r_1+r_2, \end{cases}$$

and U_K is the group of units of the ring of integers of K. The sum is over a complete system of non-zero column vectors not equivalent under the equivalence relation

$${}^tg = (g_1, \ldots, g_n) \sim (\varepsilon g_1, \ldots, \varepsilon g_n) \quad \text{for} \quad \varepsilon \in U_K.$$

The function is a special case of that considered by Tamagawa in [17] for totally real fields (where the lattice in $(K \otimes_{\boldsymbol{Q}} \boldsymbol{R})^n$ is $\Lambda = \mathfrak{a}^n$ and $O_\Lambda = \{a \in K \mid a\Lambda \subset \Lambda\} = O_K$). The need to define a slightly more general function than that of [22], that is to define $Z^{\mathfrak{a}}$ and not just Z^{O_K} results from the fact that K may not have class number one. Compare the situation for the Hilbert modular group in [14], p. 292.

Note that defining $P^{(r_2+j)} = \overline{P^{(j)}}$, $j = r_1+1, \ldots, r_1+r_2$, we have

$$\prod_{j=1}^{r_1+r_2} (P^{(j)}\{g^{(j)}\})^{e_j} = \prod_{j=1}^{m} P^{(j)}\{g^{(j)}\} = N_{K \otimes_{\boldsymbol{Q}} \boldsymbol{R}/\boldsymbol{R}}(P\{g\}). \tag{1.4}$$

Imitating Siegel's definition of an Eisenstein series for the Hilbert modular group ([14], p. 290), would lead one to consider

$$\zeta_n^{\mathfrak{a}}(P, s) = \sum_{g} \prod_{j=1}^{r_1+r_2} (P^{(j)}\{g^{(j)}\})^{-e_j s}, \tag{1.5}$$

where the sum is over a full system of nonzero column vectors g modulo units with $g = {}^t(g_1, \ldots, g_n)$ such that $\sum_{j=1}^{m} g_j O_K = \mathfrak{a}$. Here P, s and $\mathfrak{a}$ are as before. It is easy to see that if $\mathfrak{a} = c\mathfrak{b}$ for $c \in O_K$, $c \neq 0$, then

$$\zeta_n^{\mathfrak{a}}(P, s) = (Nc)^{-2s} \zeta_n^{\mathfrak{b}}(P, s). \tag{1.6}$$

Thus for an ideal class C with $\mathfrak{a}$ in C, if we define

$$S_n^C(P, s) = (N\mathfrak{a})^{2s} \zeta_n^{\mathfrak{a}}(P, s),$$

we see that S_n^C depends only on the ideal class C of $\mathfrak{a}$ in I_K, the ideal class group of K. It follows that for $\operatorname{Re} s > n/2$

$$Z_n^{O_K}(P, s) = \sum_{C \in I_K} \zeta^C(2s) S_n^C(P, s), \quad \text{where} \quad \zeta^C(s) = \sum_{\mathfrak{a} \in C} (N\mathfrak{a})^{-s}. \tag{1.7}$$

In order to prove the convergence of $Z_n^{\mathfrak{a}}(P, s)$ for $P \in \mathcal{P}_n^K$, one could imitate convergence proofs for Eisenstein series given by Siegel ([14], p. 290) or Godement ([3], p. 207). Or one could deduce convergence from bounds on theta functions, as Ramanathan does in [12], p. 54.

In order to prove the functional equation and analytic continuation of $Z_n^{\mathfrak{a}}(P, s)$, one should imitate Hecke's proof for the case $n = 1$, which is to be found in Lang ([8], pp. 255–258). Ramanathan ([12], pp. 53–59) generalizes this method to allow the n-form P to be indefinite. It would perhaps be useful to outline the method described by Lang in the pages mentioned above, with the modifications required to prove the analytic continuation and functional equation of $Z_n^{\mathfrak{a}}(P, s)$. The main idea is to note that $\Gamma(\frac{s}{2})^{r_1}\Gamma(s)^{r_2} Z_n^{\mathfrak{a}}(P, s)$ is the Mellin transform of a theta function (minus 1). The theta function in question is

$$\theta^{\mathfrak{a}}(P, t) = \sum_{g \in \mathfrak{a}^n} \exp\{-\pi \operatorname{Tr}(P\{g\}t)\}$$

where $t = (t^{(1)}, \ldots, t^{(r_1+r_2)}) \in \mathcal{P}_1^K$, with $t^{(j)} > 0$, $j = 1, \ldots, r_1+r_2$, and $P = (P^{(1)}, \ldots, P^{(r_1+r_2)}) \in \mathcal{P}_n^K$, with the first r_1 components $P^{(j)}$ being positive definite symmetric (real) $n \times n$ matrices and with the last r_2 components being $n \times n$ positive definite hermitian (complex) matrices. As usual we define $t^{(r_2+j)} = t^{(j)}$, for $j = r_1+1, \ldots, r_1+r_2$, and $P^{(r_2+j)} = \overline{P^{(j)}}$, for $j = r_1+1, \ldots, r_1+r_2$. Then

$$\operatorname{Tr}(P\{g\}t) = \sum_{j=1}^{m} P^{(j)}\{g^{(j)}\} t^{(j)}.$$

Proceeding exactly as in Lang's version of Hecke's proof of the functional equation of Dedekind's zeta function one obtains analogues of Lang's formulas ([8], pp. 256–257). For example, one can show that

$$(2^{-r_2}\pi^{-m/2})^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} Z_n^{\mathfrak{a}}(P, s)$$

$$= \int\limits_{y\in \mathscr{P}_1^K} \sum_{0\neq g\in \mathfrak{a}^n} \exp\{-\pi \operatorname{Tr}(P\{g\}y)\} N y^{s/2} \frac{dy^{(1)}}{y^{(1)}} \cdots \frac{dy^{(r_1+r_2)}}{y^{(r_1+r_2)}}$$

$$= w_K^{-1} \int\limits_{t=0}^{\infty} \int\limits_{c\in E} \{\theta(P, t^{1/m}) - 1\} J(c, t) t^{s/2} dc\, dt.$$

Here w_K denotes the order of the group of roots of unity of K. And $J(c, t)$ is $(y^{(1)} \ldots y^{(r_1+r_2)})^{-1}$ times the Jacobian of the change of variables

$$(y^{(1)}, \ldots, y^{(r_1+r_2)}) \mapsto (c^{(1)}, \ldots, c^{(r)}, t),$$

where $r = r_1 + r_2 - 1$ and $y = t^{1/m} c$, with y in $\mathscr{P}_1^K$, t in $\boldsymbol{R}^+$, and c in $(\mathscr{P}_1^K)_0$ where $(\mathscr{P}_1^K)_0$ is defined by

$$(\mathscr{P}_1^K)_0 = \Big\{c \in \mathscr{P}_1^K \mid Nc = \prod_{j=1}^{m} c^{(j)} = 1\Big\}.$$

In the above integrals $dy^{(j)}$, dt are the usual Lebesgue measures on the positive reals and $dc = dc^{(1)} \ldots dc^{(r_1+r_2)}$ with $dc^{(j)}$ being the usual Lebesgue measure on the positive reals. Thus one can compute $J(c, t)$ easily to be

$$J(c, t) = m^{-1}(tc^{(1)} \ldots c^{(r_1+r_2)})^{-1}.$$

Now we must describe E. The ideas here arise in the proof of the Dirichlet unit theorem ([8], pp. 104–110). The set $E \subset (\mathscr{P}_1^K)_0$ is a fundamental domain for an action of the unit group U_K of the field K on $(\mathscr{P}_1^K)_0$. More precisely, one embeds U_K in $(\mathscr{P}_1^K)_0$ via

$$u \in U_K \mapsto (|u^{(1)}|, \ldots, |u^{(r_1+r_2)}|),$$

where $|x|$ denotes the ordinary absolute value in the complex numbers. This embedding is to be found in [8], p. 256. Let V be the image of U_K in $(\mathscr{P}_1^K)_0$ under this mapping. Then one defines E to be a fundamental domain for $\{v^2 \mid v \in V\}$, i.e.,

$$(\mathscr{P}_1^K)_0 = \bigcup_{v\in V} v^2 E.$$

The existence and compactness of E, in fact, the existence of a change of variables mapping E onto the unit cube in r-space follows from the proof of the Dirichlet unit theorem ([8], pp. 104–110).

To complete the proof of the functional equation and analytic continuation of $Z_n^{\mathfrak{a}}(P, s)$ one applies the original method of Riemann ([8], p. 257). That is, one applies the transformation formula of the theta function defined above, proved via Poisson summation. The transformation formula is stated in Tamagawa [17], p. 260. The final result is that $Z_n^{\mathfrak{a}}(P, s)$ has an analytic continuation to all s in $\boldsymbol{C}$ with a simple pole at $s = n/2$. Moreover there is a functional equation relating $Z_n^{\mathfrak{a}}(P, s)$ and $Z_n^{\mathfrak{a}^*}(P, n/2 - s)$, where $\mathfrak{a}^* = (\mathfrak{a}\mathfrak{d}_K)^{-1}$, $\mathfrak{d}_K$ being the different of K. This result is stated in Tamagawa [17], p. 260.

The functional equation of $Z_n^{\mathfrak{a}}(P, s)$ can also be derived from the Fourier expansion which we shall prove in this section. We did this in [20] for the case $K = \boldsymbol{Q}$. The details are tedious and we omit them.

We shall need to make use of the invariance properties of the Epstein zeta function under Γ_n^K which was defined to be the group of all $n \times n$ matrices A such that both A and A^{-1} have entries in O_K, the ring of integers of K. In particular we shall use the obvious fact from definition (1.3) that

$$Z_n^{\mathfrak{a}}(P, s) = Z_n^{\mathfrak{a}}(P\{A\}, s) \quad \text{for} \quad A \text{ in } \Gamma_n^K. \tag{1.8}$$

This follows from the fact that A and A^{-1} map $\mathfrak{a}^n$ into itself, since both A and A^{-1} have entries in O_K, the ring of integers of K.

Next we obtain the Fourier expansion of $Z_n^{\mathfrak{a}}(P, s)$.

THEOREM 1. *For $1 \leqslant k \leqslant n-1$ and for P a positive n-form over K as in* (1.2)

$$Z_n^{\mathfrak{a}}(P, s) = Z_k^{\mathfrak{a}}(P_2, s) + 2^{kr_2} d_K^{-k/2} (N\mathfrak{a})^{-k} (NP_2)^{-1/2} Z_{n-k}^{\mathfrak{a}}(T, s - \tfrac{1}{2}k) \times$$
$$\times \pi^{k/2} \Gamma(s - \tfrac{1}{2}k)^{r_1} \Gamma(2s-k)^{r_2} \Gamma(s)^{-r_1} \Gamma(2s)^{-r_2} +$$
$$+ 2^{r_1 + r_2(1+2s)} d_K^{-k/2} (N\mathfrak{a})^{-k} (NP_2)^{-1/2} \pi^{ms} H_n^{\mathfrak{a}}(P, s) \Gamma(s)^{-r_1} \Gamma(2s)^{-r_2}.$$

Here

$$H_n^{\mathfrak{a}}(P, s) = \sum_{\substack{0\neq a\in \mathfrak{a}^{n-k}/U_K \\ 0\neq b\in (\mathfrak{a}^{-1}\mathfrak{d}_K^{-1})^k}} \exp\big(2\pi i \operatorname{Tr}({}^t b Q a)\big) \left|\frac{N(T\{a\})}{N(P_2^{-1}\{b\})}\right|^{\frac14 k - \frac12 s} \times$$
$$\times \prod_{j=1}^{r_1+r_2} K_{e_j(k/2-s)}\Big(2\pi e_j \sqrt{T^{(j)}\{a^{(j)}\}(P_2^{(j)})^{-1}\{b^{(j)}\}}\Big).$$

Proof. Write $g = \binom{a}{b}$, $a \in \mathfrak{a}^{n-k}$, $b \in \mathfrak{a}^k$ for the summation variable in (1.3). Then as in the proof of Theorem 1 of [20], split up the sum in (1.3) into two parts: terms with $a = 0$, and terms with $a \neq 0$. This yields, using (1.2)

$$Z_n^{\mathfrak{a}}(P, s) = Z_k^{\mathfrak{a}}(P, s) + \sum_{\substack{0\neq a\in \mathfrak{a}^{n-k}/U_K \\ b\in \mathfrak{a}^k}} N(T\{a\} + P_2\{Qa + b\})^{-s}.$$

Let S_2 be the sum over a and b on the right hand side of this equation. Now $\mathfrak{a}^k$ is a lattice in $(K\otimes_{\mathbf{Q}}\mathbf{R})^k = (\mathbf{R}^{r_1}\oplus \mathbf{C}^{r_2})^k$. So Poisson summation (as in Weil, *Basic number theory* [23], p. 106) gives

$$S_2 = \sum_{\substack{0\neq a\in\mathfrak{a}^{n-k}/U_K\\ c\in(\mathfrak{a}^k)^\perp}} f(a,c),$$

where $(\mathfrak{a}^k)^\perp$ is the dual lattice of characters of $(K\otimes_{\mathbf{Q}}\mathbf{R})^k$ trivial on $\mathfrak{a}^k$. Thus $(\mathfrak{a}^k)^\perp = (\mathfrak{a}^{-1}\mathfrak{d}_K^{-1})$ under the usual identification ([23], pp. 40–42) of $(K\otimes_{\mathbf{Q}}\mathbf{R})^k$ with its dual. And

$$f(a,c) = \int_{(K\otimes_{\mathbf{Q}}\mathbf{R})^k} N(T\{a\}+P_2\{Qa+x\})^{-s}\exp\left(2\pi i\,\mathrm{Tr}({}^tcx)\right)d\mu(x).$$

Here $d\mu$ is a measure on $(K\otimes_{\mathbf{Q}}\mathbf{R})^k$ such that $\mu\left((K\otimes_{\mathbf{Q}}\mathbf{R})^k/\mathfrak{a}^k\right)=1$. To compute $d\mu$, choose $w_1,\dots,w_m$ such that

$$\mathfrak{a} = \sum_{j=1}^{m}{}^{\oplus}\, \mathbf{Z}w_j \quad\text{and}\quad K\otimes_{\mathbf{Q}}\mathbf{R} = \sum_{j=1}^{m}{}^{\oplus}\, \mathbf{R}w_j.$$

Then writing $x=\sum_{j=1}^m x_jw_j$, it is clear that

$$d\mu(x) = \prod_{j=1}^m dx_j,$$

where dx_j = Lebesgue measure on $\mathbf{R}^k$. It is necessary to obtain another version of $d\mu$ for ease of computation. Write

$$K\otimes_{\mathbf{Q}}\mathbf{R} = \sum_{j=1}^{r_1+r_2}{}^{\oplus}\, E_j,\qquad E_j = \begin{cases} \mathbf{R}, & j=1,\dots,r_1,\\ \mathbf{C}, & j=r_1+1,\dots,r_1+r_2.\end{cases}$$

Then

$$x=\sum_{j=1}^m x_jw_j \mapsto \left(\sum_{j=1}^m x_jw_j^{(1)},\dots,\sum_{j=1}^m x_j w_j^{(r_1+r_2)}\right) = y$$

maps

$$\sum_{j=1}^{m}{}^{\oplus}\, \mathbf{R}w_j$$

isomorphically onto

$$\sum_{j=1}^{r_1+r_2}{}^{\oplus}\, E_j.$$

And the Jacobian

$$\frac{\partial\left(y_1,\dots,y_{r_1},\mathrm{Re}(y_{r_1+1}),\mathrm{Im}(y_{r_1+1}),\dots,\mathrm{Re}(y_{r_1+r_2}),\mathrm{Im}(y_{r_1+r_2})\right)}{\partial(x_1,\dots,x_m)} = d_K^{1/2}N\mathfrak{a}\,2^{-r_2}.$$

It follows that, writing

$$y=(y_1,\dots,y_{r_1+r_2})\in\sum_{j=1}^{r_1+r_2}{}^{\oplus}\, E_j^k = (K\otimes_{\mathbf{Q}}\mathbf{R})^k,$$

one has

$$f(a,c) = d_K^{-k/2}(N\mathfrak{a})^{-k}2^{kr_2}\prod_{j=1}^{r_1+r_2}\int_{E_j^k} N_{E_j/\mathbf{R}}\left(T^{(j)}\{a^{(j)}\}+P_2^{(j)}\{Q^{(j)}a^{(j)}+y_j\}\right)^{-s}\times \exp\left(2\pi i\,\mathrm{Tr}_{E_j/\mathbf{R}}({}^tc^{(j)}y_j)\right)dy_j,$$

where dy_j is Lebesgue measure on E_j^k, $j=1,\dots,r_1+r_2$.

Now one makes the change of variables

$$y_j\mapsto x_j = (T^{(j)}\{a^{(j)}\})^{-1/2}W^{(j)}(Q^{(j)}a^{(j)}+y_j),$$

where $P_2^{(j)} = {}^t\overline{W}^{(j)}W^{(j)}$. So

$$f(a,c) = d_K^{-k/2}(N\mathfrak{a})^{-k}2^{kr_2}\prod_{j=1}^{r_1+r_2}\exp\left(2\pi i\,\mathrm{Tr}_{E_j/\mathbf{R}}({}^tc^{(j)}Q^{(j)}a^{(j)})\right)\left(\det(P_2^{(j)})\right)^{-1/2}\times (T^{(j)}\{a^{(j)}\})^{\frac12 k-s} I_{E_j^k}\left(2\pi\sqrt{T^{(j)}\{a^{(j)}\}}\,({}^t\overline{W}^{(j)})^{-1}c,\ s\right),$$

where

$$I_{E_j^k}(b,s) = \int_{E_j^k}(1+{}^t\bar{x}_jx_j)^{-se_j}\exp\left(-i\,\mathrm{Tr}_{E_j/\mathbf{R}}({}^tbx_j)\right)dx_j.$$

We computed the $I_{E_j^k}$ for $j=1,\dots,r_1$ in [20]. Thus we have only to compute

$$I_{\mathbf{C}^k}(b,s) = \int_{\mathbf{C}^k}(1+{}^t\bar{x}x)^{-2s}\exp\left(-i\,\mathrm{Tr}_{\mathbf{C}/\mathbf{R}}({}^tbx)\right)dx.$$

Write $b=c+id$ and $x=u+iv$ with c,d,u,v in $\mathbf{R}^k$. Then $dx=du\,dv$ and

$$I_{\mathbf{C}^k}(b,s) = \int_{\mathbf{R}^k}\int_{\mathbf{R}^k}(1+{}^tuu+{}^tvv)^{-2s}\exp\left(-2i({}^tcu-{}^tdv)\right)du\,dv = I_{\mathbf{R}^{2k}}\left(2\binom{c}{d},\ 2s\right).$$

Thus we are again reduced to the formulas (2.4) and (2.5) of [20]. The final result is

$$f(a,0) = 2^{kr_2}d^{-k/2}(N\mathfrak{a})^{-k}NP_2^{-1/2}N(T\{a\})^{\frac12 k-s}\prod_{j=1}^{r_1+r_2}\pi^{ke_j/2}\Gamma\left(e_j(s-\tfrac12 k)\right)\Gamma(e_js)^{-1},$$

and for $c \neq 0$, we have

$$f(a, c) = 2^{kr_2} d^{-k/2} (N\mathfrak{a})^{-k} NP_2^{-1/2} N(T\{a\})^{\frac{1}{2}k-s} \exp\left(2\pi i \operatorname{Tr}({}^t c Q a)\right) \times$$
$$\times \prod_{j=1}^{r_1+r_2} 2\pi^{ke_j/2} \Gamma(e_j s)^{-1} \left(e_j \pi \sqrt{T^{(j)}\{a^{(j)}\}(P_2^{(j)})^{-1}\{c^{(j)}\}}\right)^{e_j(s-\frac{1}{2}k)} \times$$
$$\times K_{e_j(\frac{1}{2}k-s)}\left(2\pi e_j \sqrt{T^{(j)}\{a^{(j)}\}(P_2^{(j)})^{-1}\{c^{(j)}\}}\right). \quad ■$$

Note that Theorem 1 yields an analytic continuation of $Z_n^{\mathfrak{a}}(P, s)$ to all complex s and implies the functional equation, just as in [20]. The complete proof is rather complicated and we omit it.

The main problem in applying Theorem 1 to algebraic number theory is that for $n = 2$ the Dedekind zeta function does not appear in the constant term of the Fourier expansion unless K has class number one. It will be useful to invent a function with the Dedekind zeta function of K as the constant term of its Fourier expansion. Let I_K denote the ideal class group of K. For each class C in I_K choose an ideal $\mathfrak{b}_C$ in C. Define

(1.9) $$Z_n^*(P, s) = \sum_{C \in I_K} (N\mathfrak{b}_C)^{2s} Z_n^{\mathfrak{b}_C}(P, s).$$

Note that the result is independent of the choice of $\mathfrak{b}_C$ in the ideal class C. Setting $n = 2$ and $k = 1$ in Theorem 1 proves:

Corollary.

$$\Gamma(s)^{r_1}\Gamma(2s)^{r_2} Z_2^*(P, s) = (NP_2)^{-s}\zeta_K(2s)\Gamma(s)^{r_1}\Gamma(2s)^{r_2} +$$
$$+ 2^{r_2} d_K^{-1/2}(NP_2)^{-1/2}(NT)^{1/2-s}\zeta_K(2s-1)\pi^{m/2}\Gamma(s-\tfrac{1}{2})^{r_1}\Gamma(2s-1)^{r_2} +$$
$$+ 2^{r_1+r_2(1+2s)} d_K^{-1/2}(NP_2)^{-1/2}\pi^{ms} H_2(P, s),$$

where

$$H_2(P, s) = (NP)^{\frac{1}{4}-\frac{1}{2}s} \sum_{\substack{C \in I_K \\ 0 \neq a \in \mathfrak{b}_C/U_K \\ 0 \neq b \in (\mathfrak{b}_C \mathfrak{d}_K)^{-1}}} (N\mathfrak{b}_C)^{2s-1} \left|\frac{Na}{Nb}\right|^{\frac{1}{2}-s} \exp\left(2\pi i \operatorname{Tr}(qab)\right) \times$$
$$\times \prod_{j=1}^{r_1+r_2} K_{e_j(\frac{1}{2}-s)}\left(2\pi e_j \sqrt{\frac{t^{(j)}}{p_2^{(j)}}}\,|a^{(j)} b^{(j)}|\right);$$

for

$$P = \begin{pmatrix} t & 0 \\ 0 & p_2 \end{pmatrix}\left\{\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}\right\},$$

using vector notation. ■

It will be useful to simplify the formula for $H_2(P, s)$. With this in mind, we define the following generalization of the divisor function for any ideal $\mathfrak{a}$ in O_K

(1.10) $$\sigma_\nu(\mathfrak{a}) = \sum_{\mathfrak{b}|\mathfrak{a}} (N\mathfrak{b})^\nu,$$

the sum being over all ideals $\mathfrak{b}$ in O_K dividing $\mathfrak{a}$.

Proposition.

$$H_2(P, s) = (NP)^{\frac{1}{4}-\frac{1}{2}s} \sum_{0 \neq u \in \mathfrak{d}_K^{-1}} |Nu|^{s-\frac{1}{2}} \sigma_{1-2s}(u\mathfrak{d}_K) \exp\left(2\pi i \operatorname{Tr}(qu)\right) \times$$
$$\times \prod_{j=1}^{r_1+r_2} K_{e_j(s-\frac{1}{2})}\left(2\pi e_j \sqrt{\frac{t^{(j)}}{p_2^{(j)}}}\,|u^{(j)}|\right).$$

Proof. The equation $\mathfrak{a}\mathfrak{b}_C = aO_K$ defines a 1-1 map from ideals $\mathfrak{a}$ in C^{-1} onto elements $a \in \mathfrak{b}_C$ modulo U_K. Set $u = ab$ for $b \in (\mathfrak{b}_C\mathfrak{d}_K)^{-1}$. We have a map

$$(\mathfrak{b}_C/U_K) \times (\mathfrak{b}_C\mathfrak{d}_K)^{-1}/U_K \to (C^{-1}) \times (\mathfrak{d}_K^{-1}/U_K),$$
$$(a, b) \mapsto (\mathfrak{a}, ab = u).$$

Call the mapping L. Then L is 1-1, since $L(a, b) = L(c, d) = (\mathfrak{a}, u)$ implies $ab = cd$ and $\mathfrak{a}\mathfrak{b}_C = aO_K = cO_K$. Thus $a = ce$ for some e in U_K and $ab = cd$ implies that $ceb = cd$. Thus $eb = d$. It follows that $a = c$ and $b = d$, since we have chosen the a and b to be inequivalent modulo multiplication by units.

The map L is not onto. The image consists of $(\mathfrak{a}, u)$ such that $\mathfrak{a}$ divides $\mathfrak{d}_K u$. For $\mathfrak{a}\mathfrak{b}_C = aO_K$ implies that $\mathfrak{a}\mathfrak{b}_C\mathfrak{d}_K b = ab\mathfrak{d}_K = u\mathfrak{d}_K$ and thus $\mathfrak{a}$ divides $u\mathfrak{d}_K$. Conversely suppose that $\mathfrak{a}$ divides $u\mathfrak{d}_K$. Then $u\mathfrak{d}_K\mathfrak{b}_C \subset \mathfrak{a}\mathfrak{b}_C = aO_K$. So if $b = ua^{-1}$, we have $b \in (\mathfrak{d}_K\mathfrak{b}_C)^{-1}$.

To complete the proof, note that

$$(N\mathfrak{b}_C)^{2s-1}\left|\frac{Na}{Nb}\right|^{\frac{1}{2}-s} = (N\mathfrak{b}_C)^{2s-1}\left|\frac{N(a^2)}{N(ab)}\right|^{\frac{1}{2}-s}$$
$$= \left|\frac{Na}{N\mathfrak{b}_C}\right|^{1-2s}(Nu)^{s-\frac{1}{2}} = (N\mathfrak{a})^{1-2s}(Nu)^{s-\frac{1}{2}}. \quad ■$$

2. Relations between $\zeta_K(s)$ and $\zeta_K(s-1)$. Here we extend the results of [22] to all algebraic number fields K and derive a formula for the residue of the Dedekind zeta function at its pole in terms of its value at 2 and a series of modified Bessel functions of the second kind.

Theorem 2. *Let $x_1, \ldots, x_{r_1+r_2}$ denote positive real numbers and let*

$$x = \prod_{j=1}^{r_1+r_2} x_j^{e_j}.$$

Then

$$\zeta_K(2s)(x^s - x^{-s})\Gamma(s)^{r_1}\Gamma(2s)^{r_2} + \\ + d_K^{-1/2}2^{r_2}\zeta_K(2s-1)(x^{1-s}-x^{s-1})\pi^{m/2}\Gamma(s-\tfrac12)^{r_1}\Gamma(2s-1)^{r_2}$$

$$= d_K^{-1/2}2^{r_1+r_2(1+2s)}\pi^{ms}\sum_{0\neq u\in\mathfrak{d}_K^{-1}}|Nu|^{s-\frac12}\sigma_{1-2s}(u\mathfrak{d}_K)\times \\ \times\Big\{x^{-1/2}\prod_{j=1}^{r_1+r_2}K_{e_j(\frac12-s)}(2\pi e_j x_j^{-1}|u^{(j)}|) - x^{-1/2}\prod_{j=1}^{r_1+r_2}K_{e_j(\frac12-s)}(2\pi e_j x_j|u^{(j)}|)\Big\}.$$

Proof. Set

$$P^{(j)} = \begin{pmatrix} x_j & 0 \\ 0 & 1\end{pmatrix},$$

using vector notation, as usual. Now by (1.8) $Z_2^*(P, s) = Z_2^*(Q, s)$, where

$$Q^{(j)} = P^{(j)}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right\}.$$

Then apply the Corollary to Theorem 1 to each side of the equality. ■

Theorem 2 is analogous to formula (3.1) of [22]. Thus one can derive results analogous to those of [22] by the same methods. For example, we have the following

COROLLARY. *Define*

$$M_\nu(z) = K_\nu(z) + 2zK_\nu'(z), \quad \textit{where} \quad K_\nu'(z) = \frac{d}{dz}K_\nu(z).$$

Then if r_1+r_2 *is odd, we have*

$$\zeta_K(2s)s^{r_1+r_2}\Gamma(s)^{r_1}\Gamma(2s)^{r_2} + \\ + d_K^{-1/2}2^{r_2}\zeta_K(2s-1)(1-s)^{r_1+r_2}\pi^{m/2}\Gamma(s-\tfrac12)^{r_1}\Gamma(2s-1)^{r_2}$$

$$= -4^{r_2 s}d_K^{-\frac12 ms}\sum_{0\neq u\in\mathfrak{d}_K^{-1}}|Nu|^{s-\frac12}\sigma_{1-2s}(u\mathfrak{d}_K)\prod_{j=1}^{r_1+r_2}M_{e_j(\frac12-s)}(2\pi e_j|u^{(j)}|).$$

Proof. Apply

$$\frac{\partial^{r_1+r_2}}{\partial x_1\ldots\partial x_{r_1+r_2}}$$

to the equation of Theorem 2 and then set all the $x_j = 1$, $j = 1, \ldots, r_1+r_2$.

One could of course replace the differential operator of the proof of the Corollary by other operators to obtain similar formulas. It might be possible to use the Corollary to study $\zeta_K(s)$ for $0 < s < 1$. So for example in the simplest case when K is an imaginary quadratic field, let $s = \frac34$

in the Corollary to obtain the following formula for $\zeta_K(\frac12)$. Here we use the fact that $M_{1/2}(z) = -(2\pi z)^{1/2}\exp(-z)$.

For an imaginary quadratic field K,

$$\zeta_K(\tfrac12) = -\pi^{-1}\tfrac34 d_K^{1/2}\zeta_K(\tfrac32) + 16\pi\sum_{0\neq u\in\mathfrak{d}_K^{-1}}|u|\sigma_{-1/2}(u\mathfrak{d}_K)\exp(-4\pi|u|). \tag{2.1}$$

We next make use of Theorem 2 to obtain a formula for the product of the class number h_K and the regulator R_K.

THEOREM 3. *Let* $x_1, \ldots, x_{r_1+r_2}$ *be positive real numbers with*

$$x = \prod_{j=1}^{r_1+r_2}x_j^{e_j},$$

and let w_K *denote the number of roots of unity in* K. *Then*

$$h_K R_K = w_K d_K\zeta_K(2)(2\pi)^{-m}(x - x^{-1})(\log x)^{-1} - \\ - w_K d_K^{1/2}2^{r_2}(\log x)^{-1}\sum_{0\neq u\in\mathfrak{d}_K^{-1}}|Nu|^{1/2}\sigma_{-1}(u\mathfrak{d}_K)\times \\ \times\Big\{x^{-1/2}\prod_{j=1}^{r_1+r_2}K_{\frac12 e_j}(2\pi e_j x_j^{-1}|u^{(j)}|) - x^{1/2}\prod_{j=1}^{r_1+r_2}K_{\frac12 e_j}(2\pi e_j x_j|u^{(j)}|)\Big\}.$$

Proof. We let s approach 1 in the formula of Theorem 2 and use the formula for the residue of Dedekind's zeta function at $s = 1$ to find the limit of the 2nd term on the left-hand side of the formula of Theorem 2. In particular set $2s-1 = 1-v$ in the formula of Theorem 2 and let v approach 0.

Then the 2nd term on the left-hand side of the formula of Theorem 2 contains

$$\zeta_K(1-v)\Gamma\big(\tfrac12(1-v)\big)^{r_1}\Gamma(1-v)^{r_2}$$

and Lang ([8], Theorem 3, p. 260) says that

$$\zeta_K(1-v)\Gamma\big(\tfrac12(1-v)\big)^{r_1}\Gamma(1-v)^{r_2} \\ = -(2^{-r_2}d_K^{1/2}\pi^{-m/2})^{v-1}2^{r_1}w_K^{-1}h_K R_K(1-v)^{-1}v^{-1} + E(v),$$

where $E(0)$ is positive and finite. Now in the formula of Theorem 2 this quantity which we have just calculated is multiplied by

$$(x^{v/2} - x^{-v/2})d_K^{-1/2}2^{r_2}\pi^{m/2}.$$

And clearly

$$\lim_{v\to0}(x^{v/2} - x^{-v/2})v^{-1} = \log x.$$

So the 2nd term in the left-hand side of the formula of Theorem 2 has the following limit as s approaches 1 or v approaches 0:

$$-(w_K d_K)^{-1}(2\pi)^m h_K R_K \log x.$$

Theorem 2 then says that

$$\zeta_K(2)(x-x^{-1})-(w_K d_K)^{-1}(2\pi)^m h_K R_K \log x$$
$$= d_K^{-1/2} 2^{r_1+3r_2}\pi^m \sum_{0\neq u\in\mathfrak{d}_K^{-1}} |Nu|^{1/2}\sigma_{-1}(u\mathfrak{d}_K)\times$$
$$\times\Big\{x^{-1/2}\prod_{j=1}^{r_1+r_2} K_{e_j/2}(2\pi e_j x_j^{-1}|u^{(j)}|)-x^{1/2}\prod_{j=1}^{r_1+r_2} K_{e_j/2}(2\pi e_j x_j|u^{(j)}|)\Big\}.$$

Now solve for $h_K R_K$ to complete the proof. ■

It may be instructive to consider the case that K is quadratic.

If K is an *imaginary quadratic field*, then

$$\text{(2.2)}\quad h_K = w_K d_K \zeta_K(2)(2\pi)^{-2}(x-x^{-1})(\log x)^{-1} -$$
$$-w_K d_K^{1/2} 2(\log x)^{-1}\sum_{0\neq u\in\mathfrak{d}_K^{-1}} |u|\,\sigma_{-1}(u\mathfrak{d}_K)\{x^{-1}K_1(4\pi x^{-1}|u|)-xK_1(4\pi x|u|)\}.$$

If K is a *real quadratic field*, then

$$\text{(2.3)}\quad h_K R_K = \tfrac{1}{2}\pi^{-2} d_K\zeta_K(2)(x-x^{-1})(\log x)^{-1} -$$
$$-\tfrac{1}{2} d_K^{1/2}(\log x)^{-1}\sum_{0\neq u\in\mathfrak{d}_K^{-1}}\sigma_{-1}(u\mathfrak{d}_K)\times$$
$$\times\{\exp[-2\pi(|u|x_1^{-1}+|u'|x_2^{-1})]-\exp[-2\pi(|u|x_1+|u'|x_2)]\}.$$

Note that the positive numbers $x_1,\ldots,x_{r_1+r_2}$ of the preceeding formulas are arbitrary. Thus one might expect to make some statement about the size of $h_K R_K$ (or even h_K) as d_K approaches infinity while the degree of K over the rationals is held fixed (or under the hypothesis of the Brauer–Siegel theorem) by choosing the $x_1,\ldots,x_{r_1+r_2}$ wisely.

If we choose $x_1 = x$ and $x_2 = x_3 = \ldots = x_{r_1+r_2} = 1$ and we assume that x is greater than 1, then Theorem 3 yields the formula

$$\text{(2.4)}\quad h_K R_K = w_K d_K\zeta_K(2)(2\pi)^{-m}(x-x^{-1})(\log x)^{-1} -$$
$$-w_K d_K^{1/2}2^{r_2}(\log x)^{-1}\sum_{0\neq u\in\mathfrak{d}_K^{-1}}|Nu|^{1/2}\sigma_{-1}(u\mathfrak{d}_K)\Big(\prod_{j=2}^{r_1+r_2}K_{e_j/2}(2\pi e_j|u^{(j)}|)\Big)\times$$
$$\times\{x^{-1/2}K_{e_1/2}(2\pi e_1 x^{-1}|u^{(1)}|)-x^{1/2}K_{e_1/2}(2\pi e_1 x|u^{(1)}|)\}.$$

Then one possible mode of procedure is to let x approach 1, obtaining

$$\text{(2.5)}\quad h_K R_K = 2w_K d_K\zeta_K(2)(2\pi)^{-m}+w_K d_K^{1/2}2^{r_2}\sum_{0\neq u\in\mathfrak{d}_K^{-1}}|Nu|^{1/2}\sigma_{-1}(u\mathfrak{d}_K)\times$$
$$\times\Big(\prod_{j=2}^{r_1+r_2}K_{e_j/2}(2\pi e_j|u^{(j)}|)\Big)M_{e_1/2}(2\pi e_1|u^{(1)}|),$$

where $M_s(z) = K_s(z)+2z\dfrac{d}{dz}K_s(z)$, as in the corollary to Theorem 2.

Now let us suppose that K has at least one real conjugate field so that $e_1 = 1$. Since $K_{1/2}(z) = (\pi/2z)^{1/2}\exp(-z)$, we can simplify (2.5) as $M_{1/2}(z) = -(2\pi z)^{1/2}\exp(-z)$. The result is that $e_1 = 1$ implies that

$$\text{(2.6)}\quad h_K R_K = 2w_K d_K\zeta_K(2)(2\pi)^{-m} -$$
$$-w_K d_K^{1/2}2^{r_2+1}\pi\sum_{0\neq u\in\mathfrak{d}_K^{-1}}|u^{(1)}|^{1/2}|Nu|^{1/2}\sigma_{-1}(u\mathfrak{d}_K)\times$$
$$\times\Big(\prod_{j=2}^{r_1+r_2}K_{e_j/2}(2\pi e_j|u^{(j)}|)\Big)\exp\{-2\pi|u^{(1)}|\}.$$

If the field K is totally real so that all $e_j = 1$, $j = 1,\ldots,r_1 = m$, we obtain

$$\text{(2.7)}\quad h_K R_K = 2w_K d_K\zeta_K(2)(2\pi)^{-m} -$$
$$-2^{2-r_1-r_2}w_K d_K^{1/2}\pi\sum_{0\neq u\in\mathfrak{d}_K^{-1}}|u^{(1)}|\,\sigma_{-1}(u\mathfrak{d}_K)\exp\Big\{-2\pi\sum_{j=1}^{r_1+r_2}|u^{(j)}|\Big\}.$$

The upper bound on $h_K R_K$ for fields K with $e_1 = 1$ coming trivially from (2.6) is $2w_K d_K\zeta_K(2)(2\pi)^{-m}$ which is not very interesting in terms of the power of d_K appearing given the easy part of the Brauer–Siegel theorem which is proved by Lang [8], p. 261, using merely the formula for the analytic continuation of the Dedekind zeta function. Here the upper bound obtained for $h_K R_K$ is $C^m d_K^{1/2} d_K^{1/2a}(1+a)$ for all $a\geqslant 1$. Here C is a universal constant.

In the Brauer–Siegel Theorem ([8], pp. 260 ff., 321ff.) it is, of course, the lower bound on $h_K R_K$ which is difficult. For totally real fields K (2.7) implies that in order to obtain a nice lower bound for $h_K R_K$ one needs to find a good upper bound for sums of the form

$$\text{(2.8)}\quad \sum_{0\neq u\in\mathfrak{d}_K^{-1}}\exp\Big\{a\sum_{j=1}^{r_1+r_2}|u^{(j)}|\Big\},\qquad a>0.$$

We leave this question open.

Another interesting question is that of the possibility of eliminating the regulator R_K from formula (2.4). It might at first appear that this could be done as follows. Again we make the hypothesis that $e_1 = 1$ for simplicity. Then use the fact that $K_{1/2}(z) = (\pi/2z)^{1/2}\exp(-z)$ and the mean value theorem to obtain

$$x^{-1/2}K_{1/2}(2\pi x^{-1}|u^{(1)}|) - x^{1/2}K_{1/2}(2\pi x|u^{(1)}|)$$
$$= \tfrac{1}{2}|u^{(1)}|^{-1/2}\{\exp(-2\pi x^{-1}|u^{(1)}|) - \exp(-2\pi x|u^{(1)}|)\}$$
$$= \pi|u^{(1)}|^{1/2}(x - x^{-1})\exp(-2\pi|u^{(1)}|y_u),$$

where $x^{-1} < y_u < x$ and y_u depends on $u \in \mathfrak{d}_K^{-1}$. Thus (2.4) implies that when $e_1 = 1$

$$(2.9)\qquad h_K R_K = (x - x^{-1})(\log x)^{-1} \times$$
$$\times\Big\{w_K d_K \zeta_K(2)(2\pi)^{-m} - w_K d_K^{1/2} 2^{r_2}\pi \sum_{0 \neq u \in \mathfrak{d}_K^{-1}} |u^{(1)}|^{1/2}|Nu|^{1/2}\sigma_{-1}(u\mathfrak{d}_K) \times$$
$$\times \exp(-2\pi|u^{(1)}|y_u)\prod_{j=2}^{r_1+r_2} K_{e_j/2}(2\pi e_j|u^{(j)}|)\Big\}.$$

Now to eliminate any R_K greater than 2 we can choose x such that

$$R_K = f(x) = (x - x^{-1})(\log x)^{-1} = 2\sinh(\log x)(\log x)^{-1}.$$

For $f(x)$ is an increasing function for x greater than 1 and $f(1) = 2$. Also $\lim_{x\to\infty} f(x) = \infty$. Unfortunately the regulator will still be hiding in the equation (2.9) in the form of y_u. For y_u lies between x and x^{-1} and for large R_K it is possible that y_u may be small. Thus there is still much work to be done in order to obtain some concrete results from these formulas.

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Added in proof. Another reference is

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Here and in [15], I, p. 177, results like Theorem 2 are derived.

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