Now suppose that σ' consists of all 3-term geometric progressions with integral ratio. We wish to show $\underline{d}(\sigma') < \overline{d}(\sigma')$. Let U be the sequence of positive integers which have no prime divisor different from 2 or 3. Denote by σ^* the system of those progressions in σ' which have ratio in U. It is not difficult to check that

$$\max\Big\{\sum_{u\in A}\frac{1}{u} : A\ \sigma^*\text{-free},\ A\subset\{2^{r_1}3^{r_2}: r_1, r_2 = 0, 1, 2\}\Big\} = 2.$$

Thus

$$a = \lim_{n\to\infty}\lambda^U_{\sigma^*}(n) \leqslant 2\sum_{j=0}^{\infty}\frac{1}{2^{3j}}\sum_{k=0}^{\infty}\frac{1}{3^{3k}} = \frac{8}{7}\,\frac{27}{13}$$

and, by (11),

$$\underline{d}(\sigma') \leqslant \underline{d}(\sigma^*) \leqslant \Big(1-\frac{1}{2}\Big)\Big(1-\frac{1}{3}\Big)\frac{8}{7}\,\frac{27}{13} = \frac{72}{91} = 0.791\ldots$$

On the other hand the set

$$\Big(\frac{n}{32}, \frac{n}{27}\Big) \cup \Big(\frac{n}{24}, \frac{n}{12}\Big) \cup \Big(\frac{n}{9}, \frac{n}{8}\Big) \cup \Big(\frac{n}{4}, n\Big)$$

is σ'-free in $[1, n]$. Hence

$$\overline{d}(\sigma') = \tau_{\sigma'} \geqslant \frac{5}{864} + \frac{1}{24} + \frac{1}{72} + \frac{3}{4} = \frac{701}{864} = 0.811\ldots$$

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Some remarks on Goldbach's problem

by

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In this paper we shall prove by a modification of Chen's work ([3]) that every sufficiently large even integer x is written as a sum of a prime and a natural number which has at most one prime factor less than $x^{1089/2089}$.

1. Let x be a large even integer. Let $G_2(x)$ be the number of primes $p \leqslant x$ such that $x-p$ has at most two prime factors. Chen ([3]) has proved that

$$(1)\qquad G_2(x) \geqslant \frac{0.67\,xC_x}{(\log x)^2},\quad \text{where}\quad C_x = \prod_{\substack{p\mid x\\ p>2}}\frac{p-1}{p-2}\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big).$$

In fact, if we put $G_2(x, I)$ — the number of primes $p \leqslant x$ such that $x-p$ is a prime or $x-p = p_1p_2$ with primes p_1 and p_2 satisfying $p_1 \notin I$ and $p_1 \leqslant p_2$, for a subset I of $(1, x^{1/2}]$, he has proved that $G_2\big(x, (1, x^{1/10}]\big) \geqslant 0.67xC_x/(\log x)^2$. (Further Halberstam [6] or [7] has shown that 0.67 can be replaced by 0.689.) Now we wish to maximize $I \subset (1, x^{1/2}]$ such that $G_2(x, I) \geqslant AxC_x/(\log x)^2$, where A is some positive absolute constant. To study this we use the following mean value theorem which is similar to Bombieri's one.

Lemma 0. *Assume that $M+N \ll x^{1/2}$. For an arbitrarily large constant A, there exist positive constants $B = B(A)$ and $E = E(A)$ such that if $M \geqslant (\log x)^E$, and $b(m) \ll (\log x)^C$ with some positive constant C for any m in $M < m \leqslant M+N$, then*

$$\sum_{d\leqslant x^{1/2}/(\log x)^B}\ \max_{(a,d)=1}\ \max_{(M+N)^{1+\theta}<y\leqslant x}\Bigg|\sum_{\substack{m=M+1\\(m,d)=1}}^{M+N} b(m)\Bigg(\sum_{\substack{n\leqslant y/m\\ n\equiv am^*(d)}}\Lambda(n) - \frac{1}{\varphi(d)}\cdot\frac{y}{m}\Bigg)\Bigg| \ll x/(\log x)^A,$$

where θ is an arbitrarily given positive number, $n \equiv am^*(d)$ means $n \equiv am^* \pmod d$, and m^* satisfies $mm^* \equiv 1\ (d)$.

As a corollary to this we get

Corollary 1. *Assume that $M+N \ll x^{1/2}$, every prime divisor of m is $\gg x^{\varepsilon'}$ for arbitrarily given small positive ε', and that real numbers $b(m)$ satisfy $b(m) \ll (\log x)^C$ as before. Then*

$$\sum_{m=M+1}^{M+N} b(m) \sum_{\substack{n\le x/m\\ (x-mn,Q)=1}} \Lambda(n) \le \frac{8(1+\varepsilon)xC_x}{\log x}\left(\sum_{m=M+1}^{M+N}\frac{b(m)}{m}\right)+O\left(x/(\log x)^A\right),$$

where $Q = \prod_{2\le p\le x^{1/4}} p$, A is an arbitrarily large constant and ε is an arbitrarily small positive number.

We may remark (cf. Theorem 1 in [13]) here that by a slight modification of the proof of Lemma 0, Lemma 0 and Corollary 1 can be extended to the case $M+N \ll x^{1-\theta}$ such that it includes Chen's estimates (Lemma 8 of [3] and consequently (1) in this paper). As special cases of Corollary 1 we get

Corollary 2 (cf. Theorem 2 in [13]). *For $a > 2$,*

$$\begin{aligned} M_1(a) &\equiv \text{'the number of primes } p\le x \text{ such that } x-p = p_1p_2 \text{ with primes } \\ &\qquad p_1 \text{ and } p_2 \text{ satisfying } x^{1/a} < p_1 \le p_2 \le x^{1/2}\text{'} \\ &= o\left(x/(\log x)^2\right). \end{aligned}$$

Corollary 3. *For $a \ge \beta \ge 2$,*

$$\begin{aligned} M_2(a,\beta) &\equiv \text{'the number of primes } p\le x \text{ such that } x-p = p_1p_2 \text{ with } \\ &\qquad \text{primes } p_1 \text{ and } p_2 \text{ satisfying } x^{1/a} < p_1 \le x^{1/\beta} < p_2\text{'} \\ &\le 8(1+\varepsilon)\left(\log\frac{a-1}{\beta-1}\right)xC_x/(\log x)^2. \end{aligned}$$

Corollary 4. *For a and β satisfying $a \ge 2 \ge \beta \ge a/(a-1)$, 'the number of primes $p \le x$ such that $x-p = p_1p_2$ with primes p_1 and p_2 satisfying $x^{1/a} < p_1 \le x^{1/2} < p_2 \le x^{1/\beta}$'*

$$\le 8(1+\varepsilon)\left(\log\frac{1}{\beta-1}\right)xC_x/(\log x)^2.$$

Now we define the length of an interval of the form $(x^{1/a}, x^{1/\beta}]$ by

$$\delta\left((x^{1/a}, x^{1/\beta}]\right) = \log\frac{a-1}{\beta-1}.$$

If I is a finite sum of the intervals of the above type, we define

$$\delta(I) = \sum \log\frac{a-1}{\beta-1}.$$

From these corollaries and (1) with the constant 0.689, we get

Corollary 5. *For any subset I of $(x^{1/10}, x^{1/2}]$ of the above type, if $\delta(I) \le 0.0862$, then*

$$G_2\left(x, (1, x^{1/10}]\cup I\right) \ge AxC_x/(\log x)^2,$$

where A is some positive absolute constant which may depend on $\delta(I)$.

In particular, if we take $I = (x^{1/10}, x^{(1/10)+b}]$, then we get

Corollary 6. $G_2\left(x, (1, x^{1/9.26}]\right) \ge AxC_x/(\log x)^2$.

Also if we take $I = (x^{1/2-b}, x^{1/2}]$, then we get our main

Theorem. *Every sufficiently large even integer x is written as a sum of a prime and a natural number which has at most one prime factor less than $x^{1089/2089}$.*

We shall prove Lemma 0 in § 2 and corollaries in § 3. We may remark here that the above corollaries and theorem can be stated also for twin prime problem. Although we omit writing these we may mention the following

Theorem. *Let a be an arbitrarily given even integer. Then infinitely often $p+a$ has at most one prime factor less than $p^{1089/2089}$.*

2. The proof of Lemma 0

2.1. Lemmas for the proof of Lemma 0

Lemma 1. *For $T \ge 1$,*

$$\sum_{D<d\le V}\frac{1}{\varphi(d)}\sum_{\chi}{}^{*}\int_{-T}^{T}\left|\sum_{n=1}^{\infty}a(n)\chi(n)n^{-it}\right|^2\frac{dt}{\tau} \ll \sum_{n=1}^{\infty}\left(V\log T+\frac{n}{D}\right)|a(n)|^2,$$

where we put $\tau = |t|+1$, and in $\sum_{\chi}^{*}\chi$ runs over all primitive characters of the modulus d.

(Cf. Gallagher's Theorem 3 of [5].)

Lemma 2. *For $T \ge 2$,*

$$\sum_{D<d\le V}\frac{1}{\varphi(d)}\sum_{\chi}{}^{*}\int_{-T}^{T}|L'(\tfrac{1}{2}+it,\chi)|^4\,dt \ll VT(\log VT)^8.$$

Proof. Since $|L'(\frac{1}{2}+it,\chi)|^4 \ll (\log dT)^5\int_C |L(\xi,\chi)|^4|d\xi|$, we have

$$\begin{aligned}&\sum_{D<d\le V}\frac{1}{\varphi(d)}\sum_{\chi}{}^{*}\int_{-T}^{T}|L'(\tfrac{1}{2}+it,\chi)|^4\,dt \\ &\quad\ll \sum_{D<d\le V}\frac{(\log dT)^5}{\varphi(d)}\int_{1/2-(\log dT)^{-1}}^{1/2+(\log dT)^{-1}}\left(\sum_{\chi}{}^{*}\int_{-T-1}^{T+1}|L(\sigma+it,\chi)|^4\,dt\right)d\sigma,\end{aligned}$$

where C is the circle of the radius $(\log dT)^{-1}$ with the center $1/2+it$. Using Theorem 10.1 of H. L. Montgomery [9], we get our conclusion. ∎

LEMMA 3. *Let x be an even integer. If we put $\lambda_1 = 1$ and*

$$\lambda_d = \begin{cases} \dfrac{\mu(d)}{\prod\limits_{p\mid d}\dfrac{p-2}{p-1}}\left\{\sum\limits_{\substack{k\leqslant z/d\\(k,xd)=1}}\dfrac{\mu^2(k)}{f(k)}\right\}\dfrac{1}{S} & \text{for } 1<d\leqslant z,\\ 0 & \text{for } d>z,\end{cases}$$

where we put

$$S=\sum_{\substack{k\leqslant z\\(k,x)=1}}\frac{\mu^2(k)}{f(k)} \quad \text{and} \quad f(k)=\varphi(k)\prod_{p\mid k}\frac{p-2}{p-1},$$

then

(i) $$\frac{1}{S}=\sum_{\substack{d_1<z\\(d_1,x)=1}}\sum_{\substack{d_2<z\\(d_2,x)=1}}\frac{\lambda_{d_1}\lambda_{d_2}}{\varphi\left(\dfrac{d_1d_2}{(d_1,d_2)}\right)},$$

(ii) $S\geqslant(\log z)/(2C_x)+O(1)$,

(iii) $|\lambda_d|\leqslant 1$ *for* $(d,x)=1$.

(Cf. pages 162 and 171 of [3].)

2.2. The proof of Lemma 0

2.2.1. In the following A is always an arbitrarily large constant and C is appropriate not so large constant. We put $l=\log x$ and

$$\psi(y,\chi)=\sum_{n\leqslant y}\Lambda(n)\chi(n).$$

Then for $(a,d)=1$, $(m,d)=1$, and $m<y$

$$\sum_{\substack{n\leqslant y/m\\ n\equiv am^*(d)}}\Lambda(n)-\frac{1}{\varphi(d)}\cdot\frac{y}{m}=\frac{1}{\varphi(d)}\left(\sum_{n\leqslant y/m}\Lambda(n)-\frac{y}{m}\right)-$$

$$-\frac{1}{\varphi(d)}\sum_{\substack{n\leqslant y/m\\(n,d)>1}}\Lambda(n)+\frac{1}{\varphi(d)}\sum_{\chi\neq\chi_0}\overline{\chi}(a)\chi(m)\psi\left(\frac{y}{m},\chi\right)$$

$$=O\left(\frac{y}{\varphi(d)m}\left(\log\frac{y}{m}\right)^{-E}\right)+O\left(\frac{l(\log d)}{\varphi(d)}\right)+\frac{1}{\varphi(d)}\sum_{\chi\neq\chi_0}\overline{\chi}(a)\chi(m)\psi\left(\frac{y}{m},\chi\right)$$

with arbitrarily large E. If χ is induced by the primitive character χ^*, then $\psi(y/m,\chi)=\psi(y/m,\chi^*)+O(l\log d)$.

Hence for $(M+N)^{1+\theta}<y\leqslant x$,

$$\max_{(a,d)=1}\left|\sum_{\substack{m=M+1\\(m,d)=1}}^{M+N}b(m)\left(\sum_{\substack{n\leqslant y/m\\ n\equiv am^*(d)}}\Lambda(n)-\frac{y}{\varphi(d)m}\right)\right|$$

$$\ll\frac{1}{\varphi(d)}\sum_{\chi\neq\chi_0}\left|\sum_{\substack{m=M+1\\(m,d)=1}}^{M+N}b(m)\chi^*(m)\psi\left(\frac{y}{m},\chi^*\right)\right|+O\left(\frac{xl^{-E}}{\varphi(d)}\right)+O(l^Cx^{1/2}\log d).$$

Hence we get

$$\sum_{d\leqslant x^{1/2}l^{-B}}\max_{(a,d)=1}\max_{(M+N)^{1+\theta}<y\leqslant x}\left|\sum_{\substack{m=M+1\\(m,d)=1}}^{M+N}b(m)\left(\sum_{\substack{n\leqslant y/m\\ n\equiv am^*(d)}}\Lambda(n)-\frac{y}{\varphi(d)m}\right)\right|$$

$$\ll l\max_{1<b\leqslant x^{1/2}l^{-B}}N_b+O(xl^{-A})$$

by taking sufficiently large B, where we put

$$N_b=\sum_{d\leqslant x^{1/2}l^{-B}}\frac{1}{\varphi(d)}\sum_{\chi}{}^{*}\max_{y}\left|\sum_{\substack{m=M+1\\(m,b)=1}}^{M+N}b(m)\chi(m)\psi\left(\frac{y}{m},\chi\right)\right|.$$

Now, by Siegel–Walfisz theorem (pp. 134 and 144 of [8]),

$$\max_{y}\left|\sum_{\substack{m=M+1\\(m,b)=1}}^{M+N}b(m)\chi(m)\psi(y/m,\chi)\right|\ll\max_{y}\sum_{m=M+1}^{M+N}b(m)y/\left(m(\log(y/m))^{D'}\right)\ll xl^{-A}$$

for $\chi\neq\chi_0$ and $d<l^D$ with arbitrarily large constants D and D'. Hence we get

$$N_b\ll\sum_{j=1}^{J}\sum_{k=1}^{K}N_{bjk}+xl^{-A},$$

where we put

$$N_{bjk}=\sum_{Q_{j-1}<d\leqslant Q_j}\frac{1}{\varphi(d)}\sum_{\chi}{}^{*}\max_{y}\left|\sum_{\substack{M_{k-1}<m\leqslant M_k\\(m,b)=1}}b(m)\chi(m)\psi(y/m,\chi)\right|,$$

$Q_j=2^jl^D$ for $j=0,1,2,\ldots,J$, J satisfies $2^{J-1}l^D<x^{1/2}l^{-B}\leqslant 2^Jl^D$, $M_k=2^kM$ for $k=0,1,2,\ldots,K$, and K satisfies $2^{K-1}M<M+N\leqslant 2^KM$. Now for $a=1+1/\log x$ and $T=x^{C'}$ with sufficiently large C',

$$\sum_{\substack{M_{k-1}<m\leqslant M_k\\(m,b)=1}}b(m)\chi(m)\psi\left(\frac{y}{m},\chi\right)=\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{y^s}{s}\left(-\frac{L'}{L}(s,\chi)\right)f_k(s)\,ds+O(x^{\varepsilon'}),$$

where we put $f_k(s) = \sum_{\substack{M_{k-1} < m \leqslant M_k \\ (m,b)=1}} \chi(m) b(m)/m^s$ and ε' is an arbitrarily small positive number. In each interval $Q_{j-1} < d \leqslant Q_j$, we express

$$\frac{L'}{L} = L'\left(\frac{1}{L} - S\right) + L'S, \quad \text{where} \quad S = S_{H_j} = \sum_{n=1}^{H_j} \mu(n)\chi(n)/n^s.$$

Then

$$\max_y \Big| \sum_{\substack{M_{k-1} < m \leqslant M_k \\ (m,b)=1}} b(m)\chi(m)\psi(y/m, \chi)\Big|$$

$$\ll xl^2 \int_{\substack{-T \\ (\sigma=a)}}^{T} \left|\frac{1}{L} - S\right| |f_k| \frac{dt}{\tau} + x^{1/2} \int_{\substack{-T \\ (\sigma=1/2)}}^{T} |L'||S||f_k| \frac{dt}{\tau},$$

where $\tau = |t|+1$ and $\int_{\substack{-T \\ (\sigma=a)}}^{T} f\,dt$ means $\int_{-T}^{T} f(a+it)\,dt$.

2.2.2. We assume first that $Q_j < M_k$. For simplicity we denote

$$\left(\sum_{Q_{j-1} < d \leqslant Q_j} \frac{1}{\varphi(d)} \sum_{\chi}{}^{*} \int_{\substack{-T \\ (\sigma=\beta)}}^{T} |g|^{2h} \frac{dt}{\tau}\right)^{1/2h} \quad \text{by} \quad I(j, \beta, g, h).$$

Then by Schwartz's inequality, we get

$$N_{bjk} \ll xl^2 I\left(j, a, \frac{1}{L} - S, 1\right) I(j, a, f_k, 1) +$$
$$+ x^{1/2} I(j, \tfrac{1}{2}, f_k, 1) I(j, \tfrac{1}{2}, L', 2) I(j, \tfrac{1}{2}, S, 2).$$

Since $\frac{1}{L} - S = \sum_{n > H_j} \mu(n)\chi(n)/n^s$, by Lemma 1 we get

$$I\left(j, a, \frac{1}{L} - S, 1\right)^2 \ll l \sum_{n > H_j} \left(Q_j + \frac{n}{Q_j}\right) / n^{2+(2/\log x)} \ll l^2 \left(\frac{Q_j}{H_j} + \frac{1}{Q_j}\right).$$

Similarly, we get

$$I(j, a, f_k, 1)^2 \ll l^C \left(\frac{Q_j}{M_k} + \frac{1}{Q_j}\right) \quad \text{and} \quad I(j, \tfrac{1}{2}, f_k, 1)^2 \ll l^C \left(Q_j + \frac{M_k}{Q_j}\right).$$

Since $S^2 = \sum_{n=1}^{H_j^2} j(n)\chi(n)/n^s$ with $|j(n)| \leqslant \tau(n) = \sum_{l|n} 1$, we get

$$I(j, \tfrac{1}{2}, S, 2)^4 \ll l^5 \left(Q_j + \frac{H_j^2}{Q_j}\right).$$

By Lemma 2, we get $I(j, \frac{1}{2}, L', 2)^4 \ll Q_j l^C$. We choose $H_j = Q_j l^D$. Then we get $N_{bjk} \ll xl^{-A}$ by taking sufficiently large B and D.

2.2.3. Second we assume that $Q_j > M_k$. Then from 2.2.2

$$N_{bjk} \ll xl^C \left(\frac{Q_j}{H_j} + \frac{1}{Q_j}\right)^{1/2} \left(\frac{Q_j}{M_k} + \frac{1}{Q_j}\right)^{1/2} +$$
$$+ x^{1/2} I(j, \tfrac{1}{2}, S, 1) I(j, \tfrac{1}{2}, f_k, 2) I(j, \tfrac{1}{2}, L', 2).$$

By Lemma 1,

$$I(j, \tfrac{1}{2}, f_k, 2)^4 \ll l^C \left(Q_j + \frac{M_k^2}{Q_j}\right) \sum_{M_{k-1}^2 < n \leqslant M_k^2} \frac{\tau^2(n)}{n} \ll l^C \left(Q_j + \frac{M_k^2}{Q_j}\right).$$

We get also $I(j, \frac{1}{2}, S, 1)^2 \ll l^2 (Q_j + (H_j/Q_j))$. We choose $H_j = Q_j l^{2D}/M_k$. Then we get $N_{bjk} \ll xl^{-A}$ by taking sufficiently large B, D and E in $M \gg l^E$.

2.2.4. Combining 2.2.1, 2.2.2 and 2.2.3, we get $N_b \ll xl^{-A}$. Hence we get our conclusion. ■

3. Proofs of corollaries. In this paragraph ε, ε' and ε'' are always arbitrarily small positive numbers.

3.1. Proof of Corollary 1. Let λ_d be given as in Lemma 3, where we take $z = x^{1/4-\varepsilon/2}$. Then

$$\sum_{m=M+1}^{M+N} b(m) \sum_{\substack{n \leqslant x/m \\ (x-mn, Q)=1}} \Lambda(n) \leqslant \sum_{m=M+1}^{M+N} b(m) \sum_{n \leqslant x/m} \Lambda(n) \Big(\sum_{\substack{d|(x-mn, Q) \\ (d,x)=1}} \lambda_d\Big)^2$$

$$= \sum_{(d_1,x)=1} \sum_{(d_2,x)=1} \lambda_{d_1}\lambda_{d_2} \sum_{\substack{m \\ (m, d_1 d_2)=1}} b(m) \sum_{\substack{n \leqslant x/m \\ n \equiv xm^*([d_1,d_2])}} \Lambda(n)$$

$$= \left(\sum_{(d_1,x)=1} \sum_{(d_2,x)=1} \frac{\lambda_{d_1}\lambda_{d_2}}{\varphi([d_1, d_2])}\right) \left(\sum_m \frac{b(m)}{m}\right) x -$$

$$- x \sum_{(d_1,x)=1} \sum_{(d_2,x)=1} \frac{\lambda_{d_1}\lambda_{d_2}}{\varphi([d_1, d_2])} \sum_{(m, d_1 d_2) > 1} \frac{b(m)}{m} +$$

$$+ \sum_{(d_1,x)=1} \sum_{(d_2,x)=1} \lambda_{d_1}\lambda_{d_2} \sum_{\substack{m \\ (m, d_1 d_2)=1}} b(m) \Big(\sum_{\substack{n \leqslant x/m \\ n \equiv xm^*([d_1,d_2])}} \Lambda(n) - \frac{1}{\varphi([d_1, d_2])} \cdot \frac{x}{m}\Big),$$

where we put $[d_1, d_2] = d_1 d_2/(d_1, d_2)$. The second term of the last equality is

$$\ll x^{1+\varepsilon'}\sum_{d}1/d\sum_{(m,d)>1}1/m\ll x^{1+\varepsilon'}\sum_{m}1/m\sum_{p|m}\sum_{\substack{p|d\\ d\leqslant x^{1/2-\varepsilon}}}1/d\ll x^{1-\varepsilon''}.$$

The third term is

$$\ll\sum_{\substack{d\leqslant x^{1/2-\varepsilon}\\ (d,x)=1}}3^{\nu(d)}|\mu(d)|\left|\sum_{\substack{m\\ (m,d)=1}}b(m)\left(\sum_{\substack{n\leqslant x/m\\ n\equiv xm^*(d)}}\Lambda(n)-\frac{x}{\varphi(d)m}\right)\right|,$$

where $\nu(d)$ is the number of prime factors of d. By a standard argument (p. 20 of [11]) from Lemma 0, the third term becomes

$$\ll x/(\log x)^{A}.$$

Hence by Lemma 3 we get our conclusion. ∎

3.2. Proof of Corollary 2

$$M_1(\alpha)\leqslant\sum_{\substack{x^{1/\alpha}<p_1\leqslant p_2\leqslant x^{1/2}\\ (x-p_1p_2,Q)=1}}1+O(x^{1/4})$$

$$\leqslant\sum_{x^{1/\alpha}<p_1\leqslant x^{1/2}}\sum_{p_2\leqslant p_1^{1-\varepsilon}}1+\sum_{p_1=o(x^{1/2})}\sum_{p_1^{1-\varepsilon}<p_2\leqslant p_1}1+\sum_{x^{1/2}\leqslant p_1\leqslant x^{1/2}}\sum_{\substack{p_1^{1-\varepsilon}<p_2\leqslant p_1\\ (x-p_1p_2,Q)=1}}1$$

$$\leqslant o\left(x/(\log x)^2\right)+\frac{1}{1-\varepsilon}\sum_{x^{1/2}\leqslant p\leqslant x^{1/2}}\frac{1}{\log p}\sum_{\substack{n\leqslant x/p\\ (x-np,Q)=1}}\Lambda(n)$$

$$\leqslant\frac{8(1+\varepsilon)xC_x}{\log x}\left(\sum_{x^{1/2}\leqslant p\leqslant x^{1/2}}\frac{1}{p\log p}\right)+o\left(x/(\log x)^2\right)=o\left(x/(\log x)^2\right).$$

3.3. Proof of Corollaries 3 and 4

$$M_2(\alpha,\beta)=\sum_{x^{1/\alpha}<p_1\leqslant x^{1/\beta}<p_2\leqslant(x/p_1)^{1-\varepsilon}}1+\sum_{x^{1/\alpha}<p_1\leqslant x^{1/\beta}}\sum_{(x/p_1)^{1-\varepsilon}<p_2\leqslant x/p_1}1$$

$$\leqslant\frac{1}{1-\varepsilon}\sum_{x^{1/\alpha}<p\leqslant x^{1/\beta}}\frac{1}{\log(x/p)}\sum_{\substack{n\leqslant x/p\\ (x-pn,Q)=1}}\Lambda(n)+O(x^{1-\varepsilon'})$$

$$\leqslant 8(1+\varepsilon)\left(xC_x/(\log x)^2\right)\log\frac{\alpha-1}{\beta-1}.$$

Similarly we get Corollary 4.

3.4. Proof of Corollary 5

$$0.689\,xC_x/(\log x)^2\leqslant G_2\left(x,(1,x^{1/10}]\right)$$

$$=G_2\left(x,(1,x^{1/10}]\cup I\right)+\sum_{\substack{p\leqslant x\\ x-p=p_1p_2\\ p_1\in I,\,p_1\leqslant p_2\leqslant x^{1/2}}}1+\sum_{\substack{p\leqslant x\\ x-p=p_1p_2\\ p_1\in I,\,p_1<x^{1/2}\leqslant p_2}}1+O(x^{1/2}).$$

By Corollaries 2 and 3,

$$G_2\left(x,(1,x^{1/10}]\cup I\right)\geqslant\left(0.689-8(1+\varepsilon)\delta(I)\right)\frac{xC_x}{(\log x)^2}.$$

Hence if $0.689>8(1+\varepsilon)\delta(I)$, then we get our conclusion.

4. Concluding remarks

4.1. In the preparation of this paper the author was informed by Professor P. X. Gallagher and Professor H. Halberstam that at the Bordeaux Conference on Number Theory in 1974 Professor H. Halberstam [6] gave a talk about a simplification (which is due to P. M. Ross [12]) of Chen's proof [3] and an improvement (which is due to H. Halberstam [6] or [7]) of the numerical constant which we have mentioned in the introduction.

4.2. We may remark here that we can also get our Corollary 1 from the argument in [6].

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