ACTA ARITHMETICA
XXXII (1977)

Pisot sequences which satisfy no linear recurrence*

by

DAVID W. BOYD (Vancouver)

Given positive integers $a_0 < a_1$, the Pisot sequence $E(a_0, a_1)$ is the sequence of positive integers defined by

$$(1)\qquad -\frac{1}{2} < a_{n+1} - \frac{a_n^2}{a_{n-1}} \leqslant \frac{1}{2}, \qquad n = 1, 2, \ldots$$

The ratios a_{n+1}/a_n converge to a number $\theta = \theta(a_0, a_1) \geqslant 1$, and the set E of such θ contains the sets S and T of Pisot–Vijayaraghavan and Salem numbers. Pisot showed that $E(a_0, a_1)$ with $a_0 = 2$ or 3 satisfies a linear recurrence relation and in this case $\theta(a_0, a_1) \in S$. Here we shall show that there are Pisot sequences satisfying no such relation and in fact that the corresponding numbers θ are everywhere dense. Two particular such sequences are $E(14, 23)$ and $E(31, 51)$.

The paper is organized as follows. Section 1 contains background material. Section 2 contains our criteria for $E(a_0, a_1)$ to satisfy a recurrence corresponding to a Salem number and, as a corollary, the non-recurrence of infinitely many sequences $E(a_0, a_1)$. In Section 3 we consider particular pairs (a_0, a_1). The final section contains a few conjectures suggested by our results.

1. Background. In 1919, Hardy [7] posed the following question. Suppose $\lambda > 0$ and $\theta > 1$ are real numbers and that $\|x\|$ denotes the distance from the real number x to the nearest integer. In what circumstances can it be true that

$$(2)\qquad \|\lambda\theta^n\| \to 0.$$

He partially answered this by showing that if θ is algebraic then it must be an algebraic integer whose remaining conjugates all have modulus less than 1. The set S of such θ is now known as the set of Pisot–Vijayaraghavan numbers, following Salem [11]. Hardy also showed

* This work was supported in part by Canadian N. R. C. Grant A-8128.

that if $\|\lambda\theta^n\| = O(b^n)$ with $0 \leqslant b < 1$, then θ is algebraic, a fact proved independently by Thue [14].

In 1938, Pisot [10] substantially improved the latter result by showing that if $\sum_0^\infty \|\lambda\theta^n\|^2 < \infty$ then θ is algebraic, hence in S. He also noted that if one sets $\lambda\theta^n = a_n + \varepsilon_n$ with a_n the "nearest" integer to $\lambda\theta^n$, then (2) implies that (1) holds eventually, so the set of reals satisfying (2) is countable. He also investigated the set E defined above and showed that it contains the set of reals for which eventually $\|\lambda\theta^n\| \leqslant c < 1/2(\theta+1)^2$. He conjectured ([10], p. 238) that $E = S$, since both sets are countable. Salem [11] showed this to be false when he proved that S is closed and hence nowhere dense, a fact conjectured by Vijayaraghavan [15]. In fact E contains in addition to S the set T of Salem numbers introduced in [12]. These are the algebraic integers $\theta > 1$ all of whose remaining conjugates lie within or on the unit circle, at least one being on the circle. In the proofs that S and T are contained in E, the corresponding sequences $\{a_n\}$ satisfy linear recurrence relations.

Suppose then that $E(a_0, a_1)$ does satisfy a linear recurrence. Then the Fatou–Hurwitz theorem shows that it is of the form

$$a_n = g_1 a_{n-1} + \ldots + g_s a_{n-s}, \quad \text{for} \quad n \geqslant p \tag{3}$$

where the g_k are integers. Flor [5] showed that in this case the defining polynomial $P(z) = z^s - g_1 z^{s-1} - \ldots - g_s$ must have all roots but one in the closed unit disk. This implies that $P(z) = R(z)U(z)$ where R is the minimal polynomial for a member of S or T and $U(z)$ has all its roots on the unit circle and hence is cyclotomic by Kronecker's theorem [8]. Thus $\theta(a_0, a_1)$ is in S or T. In case θ is in T, R is a reciprocal polynomial and has all its roots but θ and θ^{-1} on the unit circle ([12], p. 356). We shall say that $E(a_0, a_1)$ is S-recurrent or T-recurrent in the respective cases.

A tempting conjecture is that (3) holds for all Pisot sequences. A positive answer would show that $E = S \cup T$ and answer Hardy's question about (2). However it would also imply that T is everywhere dense and hence settle in the negative a conjecture of Lehmer [9] which implies a lower bound $1 + \varepsilon_0 > 1$ for T. Cantor [2] reported that $E(4, 13)$ satisfies no linear recurrence of degree $\leqslant 100$, contrasting with Pisot's results that $E(2, a_1)$ and $E(3, a_1)$ satisfy recurrences of degree at most 3. The detailed study of $E(4, 13)$ was the starting point of our investigation.

In the next section we give some easily applied criteria for testing the T-recurrence of $E(a_0, a_1)$ provided $\theta(a_0, a_1) > \tau = (\sqrt{5}+1)/2$. By choosing a sequence a_1/a_0 converging to a point not in S, we show the existence of sequences satisfying no linear recurrence. Since there is a detailed knowledge of S only for small θ, it is not easy to give par-

ticular examples of non-recurrence. However, in Section 3 we give an "effective" but impractical method of testing this and show that $E(14, 23)$ and $E(31, 51)$ are non-recurrent.

We should point out that our result does not settle the question of whether $E = S \cup T$. Conceivably, an algebraic θ might be in E without $E(a_0, a_1)$ being recurrent. We conjecture that this is not the case and that in fact the non-recurrent $E(a_0, a_1)$ correspond to transcendental θ. We can of course state in these cases that the generating function for $E(a_0, a_1)$ is not rational and hence cannot be continued outside the unit disc, by the Polya–Carlson theorem.

We shall need the following results from [10]:

LEMMA 1. *Let a_0, a_1 be positive integers with $a_1 \geqslant a_0 + 2\sqrt{a_0}$, and let a_n be defined by* (1) *for $n = 2, 3, \ldots$ Then $a_{n+1} \geqslant a_n + 2\sqrt{a_n}$ for all n. If $\theta_n = a_{n+1}/a_n$, then $\theta_n \to \theta > 1$, and $a_n/\theta_n^n \to \lambda > 0$. Furthermore*

$$|\theta - \theta_n| \leqslant 1/2(a_{n+1} - a_n), \quad n = 0, 1, \ldots \tag{4}$$

and

$$|\lambda\theta^n - a_n| \leqslant 1/2(\theta - 1)(\varphi_n - 1), \quad n = 0, 1, \ldots, \tag{5}$$

where $\varphi_n = \sup\{\theta_m:\ m \geqslant n\}$.

If $\theta \geqslant 2$, then $|\lambda\theta^n - a_n| \leqslant 1/2$ for all n.

Proof. Up to (4), the lemma is in [10], pp. 238–241. To prove (5), write (4) as

$$|a_m\theta^{-m} - a_{m+1}\theta^{-(m+1)}| \leqslant a_m/2\theta^{m+1}(a_{m+1} - a_m) \leqslant 1/2\theta^{m+1}(\varphi_n - 1),$$

for $m \geqslant n$, and then sum from n to ∞.

If $\theta \geqslant 2$, then we claim that $a_{n+1} \geqslant 2a_n$ for all n. For, if $a_{n+1} \leqslant 2a_n - 1$ for some n, then (4) implies that $\theta \leqslant 2 - (1/a_n) + 1/2(a_{n+1} - a_n) < 2$, since $a_{n+1} > 3a_n/2$ clearly. Thus $\varphi_n \geqslant 2$, so (5) implies $|\lambda\theta^n - a_n| \leqslant 1/2$.

2. Criteria for T-recurrence. Suppose that $E(a_0, a_1)$ satisfies (3) for some $p \geqslant s$. Then

$$f(z) = \sum_{n=0}^\infty a_n z^n = B(z) + E(z)/D(z) \tag{6}$$

where B, D, E are polynomials with $\deg E < \deg D = s$. We suppose (3) is a T-recurrence so $D(1/z) = 0$ has roots $\theta, \theta^{-1}, a_3, \ldots, a_s$ with $\theta > 1$ and $|a_k| = 1$ for $k = 3, \ldots, s$. From (6) we have

$$a_n = \lambda\theta^n + \mu\theta^{-n} + \sum_{k=3}^s \beta_k a_k^n + b_n, \quad n = 0, 1, \ldots \tag{7}$$

where $b_n = 0$ for $n \geqslant p-s$. If $b_n = 0$ for all n we say the recurrence is *pure*. We shall write

$$(8)\qquad a_n = \lambda\theta^n + \varepsilon_n, \quad n = 0, 1, \ldots$$

$$(9)\qquad \delta_n = \sum_{k=3}^{s} \beta_k \alpha_k^n, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$(10)\qquad c_n = \lambda\theta^n + \mu\theta^{-n} + \delta_n, \quad n = 0, \pm 1, \ldots$$

Note that δ_n and c_n are defined for all integers.

Our results are all based on the following two observations: since c_n satisfies (3) for all n, and $g_s = \pm 1$, c_n is an integer for all n. Also, δ_n is a sum of powers of numbers of modulus one and hence is an almost periodic function of n.

THEOREM 1. *Suppose that $E(a_0, a_1)$ satisfies a T-recurrence. Let $\theta = \theta(a_0, a_1)$. Define*

$$(11)\qquad \Delta(x, y, z) = \{x\theta(2+\theta^2) - y(1+2\theta^2) + z\theta\}/\theta^2.$$

Then for all n, positive or negative, c_n satisfies

$$(12)\qquad |c_{n-1} - \Delta(c_n, c_{n+1}, c_{n+2})| \leqslant (1+\theta)/2\theta^2.$$

If $\theta \geqslant 2$, then $a_n = c_n$ for $n \geqslant 0$, so the recurrence is pure. If $(\sqrt{5}+1)/2 < \theta < 2$, then $a_n = c_n$ provided that

$$(13)\qquad a_n > 2 + 13/(\theta^2 - \theta - 1).$$

Proof. Write $\zeta_n = \theta^2\varepsilon_{n-1} - 2\theta\varepsilon_n + \varepsilon_{n+1}$, $\eta_n = \theta^2\delta_{n-1} - 2\theta\delta_n + \delta_{n+1}$ and $e_n = \theta^2 b_{n-1} - 2\theta b_n + b_{n+1}$. The condition (1) implies that

$$(14)\qquad |\lambda\theta^{n-1}\zeta_n + (\varepsilon_{n-1}\varepsilon_{n+1} - \varepsilon_n^2)| \leqslant (\lambda\theta^{n-1} + \varepsilon_{n-1})/2.$$

Thus, if $\varepsilon = \sup|\varepsilon_n|$, we have

$$(15)\qquad |\zeta_n| \leqslant (1/2) + (\varepsilon + 4\varepsilon^2)/2\lambda\theta^{n-1}.$$

Using $\varepsilon_n = \mu\theta^{-n} + \delta_n + b_n$ we see that

$$(16)\qquad \eta_n = \zeta_n - e_n - \mu\theta^{-(n-1)}(\theta - \theta^{-1})^2.$$

Let $n \to \infty$ in (16) and use $e_n = 0$ for $n > p$ and (15) to conclude that

$$(17)\qquad \limsup_{n\to\infty} |\eta_n| = \limsup_{n\to\infty} |\zeta_n| \leqslant 1/2.$$

However, η_n is an almost periodic function of n, and so (17) implies

$$(18)\qquad |\eta_n| \leqslant 1/2 \quad \text{for} \quad n = 0, \pm 1, \ldots$$

More fully, if $\alpha_j = \exp(2\pi i\omega_j)$, then by Dirichlet's theorem ([3], p. 14), there are infinitely many q so that $\|q\omega_j\| < q^{-1/s}$ for all j. Thus for any n, $|\eta_n - \eta_{n+q}|$ can be made arbitrarily small for arbitrarily large q so (17) implies (18). By (10),

$$(19)\qquad \theta^2(c_{n-1} - \Delta(c_n, c_{n+1}, c_{n+2})) = (\theta^2 c_{n-1} - 2\theta c_n + c_{n+1}) - \theta(\theta^2 c_n - 2\theta c_{n+1} + c_{n+2}) = \eta_n - \theta\eta_{n+1}.$$

Combining this with (18) proves (12).

Now calculating as in (19) but using (15) instead of (18) we obtain

$$(20)\qquad |a_{n-1} - \Delta(a_n, a_{n+1}, a_{n+2})| = \theta^{-2}|\zeta_n - \theta\zeta_{n+1}| \leqslant (1+\theta)/2\theta^2 + (\varepsilon + 4\varepsilon^2)/\lambda\theta^{n+1}.$$

We know that $a_m = c_m$ for sufficiently large m. If we suppose that $a_m = c_m$ for $m \geqslant n$, say, then (12) and (20) will show that $a_{n-1} = c_{n-1}$ provided

$$(21)\qquad (1+\theta)/\theta^2 + (\varepsilon + 4\varepsilon^2)/\lambda\theta^{n+1} < 1.$$

Using (8), (21) will hold if

$$(22)\qquad a_{n-1} > (\varepsilon + 4\varepsilon^2)/(\theta^2 - \theta - 1) + \varepsilon.$$

Now, if $\theta \geqslant 2$, then Lemma 1 shows that $\varepsilon \leqslant 1/2$, and so (22) will hold if $a_{n-1} > 2$, which is true for all $n \geqslant 1$; (we may assume $a_0 \geqslant 4$ by Pisot's results). Thus, by backwards induction, $a_n = c_n$ for $n \geqslant 0$ if $\theta \geqslant 2$.

On the other hand, if we use (4), $\varphi_n \geqslant \theta - 1/2(a_1 - a_0) \geqslant \theta - 1/4\sqrt{a_0} \geqslant \theta - 1/8$, so that (5) implies

$$(23)\qquad \varepsilon \leqslant 1/2(\theta - 1)(\theta - 9/8),$$

and if $\theta > (\sqrt{5}+1)/2$ this gives $\varepsilon + 4\varepsilon^2 < 13$. Hence (13) (with n replaced by $n-1$) implies (22) and the induction follows as before.

THEOREM 2. *Let $E(a_0, a_1)$ satisfy a T-recurrence. Then, for all n,*

$$(24)\qquad |c_{n-1} - (\theta + \theta^{-1})c_n + c_{n+1}| \leqslant (\theta + 2)/2\theta^2.$$

Proof. We have from (10) that $c_{n-1} - (\theta + \theta^{-1})c_n + c_{n+1} = \delta_{n-1} - (\theta + \theta^{-1})\delta_n + \delta_{n+1} = \gamma_n$, say. To estimate γ_n most efficiently, we write it in terms of η_n and use (18). By elementary difference calculus we have

$$(25)\qquad \gamma_n = \theta^{-2}\eta_n + (\theta^{-2} - 1)\sum_{k=1}^{\infty} \theta^{-k}\eta_{n+k},$$

from which (24) follows. (Note that this is better than the obvious use of $|\delta_n| \leqslant 1/2(\theta - 1)^2$.)

COROLLARY. *If $E(a_0, a_1)$ satisfies a T-recurrence and $\theta(a_0, a_1) > 2$, and if $c_n = \pm c_{-(n+k)}$ for some k, and for two successive values of n, then this relation holds for all n and $\lambda = \pm\mu\theta^k$.*

Proof. If $\theta > 2$ then $(\theta+2)/2\theta^2 < 1/2$. Thus, if $L(x, y)$ denotes the nearest integer to $(\theta+\theta^{-1})x-y$, we have $c_{n-1} = L(c_n, c_{n+1})$ and $c_{n+1} = L(c_n, c_{n-1})$ for all n, from which the result concerning c_n and $c_{-(n+k)}$ follows. Since as $n\to\infty$, $c_n \sim \lambda\theta^n$ and $c_{-(n+k)} \sim \mu\theta^{n+k}$, we obtain $\lambda = \pm\mu\theta^k$.

Applications. To apply Theorem 1 (or 2), we simply generate a_n for $n = 0, 1, \ldots, N$ until $a_N > 10^9$, say. Then (4) shows θ is determined to within essentially 10^{-9}. Starting with a value of n for which (13) holds ($n = 0$ if $\theta > 2$), we determine successively $c_{n-1}, c_{n-2}, \ldots, c_M$ with $M < 0$, since $(1+\theta)/2\theta^2 < 1/2$ if $\theta > (\sqrt{5}+1)/2$. The limited accuracy in θ does not cause accumulation of error because c_{n-1} is rounded to the nearest integer, so we can use (12) until c_n is nearly the same order of magnitude as a_N. We check whether $\|\Delta(c_n, c_{n+1}, c_{n+2})\| < (1+\theta)/2\theta^2$, to within the known accuracy and if this fails for any n we know $E(a_0, a_1)$ is not a T-recurrence. For example $E(4, 13)$ is not T-recurrent since in this case $\|\Delta(c_{-1}, c_0, c_1)\| = .4892 > .2022 = (1+\theta)/2\theta^2$. We postpone discussion of further examples until Section 3.

We would like to apply these theorems to sequences of (a_0, a_1) with $a_1/a_0 \to \varrho$ say. However $a_0\big(\theta(a_0, a_1)-\varrho\big)$ need not tend to zero so there is some difficulty with this direct approach. This is alleviated by Theorem 3. However this result uses only the information that c_{-1} is an integer so is not as powerful in particular cases as Theorem 1 or 2.

THEOREM 3. *If $E(a_0, a_1)$ satisfies a pure T-recurrence then*

$$\|a_0(a_0+a_2)/a_1\| \leqslant (1+\theta)/2\theta^2+10/\big((\theta-1)^3 a_1\big) \tag{26}$$

and

$$\|a_0^2/a_1\| \leqslant (1+2\theta)/2\theta^2+1/a_1. \tag{27}$$

(Remark. Note that the word "pure" is redundant if $\theta > 2$, by Theorem 1.)

Proof. Define $a_n = c_n$ for $n < 0$. From (10), using the notation of Theorem 1, we have

$$\begin{aligned}(a_2a_0-a_1^2)-(a_1a_{-1}-a_0^2) &= \lambda\eta_0-\lambda\theta^{-1}\eta_{-1}+\mu(\delta_2-2\theta^{-1}\delta_1+\theta^{-2}\delta_0)-\mu\theta(\delta_1-2\theta^{-1}\delta_0+\theta^{-2}\delta_{-1})+\\ &\quad+(\delta_0\delta_2-\delta_1^2)-(\delta_{-1}\delta_1-\delta_2^2).\end{aligned} \tag{28}$$

We shall estimate $|\eta_n| \leqslant 1/2$ by (18), and

$$|\delta_n| \leqslant 1/2(\theta-1)^2 = \delta, \tag{29}$$

which follows from (5), since $|\delta_n| \leqslant \limsup|\delta_m|$. To estimate μ, we use (15) and (14) with $n = 1$ ($e_1 = 0$) to obtain

$$|\mu(\theta-\theta^{-1})^2| \leqslant 1+(\varepsilon+4\varepsilon^2)/2\lambda. \tag{30}$$

Thus, if we use $\lambda = (a_1-\varepsilon_1)/\theta$, (28) gives

$$\begin{aligned}&|a_0(a_0+a_2)-a_1(a_{-1}+a_1)|\\ &\quad\leqslant (1+\theta)a_1/2\theta^2+(1+\theta)|\varepsilon_1|/2\theta^2+|\mu|(1+\theta)^3\delta/\theta^2+4\delta^2.\end{aligned} \tag{31}$$

Now (26) follows from (31) if we use (23), (29), (30) and $\theta \geqslant (\sqrt{5}+1)/2$, since (26) is trivial if $\theta < (\sqrt{5}+1)/2$. The constant one obtains is actually 9.621 and this can be improved by using (32) below.

The proof of (27) is similar. One starts from the expression for $a_1a_{-1}-a_0^2$ used in (28). To obtain the constant 1, (actually .955), one uses the estimate

$$|\delta_n-2\theta\delta_{n+1}+\theta^2\delta_{n+2}| \leqslant 1/2+2(\theta^2-1)/\theta^3 \tag{32}$$

valid for $\theta \geqslant \sqrt{3}$. This is obtained by expanding δ_n in terms of η_n as in the proof of Theorem 2. One could use (29) but the constant obtained would not be as good.

THEOREM 4. *There are pairs (a_0, a_1) for which $E(a_0, a_1)$ satisfies no linear recurrence. In fact, the set of numbers $\theta(a_0, a_1)$ corresponding to such pairs is dense in the interval $[\tau, \infty)$, $\tau = (\sqrt{5}+1)/2$.*

Proof. Let $I = [-1/2, 1/2)$. Let ϱ be any number in (τ, ∞) which is not in S and which is not algebraic of degree less than four. Given an integer $a_0 > 0$, write $a_0\varrho = a_1+\xi$, $a_0\varrho^2 = b+\eta$, $a_0(1+\varrho^2)/\varrho = c+\zeta$, where a_1, b, c are integers and $\xi, \eta, \zeta \in I$. By the Kronecker–Weyl theorem ([3], p. 66), (ξ, η, ζ) is uniformly distributed in I^3 as a_0 varies. Hence, if $\varepsilon > 0$ is any constant with $(1+\varrho)/2\varrho^2 < 1/2-\varepsilon = \gamma$, there are infinitely many a_0 for which

$$|\eta-2\varrho\xi| < \gamma \quad \text{and} \quad |\zeta+\xi(1+\varrho^2)/\varrho^2-\eta/\varrho| > \gamma.$$

For such a_0 we have

$$a_1^2/a_0 = (a_0\varrho+\xi)^2/a_0 = b+\eta-2\varrho\xi+\xi^2/a_0 \tag{33}$$

so that $a_2 = b$ if a_0 is sufficiently large. Furthermore,

$$\begin{aligned}a_0(a_0+a_2)/a_1 &= a_0(a_0+a_0\varrho^2-\eta)/(a_0\varrho-\xi)\\ &= \big(a_0^2(1+\varrho^2)-a_0\eta\big)(a_0\varrho)^{-1}\big(1+(\xi/a_0\varrho)+(\xi/a_0\varrho)^2+\ldots\big)\\ &= c+\zeta+\xi(1+\varrho^2)/\varrho^2-\eta/\varrho+O(1/a_0).\end{aligned} \tag{34}$$

Hence, there are infinitely many a_0 for which (26) is violated. Since, by (13) of Theorem 1, a T-recurrence is pure if a_0 is sufficiently large, we have shown that $E(a_0, a_1)$ cannot satisfy a T-recurrence for infinitely many a_0 with $a_1/a_0 \to \varrho$. But $\theta(a_0, a_1) \to \varrho \notin S$ and S is closed so $\theta(a_0, a_1) \notin S$ for a_0 sufficiently large so $E(a_0, a_1)$ can satisfy an S-recurrence for only finitely many such a_0. Since these are the only possibilities, our proof is complete.

3. Specific examples. Although in Theorem 4 we used (a_0, a_1) with $a_1/a_0 \to \varrho$, where ϱ is not algebraic of degree less than four, most of the specific examples we have considered were obtained by taking a_1/a_0 to be a good approximation to a quadratic integer not in S. Cantor's example $(4, 13)$ can be considered as the first in the sequence $(4, 13)$, $(17, 55)$, $(72, 233)$, ... in which a_1/a_0 is a convergent in the continued fraction expansion of $\varrho = \sqrt{5}+1$. In this case the approximation is so good that $E(a_0, a_1)$ follows the recurrence $a_n = 2a_{n-1}+4a_{n-2}$ for a large number of terms and this in turn forces

$$|\theta(a_0, a_1)-\varrho| < a_0^{-\varkappa}, \quad \text{where} \quad \varkappa = \log 4/\log(\varrho/4) = 6.54112\ldots$$

Using this, one can show that (12) is violated for all convergents since

$$(1+\varrho)/2\varrho^2 = .202\ldots \quad \text{while} \quad \|a_0\varrho^{-1}\| = 1/4+O(1/a_0).$$

The convergents to $2+\tau = (5+\sqrt{5})/2$ provide another interesting set of examples: $(2, 7)$, $(3, 11)$, $(5, 18)$, $(8, 29)$, ... none of which is T-recurrent. By Pisot's results $E(2, 7)$ and $E(3, 11)$ are S-recurrent and it can be shown that $E(5, 18)$ is also S-recurrent. We know that this can be true for only a finite number of convergents since $2+\tau \notin S$, but to determine these explicitly would require a good estimate of $\operatorname{dist}(2+ +\tau, S)$ and this seems difficult to determine. (There are points of S within $.0001\ldots$ of $2+\tau$).

We should note that $\sqrt{5}+1$ and $(\sqrt{5}+5)/2$ are not in E as the following result shows ([10], p. 236). Note that P need not be the minimal polynomial.

LEMMA 2. *Suppose that θ is a root of the polynomial $P(z) = c_0z^m+ +\ldots+c_m$ with integer coefficients, and suppose that $L(P) = |c_0|+\ldots+ +|c_m| < 2(\theta-1)^2$. If θ is in E, then θ is in S or T.*

Proof. If $\theta = \theta(a_0, a_1)$ in E, then by Lemma 1, $\limsup|\lambda\theta^n - a_n| \leqslant 1/2(\theta-1)^2$. Thus if $d_n = c_0a_n+\ldots+c_ma_{n-m}$, then $|d_n| < 1$ for n sufficiently large. Hence $d_n = 0$, being an integer. But then $E(a_0, a_1)$ satisfies a linear recurrence so θ is in S or T.

A great deal is known about the set S for $\theta \leqslant 2$, so this is a natural place to seek $\theta(a_0, a_1) \notin S$. Siegel [13] determined the smallest two elements in S. Dufresnoy and Pisot [4] found all θ in S less than $\hat{\theta}_{15} = 1.61836\ldots$, and showed that $\tau = 1.61803\ldots$ is the smallest limit point of S. Grandet-Hugot [6] showed that 2 is the minimum of S'' and Amara [1] has found all points in S' less than 2. Unfortunately, if $\theta < \hat{\theta}_{15}$, we have $(1+\theta)/2\theta^2 > .49986$, so Theorem 1 is not very useful here. As a compromise we sought examples among the convergents to $\sqrt{7}-1 = 1.6457\ldots$ Theorem 1 applies to $E(14, 23)$ since $\|\Delta(c_{-11}, c_{-10}, c_{-9})\| > (1+\theta)/2\theta^2$;

it also applies to $(17, 28)$ and $(31, 51)$ but fails for $(48, 79)$, the next convergent.

Our criterion for $\theta \notin S$ is based on the ideas in [11] and [13]. If $\theta \in S$, then there is a sequence of positive integers $\{b_n\}$ such that ([13], p. 598)

$$b_0^2+\sum_{n=1}^{\infty}(b_n-\theta b_{n-1})^2 \leqslant 1+\theta^2. \tag{35}$$

Conversely, if $\theta > 1$ and (35) holds for some sequence $\{b_n\}$ then Theorem B of [12] shows that θ is in S. We observe that the finite sequences $(b_0, \ldots, b_N)$ satisfying $b_0 \geqslant 1$ and

$$b_0^2+\sum_{n=1}^{N}(b_n-\theta b_{n-1})^2 \leqslant 1+\theta^2 \tag{36}$$

can be arranged in a tree in a natural way since if $(b_0, \ldots, b_N)$ satisfies (36) then so does $(b_0, \ldots, b_{N-1})$. Furthermore if $(b_0, \ldots, b_N)$ satisfies (36) then there are only a finite number of integers b_{N+1} so that $(b_0, \ldots, b_{N+1})$ satisfies (36). Comparing with (35), we see that θ is not in S if and only if this tree is finite. The finiteness can be checked with a finite amount of computation in a standard way ("backtrack"). In practice, one is limited by the size of the tree and the accuracy required for θ so that $b_N\theta$ can be determined with sufficient accuracy. There are some obvious economies that can be effected. Using slightly less than one second of CPU time on an IBM370/168, we were able to check in this way that $\theta(14, 23)$ and $\theta(31, 51)$ are not in S.

One can construct an amusing proof that the complement of S is open by the association between these θ and finite trees of the above type.

4. Conjectures. The following are at least partially suggested by our results:

1. For almost all pairs (a_0, a_1), $E(a_0, a_1)$ satisfies no linear recurrence.
2. If θ is in S or T and $\theta = \theta(a_0, a_1) \in E$, then $E(a_0, a_1)$ satisfies a linear recurrence.
3. If $\theta \in E$ is algebraic then θ is in S or T.
4. The set $S \cup T$ is closed.

References

[1] M. M. Amara, *Ensembles fermés de nombres algébriques*, Ann. Sc. Éc. Norm. Sup. (3) 83 (1966), pp. 215–270.

[2] D. G. Cantor, *Investigation of T-numbers and E-sequences*, in *Computers in number theory*, ed. A. O. L. Atkins and B. J. Birch, Academic Press, N. Y. 1971.

[3] J. W. S. Cassels, *An introduction to diophantine approximation*, Cambridge University Press, 1957.

[4] J. Dufresnoy et Ch. Pisot, *Étude de certaines fonctions méromorphes bornées sur le cercle unité, application a un ensemble fermé d'entiers algébriques*, Ann. Sc. Éc. Norm. Sup. (3) 72 (1955), pp. 69–92.

[5] P. Flor, *Über eine Klasse von Folgen natürlicher Zahlen*, Math. Annalen 140 (1960), pp. 299–307.

[6] M. Grandet-Hugot, *Ensembles fermés d'entiers algébriques*, Ann. Sc. Éc. Norm. Sup. (3) 82 (1965), pp. 1–35.

[7] G. H. Hardy, *A problem of diophantine approximation*, Journ. Ind. Math. Soc. 11 (1919), pp, 162–166; *Collected works I*, pp. 124–129.

[8] L. Kronecker, *Zwei Sätze über Gleichungen mit Ganzzahligen Coefficienten*, J. Reine Angew. Math. 53 (1857), pp. 173–175.

[9] D. H. Lehmer, *Factorization of certain cyclotomic functions*, Ann. Math. 34 (1933), pp. 461–479.

[10] Ch. Pisot, *La repartition modulo 1 et les nombres algébriques*, Ann. Scuola Norm. Sup. Pisa 7 (1938), pp. 205–248.

[11] R. Salem, *A remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan*, Duke Math. J. 11 (1944), pp. 103–107.

[12] — *Power series with integral coefficients*, ibid., 12 (1945), pp. 153–171.

[13] C. L. Siegel, *Algebraic integers whose conjugates lie in the unit circle*, ibid., 11 (1944), pp. 597–602.

[14] A. Thue, *Über eine Eigenshaft die keine transzendente Grösse haben kann*, Skrifter Vidensk. I. Kristiania 2 (1912), No. 20, pp. 1–15.

[15] T. Vijayaraghavan, *On the fractional parts of the powers of a number (II)*, Proc. Camb. Phil. Soc. 37 (1941), pp. 349–357.

THE UNIVERSITY OF BRITISH COLUMBIA
Vancouver, Canada

Received on 25. 6. 1975 (732)

ACTA ARITHMETICA
XXXII (1977)

Suites à spectre vide et suites pseudo-aléatoires

par

J. Coquet (Valenciennes) et M. Mendès-France (Talence)

1. Introduction. Soit $F\colon \boldsymbol{N}\to\boldsymbol{C}$ une suite infinie. On appelle spectre (de Fourier–Bohr) de F l'ensemble

$$\operatorname{sp}(F) = \Big\{\alpha\in \boldsymbol{R}/\boldsymbol{Z} \;\Big|\; \limsup_{n\to\infty} \frac{1}{n}\Big|\sum_{k=0}^{n-1} F(k)\,e(-\alpha k)\Big| > 0\Big\}$$

(la notation $e(x)$ représente $\exp 2i\pi x$).

On dit que F est pseudo-aléatoire si les deux conditions suivantes sont remplies (voir [1] et [2]):

(i) Pour tout entier p, la limite $\gamma(p)$ de la quantité

$$\frac{1}{n}\sum_{k=0}^{n-1} \overline{F}(k)F(k+p)$$

existe quand n croît indéfiniment (γ s'appelle la corrélation de F);

(ii) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|\gamma(k)|^2 = 0.$

Une des propriétés remarquables des suites pseudo-aléatoires est qu'elles sont à spectre vide (dans la théorie de l'équirépartition (mod 1), cette propriété porte le nom de „théorème de Van der Corput"). La réciproque est fausse: la suite $n\mapsto e(\sqrt{n})$ est à spectre vide, mais elle n'est pas pseudo-aléatoire.

Dans [2], J.-P. Bertandias précise les différences (et les ressemblances) entre suite pseudo-aléatoire et suite à spectre vide. Dans notre article, nous nous proposons de montrer que pour certaines classes de suites, il y a équivalence entre les deux concepts „spectre vide" et „pseudo-aléatoire".

2. Les suites q-multiplicatives. Soit $q\geqslant 2$ un entier donné. On dit