where

$$I = \{i|\ \mu_i > 0\}$$

and the C_i are the non-overlapping Z^n-cones defined by (1).

For each i, the index of C_i with respect to Z^n is

$$|\det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{a}, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_n)|,$$

and, by (3) and (4), this is

$$\frac{\mu_i}{m}|\det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)| = \mu_i \leqslant m-1.$$

Hence by our inductive hypothesis there is a subdivision

$$C_i = \bigcup_{i\in J_i} C_{ij}$$

of each C_i into at most n^{m-2} non-overlapping basic Z^n-cones. Since the C_i are also non-overlapping it follows that the C_{ij}, $i\in I$, $j\in J_i$ are non-overlapping. Hence

$$C = \bigcup_{\substack{i\in I\\ j\in J_i}} C_{ij}$$

is a subdivision of C into at most n^{m-1} non-overlapping basic Z^n-cones, and the theorem follows by induction.

References

[1] W. M. Schmidt, *A problem of Schinzel on lattice points*, Acta Arith. 15 (1969), pp. 199–203.

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On products of special linear forms with algebraic coefficients

by

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Dedicated to Professor Dr. Theodor Schneider on his 65th birthday

1. Introduction. In a recent paper [6] I generalized the results of W. M. Schmidt [8] on real linear forms with algebraic coefficients to include the p-adic case. By means of the results of [6] we shall derive in this paper theorems on rational diophantine approximation, considering special linear forms.

Suppose n is a natural number, $t_0, \ldots, t_{n+1}$ are non-negative integers and $p_{10}, \ldots, p_{t_00}, \ldots, p_{1,n+1}, \ldots, p_{t_{n+1},n+1}$ form a fixed system of primes distinct in pairs. Let further $\alpha_{10}, \ldots, \alpha_{n0}$ be real algebraic numbers and let $\alpha_{1\tau}, \ldots, \alpha_{n\tau}$ be $p_{\tau0}$-adic algebraic numbers $(1\leqslant \tau\leqslant t_0)$; writing $\|\mathfrak{s}\| = \max\{|s_1|, \ldots, |s_{n+1}|\}$ for any $\mathfrak{s} = (s_1, \ldots, s_{n+1})\in \boldsymbol{Z}^{n+1}$ we obtain

THEOREM 1.1. *Let $\varepsilon > 0$ be any real number. Then the inequality*

$$(1.1)\qquad 0 < |s_1\alpha_{10}+\ldots+s_n\alpha_{n0}+s_{n+1}|\prod_{\tau=1}^{t_0}|s_1\alpha_{1\tau}+\ldots+s_n\alpha_{n\tau}+s_{n+1}|_{p_{\tau0}}\times$$
$$\times\prod_{i=1}^{n+1}\Big(\prod_{\tau=1}^{t_i}|s_i|_{p_{\tau i}}\Big)|s_1|\cdot\ldots\cdot|s_n| \leqslant \|\mathfrak{s}\|^{-\varepsilon}$$

is satisfied by at most a finite number of $\mathfrak{s} = (s_1, \ldots, s_{n+1})\in \boldsymbol{Z}^{n+1}\setminus\{0\}$.

COROLLARY 1.1. *Let in addition to the hypotheses of Theorem 1.1 $\varkappa_1, \ldots, \varkappa_{n+1}$ be real numbers with*

$$(1.2)\qquad 0\leqslant \varkappa_i\leqslant 1 \quad (1\leqslant i\leqslant n+1).$$

Let $s_1, \ldots, s_{n+1}$ be restricted to integers of the form

$$(1.3)\qquad s_i = s_i^* p_{1i}^{\varrho_{1i}}\cdot\ldots\cdot p_{t_ii}^{\varrho_{t_ii}} \quad (1\leqslant i\leqslant n+1),$$

where $\varrho_{1i}, \ldots, \varrho_{t_ii}$ are non-negative integers and s_i^* are integers satisfying

$$(1.4)\qquad 0 < |s_i^*|\leqslant c|s_i|^{\varkappa_i} \quad (1\leqslant i\leqslant n+1),$$

c being an absolute constant > 0. Then to every $\varepsilon > 0$ there are only a finite number of $\mathfrak{s} = (s_1, \ldots, s_{n+1}) \in \mathbf{Z}^{n+1}$ with

$$(1.5) \qquad 0 < \left| \frac{s_1}{s_{n+1}} \alpha_{10} + \ldots + \frac{s_n}{s_{n+1}} \alpha_{n0} + 1 \right| \leqslant \|\mathfrak{s}\|^{-(\varkappa_1 + \ldots + \varkappa_{n+1}) - \varepsilon}.$$

THEOREM 1.2. *Let us suppose the hypotheses are the same as in Theorem 1.1. Let in addition $1, \alpha_{1\tau}, \ldots, \alpha_{n\tau}$ be linearly independent over the field $\mathbf{Q}$ of rationals for every τ $(0 \leqslant \tau \leqslant t_0)$. Then for every $\varepsilon > 0$ there are only a finite number of $\mathfrak{s} = (s_1, \ldots, s_{n+1}) \in \mathbf{Z}^{n+1}$ with for all i $(1 \leqslant i \leqslant n+1)$ $s_i \neq 0$, such that the inequality*

$$(1.6) \qquad \prod_{i=1}^{n} \left\{ \min\left(1, \left| \frac{s_i}{s_{n+1}} - \alpha_{i0} \right| \right) \prod_{\tau=1}^{t_0} \min\left(1, \left| \frac{s_i}{s_{n+1}} - \alpha_{i\tau} \right|_{p_{\tau 0}} \right) \right\} \times \\ \times \prod_{i=1}^{n+1} \left(\prod_{\tau=1}^{t_i} |s_i|_{p_{\tau i}} \right) \leqslant \|\mathfrak{s}\|^{-n-1-\varepsilon}$$

is satisfied.

COROLLARY 1.2. *Let the hypotheses be the same as in Theorem 1.2 and Corollary 1.1. Then for every $\varepsilon > 0$ the simultaneous inequalities*

$$(1.7) \qquad \left| \frac{s_i}{s_{n+1}} - \alpha_{i0} \right| \leqslant \|\mathfrak{s}\|^{-\left(\varkappa_i + \frac{\varkappa_{n+1}}{n}\right) - \varepsilon} \qquad (1 \leqslant i \leqslant n)$$

have only a finite number of solutions $\mathfrak{s} = (s_1, \ldots, s_{n+1}) \in \mathbf{Z}^{n+1}$ with $s_{n+1} \neq 0$.

The case $n = 1$ of Theorems 1.1 and 1.2 is a result of Mahler [1] (p. 134 second approximation theorem (I)). When for all i $(0 \leqslant i \leqslant n+1)$ $t_i = 0$, the theorems are identical with the theorems of Schmidt [7]; in particular they include for $n = 1$ the theorem of Roth [4]. There even is a certain advantage as compared with the result of [7], for in Theorem 1.1 there is no hypothesis on linear independence. For proving Theorem 1.1 by means of the subspace theorem of [6], we do not need anything such as "proper systems" (cf. [7]). When $t_1 = \ldots = t_{n+1} = 0$ we obtain the results of [5]. If in addition $n = 1$, we have the theorem of Ridout [3]. Corollaries 1.1 and 1.2 are analogous to results of Ridout [2] in the n-dimensional case.

2. Weakly-general p-adic Roth systems. In [6], Definition 1.2, we said what we shall understand by general p-adic Roth system.

When, however, we are only interested in solutions $\mathfrak{x} \in \mathbf{Z}^n$ of certain simultaneous linear inequalities, for which none of the real and p-adic linear forms vanish, we may prefer the following conception of a weakly-general p-adic Roth system instead of that of a general p-adic Roth system.

DEFINITION 2.1. Let n be a natural number, t a non-negative integer, $p_1, \ldots, p_t$ primes distinct in pairs, $L_{10}, \ldots, L_{n0}$ real linear forms with algebraic coefficients in $\mathfrak{x} = (x_1, \ldots, x_n)$ and for τ $(1 \leqslant \tau \leqslant t)$ $L_{1\tau}, \ldots, L_{n\tau}$ p_τ-adic linear forms with algebraic coefficients in the same variables as the real forms. Suppose that $c_{10}, \ldots, c_{n0}, \ldots, c_{1t}, \ldots, c_{nt}$ are reals with

$$(2.1) \qquad \sum_{i=1}^{n} \sum_{\tau=0}^{t} c_{i\tau} = 0 \quad \text{and} \quad c_{i\tau} \leqslant 0 \quad (1 \leqslant i \leqslant n,\ 1 \leqslant \tau \leqslant t).$$

Then we call the system $(L_{10}, \ldots, L_{nt};\ c_{10}, \ldots, c_{nt})$ *a weakly-general p-adic Roth system* iff for every $\varepsilon > 0$ there is a

$$Q_1 = Q_1(\varepsilon, L_{10}, \ldots, L_{nt}, c_{10}, \ldots, c_{nt})$$

such that the simultaneous inequalities

$$(2.2) \qquad 0 < |L_{i0}(\mathfrak{x})| \leqslant Q^{c_{i0} - \varepsilon} \quad (1 \leqslant i \leqslant n),$$

$$(2.3) \qquad 0 < |L_{i\tau}(\mathfrak{x})|_{p_\tau} \leqslant Q^{c_{i\tau}} \quad (1 \leqslant i \leqslant n,\ 1 \leqslant \tau \leqslant t)$$

have no integer solution $\mathfrak{x}$ if $Q > Q_1$.

From now on we shall assume that

$$(2.4) \qquad c_{i\tau} \leqslant c_{i+1,\tau} \quad \text{for all } i, \tau \quad (1 \leqslant i \leqslant n-1,\ 0 \leqslant \tau \leqslant t).$$

Let S^d be a rational subspace of dimension $d > 0$ of the n-dimensional space. Suppose the forms $L_{1\tau}, \ldots, L_{n\tau}$ have rank r_τ on S^d. Now construct the number $c(S^d)$ in the following way:

When $r_0 < d$ put $c(S^d) = 1$.

When $r_0 = d$, let $s_{\tau 1}$ be the smallest integer such that $L_{s_{\tau 1}\tau} \not\equiv 0$ on S^d, i.e., that $L_{s_{\tau 1}\tau}$ has rank 1 on S^d. Let $s_{\tau 2}$ be the smallest integer, such that $L_{s_{\tau 1}\tau}, L_{s_{\tau 2}\tau}$ have rank 2 on S^d, and so on. In this way we obtain for each τ $(0 \leqslant \tau \leqslant t)$ r_τ integers $s_{\tau 1}, \ldots, s_{\tau r_\tau}$. (In particular for $\tau = 0$ d integers $s_{01}, \ldots, s_{0d}$).

Put

$$(2.5) \qquad c(S^d) = \sum_{\tau=0}^{t} \sum_{i=1}^{r_\tau} c_{s_{\tau i}\tau}.$$

Satz 1.2 [6] gives us by means of the number $c(S^d)$ a necessary and sufficient condition for general p-adic Roth systems.

Let us have a look at the proof of Satz 1.2 in Section 7 of [6]. We recognize immediately that for weakly-general p-adic Roth systems we have to consider only such rational subspaces S^d for which none of the forms vanish identically. According to Satz 1.2 [6] we obtain the following

THEOREM 2.1. *Suppose that $L_{10}, \ldots, L_{nt}$ are as in Definition 2.1 and that $c_{10}, \ldots, c_{nt}$ are constants subject to (2.1) and (2.4). Let us suppose further that for $(L_{10}, \ldots, L_{nt}; c_{10}, \ldots, c_{nt})$ we have*

$$c(S^d) \leqslant 0 \tag{2.6}$$

for every rational subspace S^d of dimension $d > 0$, on which none of the forms $L_{i\tau}$ vanish identically $(1 \leqslant i \leqslant n,\ 0 \leqslant \tau \leqslant t)$. Then $(L_{10}, \ldots, L_{nt}; c_{10}, \ldots, c_{nt})$ is a weakly-general p-adic Roth system.

Applying Theorem 2.1 we shall derive a corollary, which will enable us to prove Theorem 1.1.

Let $t_0, \ldots, t_{n+1}$ be non-negative integers and $p_{10}, \ldots, p_{t_0 0}, p_{11}, \ldots, p_{t_1 1}, \ldots, p_{1,n+1}, \ldots, p_{t_{n+1},n+1}$ be a fixed system of primes distinct in pairs. For these primes we consider systems of $p_{\tau i}$-adic linear forms $(0 \leqslant i \leqslant n+1,\ 1 \leqslant \tau \leqslant t_i)$:

When $i = 0$ let $a_{1\tau}, \ldots, a_{n\tau}$ be $p_{\tau 0}$-adic algebraic numbers $(1 \leqslant \tau \leqslant t_0)$ and let us write

$$\begin{aligned} L_{j\tau}^{(0)}(\mathfrak{x}) &= x_j \quad (1 \leqslant j \leqslant n,\ 1 \leqslant \tau \leqslant t_0),\\ L_{n+1,\tau}^{(0)}(\mathfrak{x}) &= a_{1\tau}x_1 + \ldots + a_{n\tau}x_n + x_{n+1} \quad (1 \leqslant \tau \leqslant t_0). \end{aligned} \tag{2.7}$$

When $1 \leqslant i \leqslant n+1$ is given for all τ $(1 \leqslant \tau \leqslant t_i)$

$$L_{j\tau}^{(i)}(\mathfrak{x}) = x_j \quad (1 \leqslant j \leqslant n+1). \tag{2.8}$$

Let us suppose in addition that $a_{10}, \ldots, a_{n0}$ are real algebraic numbers and write

$$\begin{aligned} L_{j0}(\mathfrak{x}) &= x_j \quad (1 \leqslant j \leqslant n),\\ L_{n+1,0}(\mathfrak{x}) &= a_{10}x_1 + \ldots + a_{n0}x_n + x_{n+1}. \end{aligned} \tag{2.9}$$

Let $c_{10}, \ldots, c_{n+1,0}, c_{11}^{(0)}, \ldots, c_{n+1,1}^{(0)}, \ldots, c_{1t_0}^{(0)}, \ldots, c_{n+1,t_0}^{(0)}, \ldots, c_{11}^{(n+1)}, \ldots, c_{n+1,1}^{(n+1)}, \ldots, c_{1t_{n+1}}^{(n+1)}, \ldots, c_{n+1,t_{n+1}}^{(n+1)}$ be real constants subject to the following conditions

$$\sum_{j=1}^{n+1}\Big(c_{j0} + \sum_{i=0}^{n+1}\sum_{\tau=1}^{t_i} c_{j\tau}^{(i)}\Big) = 0, \tag{2.10}$$

$$c_{j0} > 0 \quad (1 \leqslant j \leqslant n), \tag{2.11}$$

$$c_{j\tau}^{(0)} = 0 \quad (1 \leqslant j \leqslant n,\ 1 \leqslant \tau \leqslant t_0), \tag{2.12}$$

$$c_{n+1,\tau}^{(0)} \leqslant 0 \quad (1 \leqslant \tau \leqslant t_0), \tag{2.13}$$

$$c_{j\tau}^{(i)} = 0 \quad (1 \leqslant i, j \leqslant n+1,\ j \neq i,\ 1 \leqslant \tau \leqslant t_i), \tag{2.14}$$

$$c_{i\tau}^{(i)} \leqslant 0 \quad (1 \leqslant i \leqslant n+1,\ 1 \leqslant \tau \leqslant t_i). \tag{2.15}$$

Now order the forms (2.7)–(2.9) in such a way, that in the real as well as in the $p_{\tau i}$-adic case the corresponding constants c_{j0}, $c_{j\tau}^{(i)}$ respectively are an increasing sequence.

Let S^d be a rational subspace of dimension $d > 0$, on which none of the forms (2.7)–(2.9) vanish identically.

Looking at the conditions (2.10)–(2.15) we recognize without any difficulty that for such a subspace we have

$$c(S^d) \leqslant 0. \tag{2.16}$$

In view of Theorem 2.1, formula (2.16) implies

COROLLARY 2.1. *The forms (2.7)–(2.9) together with real constants c_{j0}, $c_{j\tau}^{(i)}$ subject to the conditions (2.10)–(2.15) are a weakly-general p-adic Roth system.*

For the proof of Theorem 1.1 it will be more suitable to have another formulation of Corollary 2.1:

Let $\varepsilon > 0$ be any real number. Let

$$\begin{aligned} &c_{10}, \ldots, c_{n+1,0}, c_{11}^{(0)}, \ldots, c_{n+1,1}^{(0)}, \ldots, c_{1t_0}^{(0)}, \ldots, c_{n+1,t_0}^{(0)}, \ldots,\\ &\qquad c_{11}^{(n+1)}, \ldots, c_{n+1,1}^{(n+1)}, \ldots, c_{1t_{n+1}}^{(n+1)}, \ldots, c_{n+1,t_{n+1}}^{(n+1)} \end{aligned} \tag{2.17}$$

be real constants with (2.12)–(2.15).

Instead of (2.10) and (2.11) suppose,

$$\sum_{j=1}^{n+1}\Big(c_{j0} + \sum_{i=0}^{n+1}\sum_{\tau=1}^{t_0} c_{j\tau}^{(i)}\Big) \leqslant -\varepsilon \tag{2.18}$$

and

$$c_{j0} \geqslant 0 \quad (1 \leqslant j \leqslant n). \tag{2.19}$$

Then we have

COROLLARY 2.2. *The simultaneous inequalities*

$$0 < |s_j| \leqslant \|\mathfrak{s}\|^{c_{j0}} \quad (1 \leqslant j \leqslant n), \tag{2.20}$$

$$0 < |s_1 a_{10} + \ldots + s_n a_{n0} + s_{n+1}| \leqslant \|\mathfrak{s}\|^{c_{n+1,0}}, \tag{2.21}$$

$$0 < |s_j|_{p_{\tau 0}} \leqslant \|\mathfrak{s}\|^0 \quad (1 \leqslant j \leqslant n,\ 1 \leqslant \tau \leqslant t_0), \tag{2.22}$$

$$0 < |s_1 a_{1\tau} + \ldots + s_n a_{n\tau} + s_{n+1}|_{p_{\tau 0}} \leqslant \|\mathfrak{s}\|^{c_{n+1,\tau}^{(0)}} \quad (1 \leqslant \tau \leqslant t_0), \tag{2.23}$$

$$0 < |s_j|_{p_{\tau i}} \leqslant \|\mathfrak{s}\|^0 \quad (1 \leqslant i,\ j \leqslant n+1,\ j \neq i,\ 1 \leqslant \tau \leqslant t_i), \tag{2.24}$$

$$0 < |s_i|_{p_{\tau i}} \leqslant \|\mathfrak{s}\|^{c_{i\tau}^{(i)}} \quad (1 \leqslant i \leqslant n+1,\ 1 \leqslant \tau \leqslant t_i) \tag{2.25}$$

have only finitely many solutions $\mathfrak{s} \in \mathbf{Z}^{n+1}$.

3. Proof of Theorem 1.1 and Corollary 1.1. Let L be a linear form in $n+1$ variables with real algebraic coefficients. Let $\mathfrak{s}\in \mathbf{Z}^{n+1}$ be such that $L(\mathfrak{s})\neq 0$. Using the norm, we see that there are positive constants c_1, c_2, c_3 independent of $\mathfrak{s}$ such that

$$c_3\|\mathfrak{s}\|^{-c_4}\leqslant |L(\mathfrak{s})|\leqslant c_5\|\mathfrak{s}\|. \tag{3.1}$$

Naturally a result analogous to (3.1) is true in the p-adic case.

In proving Theorem 1.1 we may assume since $\varepsilon>0$ is arbitrary that all algebraic numbers in consideration are algebraic integers.

Now assume that the inequality (1.1) has an infinity of solutions $\mathfrak{s}\in \mathbf{Z}^{n+1}$. From this assumption we shall derive a contradiction to Corollary 2.2. To each solution $\mathfrak{s}$ of (1.1) there are numbers

$$c_{10},\ldots,c_{n+1,0},c^{(0)}_{n+1,1},\ldots,c^{(0)}_{n+1,t_0},c^{(1)}_{11},\ldots,c^{(1)}_{1t_1},\ldots,c^{(n+1)}_{n+1,1},\ldots,c^{(n+1)}_{n+1,t_{n+1}} \tag{3.2}$$

satisfying the conditions (2.13), (2.15), (2.18), (2.19).

Complete the numbers (3.2) to a system (2.17) by putting the other numbers in (2.17) equal to zero. Let the numbers (3.2) be such that $\mathfrak{s}$ is solution of inequalities analogous to (2.20)–(2.25). We notice that the restriction of the conditions (2.13) is immaterial, since the algebraic numbers are supposed to be algebraic integers.

By (3.1) we may assume that the numbers (3.2) are uniformly bounded Let us suppose without loss of generality that 1 is a bound. Take an integer N large enough that

$$\frac{1}{N}<\frac{\varepsilon}{2\left(n+1+\sum\limits_{i=0}^{n+1}t_i\right)}. \tag{3.3}$$

We divide the interval $[-1-1/N, 1+1/N]$ into $2N+2$ subintervals of length $1/N$. To the numbers (3.2) corresponding to a solution $\mathfrak{s}$ of (1.1) choose the subintervals of our partition, in which they lie. In this way we obtain:

$$c''_{j0}<c_{j0}\leqslant c'_{j0}\quad\text{and}\quad c'_{j0}-c''_{j0}=\frac{1}{N}\qquad (1\leqslant j\leqslant n+1), \tag{3.4}$$

$$c^{(0)''}_{n+1,\tau}<c^{(0)}_{n+1,\tau}\leqslant c^{(0)'}_{n+1,\tau}\quad\text{and}\quad c^{(0)'}_{n+1,\tau}-c^{(0)''}_{n+1,\tau}=\frac{1}{N}\qquad (1\leqslant \tau\leqslant t_0), \tag{3.5}$$

$$c^{(i)''}_{i\tau}<c^{(i)}_{i\tau}\leqslant c^{(i)'}_{i\tau}\quad\text{and}\quad c^{(i)'}_{i\tau}-c^{(i)''}_{i\tau}=\frac{1}{N}\qquad (1\leqslant i\leqslant n+1,\ 1\leqslant \tau\leqslant t_i). \tag{3.6}$$

By our construction of the subintervals the numbers $c'_{j0}, c^{(0)'}_{n+1,\tau}, c^{(i)'}_{i\tau}$ satisfy conditions (2.13), (2.15) and (2.19). Because of (2.18), and (3.3)–(3.6) we further have

$$\sum_{\tau=1}^{t_0}c^{(0)'}_{n+1,\tau}+\sum_{j=1}^{n+1}\left(c'_{j0}+\sum_{\tau=1}^{t_j}c^{(j)'}_{j\tau}\right)<-\varepsilon/2. \tag{3.7}$$

If $\mathfrak{s}$ is a solution of (2.20)–(2.25) with exponents (3.2), then *a fortiori* $\mathfrak{s}$ is a solution of (2.20)–(2.25) with exponents $c'_{j0}, c^{(0)'}_{n+1,\tau}, c^{(i)'}_{i\tau}$. Therefore, we may apply Corollary 2.2. The conclusion follows now by the pigeon hole principle.

From Theorem 1.1 we shall derive easily Corollary 1.1.

Using (1.3) we obtain

$$\prod_{\tau=1}^{t_i}|s_i|^{-1}_{p_{\tau i}}\cdot|s_i^*|\geqslant|s_i|\qquad (1\leqslant i\leqslant n+1). \tag{3.8}$$

Combined with (1.4) this implies

$$c\cdot|s_i|^{\varkappa_i}\geqslant|s_i|\prod_{\tau=1}^{t_i}|s_i|_{p_{\tau i}}. \tag{3.9}$$

If we replace in (1.1) terms corresponding to the right-hand side of (3.9) by the left-hand side of (3.9) we obtain, since $\varepsilon>0$ is arbitrary, the conclusion.

4. A generalized version of the subspace theorem. The Theorems 1.1 and 1.2 are dual to each other. However, there seems to be no possibility of proving Theorem 1.2 in the same way as Theorem 1.1:

For the rational subspaces S^d, in which we are interested, it is relatively easy to estimate $c(S^d)$ in the situation of Theorem 1.1. But this method does not work for Theorem 1.2. In proving Theorem 1.2 we apply a method very similar to that of W. M. Schmidt in [10]. In [9] (p. 535–537) Schmidt proved several versions of the real case of the subspace theorem. By using the p-adic subspace theorem ([6], Satz 2.1) these results may be generalized without any difficulty. We obtain

THEOREM 4.1. *Let n be a natural number $\geqslant 2$, t an integer $\geqslant 0$, and $p_1,\ldots,p_t$ primes distinct in pairs. Let $L_{10},\ldots,L_{n0}$ be independent linear forms with real algebraic coefficients.*

For every τ $(1\leqslant\tau\leqslant t)$ let v_τ be an integer with $1\leqslant v_\tau\leqslant n$. Suppose that $L_{1\tau},\ldots,L_{v_\tau\tau}$ are independent linear forms with p_τ-adic algebraic coefficients $(1\leqslant\tau\leqslant t)$ in the same variables as the real forms. Then for every $\varepsilon>0$ there are a finite number of proper rational subspaces $T_1,\ldots,T_r$ such that every point $\mathfrak{x}\in\mathbf{Z}^n\setminus\{0\}$ satisfying

$$\prod_{i=1}^{n}|L_{i0}(\mathfrak{x})|\prod_{\tau=1}^{t}\left(\prod_{j=1}^{v_\tau}|L_{j\tau}(\mathfrak{x})|_{p_\tau}\right)\leqslant\|\mathfrak{x}\|^{-\varepsilon} \tag{4.1}$$

lies in one of these subspaces.

For the proof of this theorem we have only to observe, that to each solution $\mathfrak{x}$ of (4.1) there are suitable reals c_{i0} $(1 \leqslant i \leqslant n)$, $c_{j\tau}$ $(1 \leqslant \tau \leqslant t, 1 \leqslant j \leqslant v_\tau)$ satisfying

$$(4.2) \qquad \sum_{i=1}^{n} c_{i0} + \sum_{\tau=1}^{t} \sum_{j=1}^{v_\tau} c_{j\tau} \leqslant -\varepsilon, \quad c_{j\tau} \leqslant 0 \quad (1 \leqslant \tau \leqslant t,\ 1 \leqslant j \leqslant v_\tau),$$

such that $\mathfrak{x}$ is solution of the simultaneous inequalities

$$(4.3) \qquad |L_{i0}(\mathfrak{x})| \leqslant \|\mathfrak{x}\|^{c_{i0}} \quad (1 \leqslant i \leqslant n),$$

$$(4.4) \qquad |L_{j\tau}(\mathfrak{x})|_{p_\tau} \leqslant \|\mathfrak{x}\|^{c_{j\tau}} \quad (1 \leqslant \tau \leqslant t,\ 1 \leqslant j \leqslant v_\tau).$$

It may be excluded that one of the forms in consideration vanishes in $\mathfrak{x}$, since there are a finite number of proper rational subspaces containing all such $\mathfrak{x}$. On the other hand, the condition $c_{j\tau} \leqslant 0$ is no serious disadvantage, since we may restrict ourselves to the case of algebraic integers. In Satz 2.1 [6] we have for every τ n independent p_τ-adic forms. Therefore we complete for every τ the independent forms $L_{1\tau}, \ldots, L_{v_\tau \tau}$ by suitable forms between $X_1, \ldots, X_n$ to have a system of n independent forms. By adding some of the trivial relations

$$(4.5) \qquad |x_i|_{p_\tau} \leqslant \|\mathfrak{x}\|^0$$

to the inequalities (4.4) we have a situation similar to that of Satz 2.1 [6] Then we may proceed exactly as in [9].

5. Proof of Theorem 1.2 and Corollary 1.2. We shall denote the prime corresponding to the absolute value by p_{00}. Let $\sigma_0, \ldots, \sigma_{t_0}$ be subsets of $\{1, 2, \ldots, n\}$. It will be sufficient to prove that the inequality

$$(5.1) \qquad \prod_{\tau=0}^{t_0} \prod_{i \in \sigma_\tau} \left| \frac{s_i}{s_{n+1}} - \alpha_{i\tau} \right|_{p_{\tau 0}} \prod_{j=0}^{n+1} \prod_{\tau=1}^{t_j} |s_j|_{p_{\tau j}} \leqslant \|\mathfrak{s}\|^{-n-1-\varepsilon}$$

has for arbitrarily taken $\varepsilon > 0$ and any choice of $\sigma_0, \ldots, \sigma_{t_0}$ only a finite number of solutions $\mathfrak{s} \in \mathbf{Z}^{n+1}$ with the desired properties. We shall prove this by induction on $|\sigma_0| + \ldots + |\sigma_{t_0}|$. The case $\sigma_\tau = \emptyset$ for all τ $(0 \leqslant \tau \leqslant t_0)$ is trivial.

We call a subspace S of $\mathbf{Q}^{n+1}$ a solution space, if there is a solution $\mathfrak{s}$ of (5.1) with $\mathfrak{s} \in \mathbf{Z}^{n+1} \cap S$. We call S a reducible solution space, if there is a finite number of proper subspaces of S, such that every solution $\mathfrak{s} \in \mathbf{Z}^{n+1} \cap S$ of (5.1) lies in one of these subspaces. Otherwise we call S an irreducible solution space.

LEMMA 5.1 (Schmidt [10]). *There are a finite number of irreducible solution spaces $S_1, \ldots, S_k$, such that every solution of (5.1) lies in one of $S_1, \ldots, S_k$.*

Pick one of the irreducible solutionspaces of Lemma 5.1, say S. We may distinguish two cases:

(i) There is τ $(0 \leqslant \tau \leqslant t_0)$ such that the $|\sigma_\tau| + 1$ coordinates s_{n+1}, s_i $(i \in \sigma_\tau)$ are linearly dependent in S.

(ii) For every τ $(0 \leqslant \tau \leqslant t_0)$ the coordinates s_{n+1}, s_i $(i \in \sigma_\tau)$ are linearly independent in S respectively.

As for (i), we may proceed in exactly the same way as Schmidt [10] (p. 64 f.).

As for (ii), we shall consider linear forms corresponding to the terms $\alpha_{i\tau} - \frac{s_i}{s_{n+1}}$, i.e. the forms $s_{n+1}\alpha_{i\tau} - s_i$ $(1 \leqslant i \leqslant n+1,\ 0 \leqslant \tau \leqslant t_0)$. We now choose for every τ $(0 \leqslant \tau \leqslant t_0)$ $q = \dim S$ p_τ-adic linear forms $L_{1\tau}, \ldots, L_{q\tau}$ with algebraic coefficients, which are linearly independent in S. The hypothesis on the sets $\sigma_0, \ldots, \sigma_{t_0}$ implies that in particular the forms $s_{n+1}\alpha_{i\tau} - s_i$ with $i \in \sigma_\tau$ $(0 \leqslant \tau \leqslant t_0)$ are linearly independent in S. Choose for every τ the q independent forms in such a way that they include all forms $s_{n+1}\alpha_{i\tau} - s_i$ with indices $i \in \sigma_\tau$. By (5.1) we obtain.

$$(5.2) \qquad \prod_{\tau=0}^{t_0} \prod_{i=1}^{q} |L_{i\tau}(\mathfrak{s})|_{p_{\tau 0}} \cdot \prod_{j=1}^{n+1} \prod_{\tau=1}^{t_j} |s_j|_{p_{\tau j}} \leqslant c_2 \|\mathfrak{s}\|^{q-n-1-\varepsilon}.$$

(5.2) implies

$$(5.3) \qquad \prod_{\tau=0}^{t_0} \prod_{i=0}^{q} |L_{i\tau}|_{p_{\tau 0}} \prod_{j=1}^{q} \prod_{\tau=1}^{t_j} |s_j|_{p_{\tau j}} \leqslant c_2 \|\mathfrak{s}\|^{-\varepsilon}.$$

Now we are in exactly the same situation as Schmidt [10], (21). Assuming that (5.1) has an infinite number of solutions with (ii), we may construct in considering (5.3) a contradiction between Theorem 4.1 and Lemma 5.1. This proves Theorem 1.2.

Corollary 1.2 is now proved as follows: First we note that the components of solutions $\mathfrak{s} = (s_1, \ldots, s_{n+1})$ of (1.7) have the same order of size. On the other hand applying Theorem 1.2 we recognize that the simultaneous inequalities

$$(5.4) \qquad \left| \frac{s_i}{s_{n+1}} - \alpha_{i0} \right| \prod_{\tau=1}^{t_i} |s_i|_{p_{\tau i}} \left(\prod_{\tau=1}^{t_{n+1}} |s_{n+1}|_{p_{\tau,n+1}} \right)^{1/n} \leqslant \|\mathfrak{s}\|^{-1-1/n-\varepsilon} \quad (1 \leqslant i \leqslant n)$$

have only a finite number of solutions $\mathfrak{s} \in \mathbf{Z}^{n+1}$ with $s_i \neq 0$ for all i $(1 \leqslant i \leqslant n+1)$.

(1.3) implies

$$(5.5) \qquad |s_i|^{-1} |s_i^*| \geqslant \prod_{\tau=1}^{t_i} |s_i|_{p_{\tau i}} \quad (1 \leqslant i \leqslant n+1).$$

Combining (1.4) and (5.5) we obtain

$$c|s_i|^{\varkappa_i-1} \geqslant \prod_{\tau=1}^{t_i} |s_i|_{p_{\tau i}} \quad (1 \leqslant i \leqslant n+1). \tag{5.6}$$

Since the components s_i have the same order of size, we may conclude by (5.6)

$$c_5\|\mathfrak{s}\|^{\varkappa_i-1} \geqslant \prod_{\tau=1}^{t_i} |s_i|_{p_{\tau i}} \quad (1 \leqslant i \leqslant n+1), \tag{5.7}$$

and the corollary follows.

References

[1] K. Mahler, *Lectures on diophantine approximations*, University of Notre Dame, 1961.

[2] D. Ridout, *Rational approximations to algebraic numbers*, Mathematika 4 (1957), pp. 125–131.

[3] — *The p-adic generalization of the Thue–Siegel–Roth Theorem*, Mathematika 5 (1958), pp. 40–48.

[4] K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika 2 (1955), pp. 1–20.

[5] H. P. Schlickewei, *Die p-adische Verallgemeinerung des Satzes von Thue–Siegel–Roth–Schmidt*, J. Reine Angew. Math. (to appear).

[6] — *Linearformen mit algebraischen Koeffizienten*, Manuscripta Math. 18 (1976), pp. 147–185.

[7] W. M. Schmidt, *Simultaneous approximation to algebraic numbers by rationals*, Acta Math. 125 (1970), pp. 189–201.

[8] — *Linear forms with algebraic coefficients I*, J. Number Theory 3 (1971), pp. 253–277.

[9] — *Norm form equations*, Ann. of Math. 96 (1972), pp. 526–551.

[10] — *Simultaneous approximation to algebraic numbers by elements of a number field*, Monatsh. Math. 79 (1975), pp. 55–66.

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Factorizations of distinct lengths in algebraic number fields

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1. Let K be an algebraic number field. We shall denote by R_K its ring of integers, by P the set of all prime ideals of R_K, by H the classgroup of K and by h the classnumber.

It is known (L. Carlitz [1]) that in the case $h \geqslant 3$ some elements of R_K have factorizations into irreducibles of distinct lengths. In this paper we shall study the asymptotic distribution of numbers with factorizations of $m \geqslant 1$ distinct lengths. The set of all such numbers will be denoted by $G_m(K)$. In the case $m = 1$ we shall write also $G_1(K) = G(K)$.

Let $G_m(x)$ be the number of non-associated integers a in $G_m(K)$ with $|N(a)| \leqslant x$. We shall determine the asymptotic behaviour of $G_m(x)$ (Theorem 4) and in particular we shall prove that

$$G_1(x) = \big(C(K)+o(1)\big)\frac{x(\log\log x)^a}{(\log x)^{1-\frac{t}{h}}},$$

where $C(K) > 0$, a is a non-negative integer and $t = t(H)$ is a positive integer, which has a combinatorial meaning. We shall also obtain a similar result for natural numbers $\leqslant x$ lying in $G_m(K)$ (Theorem 5).

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2. To begin with we define two combinatorial constants attached to a given finite abelian group A which we shall write multiplicatively.

If $g_1, \ldots, g_k \in A$, $n_1, \ldots, n_k \in Z$ and

$$g_1^{n_1} \ldots g_k^{n_k} = 1 \tag{1}$$

then (1) will be called a *minimal equality*, provided

1° $0 \leqslant n_i \leqslant r_i =$ order of g_i $(i = 1, \ldots, k)$ and

$$\langle n_1, \ldots, n_k\rangle \neq \langle 0, \ldots, 0\rangle.$$