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Three restricted product-sum partition functions

by

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In [4], Chawla and Maxfield defined and studied the product-sum partition function $p(n, m)$, the number of otherwise unrestricted partitions of n such that the product of the summands is m. They also studied the allied arithmetic function $C_p(n)$, which is the cardinality of the set C_P of integers m such that $p(n, m) \geqslant 1$. In the present paper, we study three restricted product-sum partition functions $P_S(n, m)$, $P_R(n, m)$ and $P_Q(n, m)$, namely, the number of partitions of n such that the product of the summands is m and further for $P_S(n, m)$ the summands come from the set S of non-negative integer powers of primes; for $P_R(n, m)$ the summands are pairwise relatively prime; and for $P_Q(n, m)$ at least one of the summands divides all the others. As in [2], p. 263, let $f(x)$ be the sum of the prime factors of x counting multiplicities, and let $q(n)$ be the number of positive integers x such that $f(x) = n$. Chawla and Maxfield ([4], pp. 104–105) obtained the results that $p(n, m) \geqslant 1$ if and only if $f(m) \leqslant n$; $C_p(n) = q(0)+q(1)+\ldots+q(n)$; and $C_p = \{m \mid f(m) = j, j = 0, 1, \ldots, n\}$. In this paper we derive similar results for our partition functions $P_S(n, m)$, $P_R(n, m)$ and $P_Q(n, m)$, and also obtain upper bounds for the first two of these functions.

First recall the following definitions and notations: The pair of functions $f(x)$ and $q(n)$ were defined in [2], p. 263, as follows:

$$f(1) = 0; \quad f(p_1^{a_1}, \ldots, p_r^{a_r}) = a_1 p_1 + \ldots + a_r p_r;$$
$$q(n) = \sum_{f(x)=n} 1,$$

that is to say, $q(n)$ is the number of positive integers x which satisfy the equation $f(x) = n$. Note that $q(n)$ is also the number of partitions of n into primes, not necessarily distinct.

The pair of functions $h(x)$ and $\delta(n)$ were defined in [5], p. 103, as follows:

$$h(1) = 0; \quad h(p_1^{a_1} \cdot \ldots \cdot p_r^{a_r}) = p_1^{a_1} + \ldots + p_r^{a_r};$$
$$\delta(n) = \sum_{h(x)=n} 1.$$

Note that $\delta(n)$ is the number of partitions of n into powers of distinct primes.

Let P be the set of all primes, N the set of all positive integers, W the set of all non-negative integers, and

$$S = \{p^a \mid p \in P,\ a \in W\},$$

that is, S is the set of all powers of primes, together with 1.

Throughout this paper we use the following canonical representation

$$m = q_1^{b_1} \ldots q_s^{b_s} \quad \text{when} \quad m \geqslant 2.$$

1. Let $P_S(n)$ be the number of partitions of n with summands in S, and let $P_S(n, m)$ be the number of partitions of n over S such that the product of the summands in each partition is m.

Let

$$M(S, n) = \{m \in N \mid P_S(n, m) \geqslant 1\};$$

then

$$|M(S, n)| = \sum_{P_S(n,m) \geqslant 1} 1,$$

that is to say, $|M(S, n)|$ is the number of distinct integers m such that $P_S(n, m) \geqslant 1$.

It is evident that

$$P_S(n) = \sum_{m \in M(S,n)} P_S(n, m),$$

that is to say the set of all the partitions counted by $P_S(n)$ is classified into $|M(S, n)|$ disjoint classes.

LEMMA 1.1. (i) *$P_S(n, m) \geqslant 1$ if and only if $f(m) \leqslant n$;*
(ii) *$P_S(n, m) = P_S(m, m)$ if $1 \leqslant m \leqslant n$;*
(iii) *$P_S(n, m) = 0$ if m is a prime $> n$;*
(iv) *$P_S(n, m) = 1$ if m is a prime $\leqslant n$.*

To prove (i), first assume that $x_1 + \ldots + x_k = n$ and $x_1 \cdot \ldots \cdot x_k = m$. Then

$$f(m) = f(x_1 \cdot \ldots \cdot x_k) = f(x_1) + \ldots + f(x_k) \leqslant x_1 + \ldots + x_k = n.$$

Conversely if $f(m) = b_1 q_1 + \ldots + b_s q_s \leqslant n$, then n can be partitioned in at least one way as

$$n = 1 + \ldots + 1 + q_1 + \ldots + q_1 + q_2 + \ldots + q_2 + \ldots + q_s + \ldots + q_s,$$

where the product of all the summands is m and their sum is n. This proves (i).

To prove (ii), assume that $1 \leqslant m \leqslant n$. Since $f(m) \leqslant m \leqslant n$, it follows from (i) that $P_S(n, m) \geqslant 1$ and $P_S(m, m) \geqslant 1$. Now, every partition counted by $P_S(n, m)$ is of the form $n = 1 + \ldots + 1 + x_t + \ldots + x_k$ for some $t \geqslant 1$, where $x_j \geqslant 2$ for $t \leqslant j \leqslant k$ and where $x_t \cdot \ldots \cdot x_k = m$. It follows that $x_t + \ldots + x_k \leqslant m$, whence this sum can be extended to a partition of m by adding as many 1's as are needed; this partition of m will be among those counted by $P_S(m, m)$. Thus every partition counted by $P_S(n, m)$ corresponds to a unique partition counted by $P_S(m, m)$; and since the converse is clearly true, (ii) follows. If m is prime then the only sum for which the product of the terms is m is of the form $1 + \ldots + 1 + m$; this gives a partition of n if and only if $m \leqslant n$. This proves (iii) and (iv), since 1 and m are in S.

THEOREM 1.2. (i) *$M(S, n) = \{x \in N \mid f(x) = j,\ j = 0, 1, \ldots, n\}$;*
(ii) *$|M(S, n)| = q(0) + q(1) + \ldots + q(n)$.*

Alternatively, $M(S, n)$ is the set of all integers m obtained by multiplying the summands in each of the partitions counted by $q(j)$, $0 \leqslant j \leqslant n$, with the convention that $q(0)$ gives $m = 1$ and $q(1)$ gives no value of m.

This is an immediate consequence of Lemma 1.1 (i), and is an analogue of Theorem 1.2 in [4], p. 105.

We note that for all $m, f(m) \leqslant h(m) \leqslant m$. Hence if $h(m) \leqslant n$ then $f(m) \leqslant n$ and $P_S(n, m) \geqslant 1$. A stronger result than this is the following theorem, in which $p(b)$ is the number of unrestricted partitions of a positive integer b.

THEOREM 1.3. *Let $m = q_1^{b_1} \cdot \ldots \cdot q_s^{b_s}$. Then*

$$P_S(n, m) \leqslant p(b_1) \cdot \ldots \cdot p(b_s),$$

with equality holding if and only if $h(m) \leqslant n$.

To prove this, let $(b_{i1}, \ldots, b_{ik_i})$ be a partition of b_i, $1 \leqslant i \leqslant s$. Then every sum of terms in $S - \{1\}$ whose product is m can be written in the form

$$q_1^{b_{11}} + q_1^{b_{12}} + \ldots + q_1^{b_{1k_1}} + \ldots + q_s^{b_{s1}} + q_s^{b_{s2}} + \ldots + q_s^{b_{sk_s}}. \tag{1}$$

The largest of these is of course $q_1^{b_1} \cdot \ldots \cdot q_s^{b_s} = h(m)$. If $h(m) \leqslant n$, then this sum can be extended to a partition of n by adding an appropriate number of 1's, as can all of the other sums referred to above. If $h(m) > n$, then this sum does not yield a partition of n. Since the number of sums of the form (1) is clearly equal to $p(b_1) \cdot \ldots \cdot p(b_s)$, the theorem is proved.

COROLLARY 1.4. *$P_S(m, m) = p(b_1) \cdot \ldots \cdot p(b_s)$.*

2. Let $P_R(n)$ be the number of partitions of n whose summands are pairwise relatively prime. Let $P_R(n, m)$ be the number of partitions of n of the type

$$x_1 + \ldots + x_k = n \quad \text{and} \quad x_1 \cdot \ldots \cdot x_k = m, \quad \text{where} \quad (x_i, x_j) = 1 \quad \text{for } i \neq j.$$

Let
$$M(R,n) = \{m \in N \mid P_R(n,m) \geqslant 1\};$$
then
$$|M(R,n)| = \sum_{P_R(n,m)\geqslant 1} 1,$$
that is, $|M(R,n)|$ is the number of distinct integers m such that $P_R(n,m) \geqslant 1$. Clearly, $P_R(n) = \sum P_R(n,m)$, the sum being taken over all m in $M(R,n)$; that is to say, the set of all the partitions counted by $P_R(n)$ is classified into $|M(R,n)|$ disjoint classes.

LEMMA 2.1. (i) *$P_R(n,m) \geqslant 1$ if and only if $h(m) \leqslant n$;*
(ii) *$P_R(n,m) = P_R(m,m)$ if $1 \leqslant m \leqslant n$;*
(iii) *$P_R(n,m) = 0$ if m is a prime $> n$;*
(iv) *$P_R(n,m) = 1$ if m is a prime $\leqslant n$.*

To prove (i), first assume that $x_1 + \ldots + x_k = n$, $x_1 \cdot \ldots \cdot x_k = m$, and $(x_i, x_j) = 1$ for $i \neq j$. Then $h(m) = h(x_1 \cdot \ldots \cdot x_k) = h(x_1) + \ldots + h(x_k) \leqslant x_1 + \ldots + x_k = n$, where the second equality follows from the fact that the function h is strictly additive. Conversely, if $h(m) \leqslant n$, then one partition of n of the desired form is $1 + \ldots + 1 + q_1^{b_1} + \ldots + q_s^{b_s}$, where the number of 1's added on is $n - h(m)$. This proves (i).

The proofs of (ii), (iii) and (iv) are analogous to those for Lemma 1.1 (ii), (iii) and (iv), and are therefore omitted.

An immediate consequence of Lemma 2.1 (i) is the following theorem.

THEOREM 2.2. (i) $M(R,n) = \{x \in N \mid h(x) = j,\ j = 0, 1, \ldots, n\}$;
(ii) $|M(R,n)| = \delta(0) + \ldots + \delta(n)$.

Alternatively, $M(R,n)$ is the set of all integers m obtained by multiplying the summands in each of the partitions counted by $\delta(j)$, $0 \leqslant j \leqslant n$, with the convention that $\delta(0)$ gives $m = 1$, and $\delta(1)$ and $\delta(6)$ give no values of m.

In the next theorem we show that an upper bound for $P_R(n,m)$ is $A(s)$, where $A(n)$ is the number of partitions of a set with n distinguishable elements. (A partition of a set is any collection of non-empty subsets, pairwise disjoint, whose union is the whole set.) It is known ([1], p. 97, exercise 5) that
$$A(n+1) = \sum_{i=0}^{n} \binom{n}{i} A(i).$$

THEOREM 2.3. *Let $m = q_1^{b_1} \cdot \ldots \cdot q_s^{b_s}$. Then $P_R(n,m) \leqslant A(s)$, with equality holding if and only if $m \leqslant n$.*

To show this, first assume that $n = 1 + \ldots + 1 + x_t + \ldots + x_k$, where $x_j \geqslant 2$ for $t \leqslant j \leqslant k$, where $x_t \cdot \ldots \cdot x_k = m$, and $(x_i, x_j) = 1$ for $i \neq j$.

Then each of the x_j's must be some one of the $q_i^{b_i}$, or else a product of two or more of them. Thus to each such partition of n there corresponds a partition of the set of terms $\{q_1^{b_1}, \ldots, q_s^{b_s}\}$; namely, the jth subset consists of those terms $q_i^{b_i}$ which are factors of x_j, $t \leqslant j \leqslant k$. Conversely, to each partition of the set $\{q_1^{b_1}, \ldots, q_s^{b_s}\}$ there corresponds a partition of n if n is big enough, that is, if n is greater than or equal to the sum $\sum(\prod q_i^{b_i})$, where the sum is taken over the different subsets in the partition and the products are over terms within each subset. Clearly, the partition of this set which yields the largest sum is the trivial partition with only one subset; this corresponds to the sum consisting of the single term $q_1^{b_1} \cdot \ldots \cdot q_s^{b_s} = m$. The total number of such sums is therefore equal to $A(s)$, but only those sums which are less than or equal to n will yield a partition of n. This completes the proof of the theorem.

COROLLARY 2.4. $P_R(m,m) = A(s)$.

3. Let $P_Q(n)$ be the number of partitions of n such that in each partition there is at least one summand which divides the rest. Since every such partition is of the type
$$x_1 + \ldots + x_k = n, \quad x_1 \mid x_i, \quad 1 \leqslant i \leqslant k,$$
it follows ([3], p. 95) that
$$P_Q(n) = \sum_{d \mid n} p(d-1).$$

Let $P_Q(n,m)$ be the number of partitions of the type
$$x_1 + \ldots + x_k = n \quad \text{and} \quad x_1 \cdot \ldots \cdot x_k = m, \quad \text{where} \quad x_1 \mid x_i, \quad 1 \leqslant i \leqslant k;$$
that is to say, $P_Q(n,m)$ is the number of partitions counted by $P_Q(n)$ such that the product of the summands is m.

Let
$$M(Q,n) = \{m \in N \mid P_Q(n,m) \geqslant 1\};$$
and, as before, let $|M(Q,n)|$ be the number of distinct integers in $M(Q,n)$. Clearly, $P_Q(n) = \sum P_Q(n,m)$, the sum being taken over all m in $M(Q,n)$.

LEMMA 3.1. (ia) *$P_Q(n,m) \geqslant p(n-1, m) \geqslant 1$ if $f(m) \leqslant n-1$;*
(ib) *$P_Q(n,m) = 0$ if $f(m) > n$;*
(ic) *$P_Q(n,m) = 0$ if $f(m) = n$ and m has more than one prime factor;*
(id) *$P_Q(n,m) = 1$ if $f(m) = n$ and $m = p^a$ for some prime $p \geqslant 3$;*
(ie) *$P_Q(n,m) = 1 + [a/2]$ if $f(m) = n$ and $m = 2^a$;*
(ii) *$P_Q(n,m) = P_Q(m,m)$ if $1 \leqslant m \leqslant n$;*
(iii) *$P_Q(n,m) = 0$ if m is a prime $> n$;*
(iv) *$P_Q(n,m) = 1$ if m is a prime $\leqslant n$.*

To prove (ia), we note that if $f(m) \leqslant n-1$ we can take $x_1 = 1$ and then adjoin any of the partitions counted by $p(n-1, m)$; such a partition will clearly satisfy the condition that $x_1 | x_i$, $1 \leqslant i \leqslant k$. (ib) follows from the fact that $P_Q(n, m) \leqslant p(n, m)$ for all n and m, and $p(n, m) = 0$ if $f(m) > n$. Now suppose that $f(m) = n$ and $m = q_1^{b_1} \cdot \ldots \cdot q_s^{b_s}$. Then the only partitions of n such that the product of the summands is m are of the form

$$n = q_1 + \ldots + q_1 + q_2 + \ldots + q_2 + \ldots + q_s + \ldots + q_s,$$

together with any partition obtained from this by replacing two 2's by a 4 one or more times in this sum, if possible. Now, if m has more than one prime factor, there is no choice of x_1 which will divide all the other summands, whence $P_Q(n, m) = 0$. If $m = p^a$ and $p \geqslant 3$, then $n = p + \ldots + p$ is the only partition counted by $P_Q(n, m)$. If $m = 2^a$, then we have not only the partition $n = 2 + \ldots + 2$, but also the partitions obtained from this by replacing up to $[a/2]$ pairs of 2's by 4's. This concludes the proof of (i). The proofs of (ii), (iii) and (iv) are essentially the same as for Lemma 1.1. An immediate consequence of (i) is the following theorem.

THEOREM 3.2.

(i) $M(Q, n) = \{x \in N \mid f(x) = j,\ j = 0, 1, \ldots, n-1\} \cup \{x \in S \mid f(x) = n\}$;

(ii) $|M(Q, n)| = q(0) + q(1) + \ldots + q(n-1) + \lambda$, *where* $\lambda = \mathrm{card}\{m \in S \mid f(m) = n\}$.

To get an exact expression for $P_Q(n, m)$, we first introduce the length, restricted product-sum partition function $p(n, m, k)$, namely the numbe- of partitions of n as a sum of *exactly* k integers whose product is m. Clearlyr

$$p(n, m) = \sum_{k=1}^{n} p(n, m, k).$$

THEOREM 3.3. *Let* $m = q_1^{b_1} \cdot \ldots \cdot q_s^{b_s}$. *Then*

$$P_Q(n, m) = p(n-1, m) + p(n, m, 1) + \sum_{1<d|n} \sum_{d^k|m} p(n/d-1, m/d^k, k-1).$$

To show this, let d be an arbitrary divisor of n, and let us count the number of partitions for which $x_1 = d$. If $d = 1$, then the remaining terms need only add up to $n-1$ and have product equal to m; there are $p(n-1, m)$ of these partitions. If $d > 1$ and $k > 1$, then each partition with k terms for which $x_1 = d$ must be of the form

$$n = d + d \cdot y_2 + \ldots + d \cdot y_k,$$

where the y's satisfy the conditions

$$y_2 + \ldots + y_k = n/d - 1 \quad \text{and} \quad y_2 \cdot \ldots \cdot y_k = m/d^k;$$

the number of these partitions is clearly equal to $p(n/d-1, m/d^k, k-1)$. There is also a partition with just one term if $n = m$; $p(n, m, 1)$ is 1 in this case, and 0 otherwise. This concludes the proof of the theorem.

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