[3] P. Erdös, *Some asymptotic formulas in number theory*, J. Indian Math. Soc. 12 (1948), p. 75–78.
[4] P. Erdös and G. G. Lorentz, *On the probability that n and g(n) are relatively prime*, Acta Arith. 5 (1958), p. 35–44.
[5] T. Estermann, *On the number of primitive lattice points in a parallelogramm*, Canad. J. Math. 5 (1953), p. 456–459.
[6] A. S. Fainleib, *On the relative primality of n and f(n)*, Math. Notes 11 (1972), p. 163–168.
[7] A. O. Gelfond, *Sur les nombres qui ont des propriétes additives et multiplicatives données*, Acta Arith. 13 (1968), p. 259–265.
[8] R. R. Hall, *On the probability that n and f(n) are relatively prime*, Acta Arith. 17 (1970), p. 169–183.
[9] — *On the probability that n and f(n) are relatively prime, II*, Acta Arith. 19 (1971), p. 175–184.
[10] — *On the probability that n and f(n) are relatively prime, III*, Acta Arith. 20 (1972), p. 267–289.
[11] M. Mendès-France, *Suites à spectre vide et repartition modulo* 1, J. Number Theory 5 (1973), p. 1–15.
[12] D. J. Newman, *On the number of binary digits in a multiple of three*, Proc. Amer. Math. Soc. 21 (1969), p. 719–721.
[13] E. J. Scourfield, *An asymptotic formula for the property $(n, f(n)) = 1$ for a class of multiplicative functions*, Acta Arith. 29 (1976), p. 401–423.
[14] G. L. Watson, *On integers n relatively prime to* [αn], Canad. J. Math. 5 (1953), p. 451–455.

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Reçu le 28. 4. 1975
et dans la forme modifiée 2. 7. 1975 (697)

ACTA ARITHMETICA
XXXI (1976)

A problem of Schinzel on lattice points

by

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1. Introduction. Let L be a lattice in R^n, and for any basis $B = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ of R^n, let $c(B)$ be the cone with basis B, that is,

$$c(B) = c(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = \Big\{\sum_{i=1}^{n} \lambda_i \boldsymbol{a}_i \Big| \ \lambda_i \in R,\ \lambda_i \geqslant 0 \ \ \forall i\Big\}.$$

By an *L-cone* we shall mean a cone C of the form $C = c(B)$ such that B is a basis of R^n and $B \subseteq L$.

Any L-cone C has a unique such basis B which satisfies the further condition that each vector $\boldsymbol{a}_i$ in B is primitive (i.e., $\boldsymbol{a}_i \neq s\boldsymbol{x}$ for integral s greater than 1, and $\boldsymbol{x}$ in L); and we define the *index* of an L-cone C (with respect to L) as the index in L of the sublattice generated by this primitive basis of C. We shall call an L-cone C *basic* (with respect to L) if its index is 1, that is, if $C = c(B)$ for some basis B of the lattice L.

W. M. Schmidt [1] (reviewed MR #1408, Vol. 39, Feb., 1970) showed that if L is a sublattice of the integer lattice Z^n then the non-negative orthant

$$E^+ = \{\boldsymbol{x} \in R^n \mid x_i \geqslant 0 \ \ \forall i\}$$

can be written as

$$E^+ = \bigcup_{i=1}^{N} c(B_i),$$

where each B_i is a basis of L, that is, E^+ is a finite union of basic L-cones. This result had been conjectured by Schinzel, and Schmidt's proof makes use of a compactness argument. In this paper, I shall give a simple constructive proof of the following theorem, from which Schmidt's theorem follows.

THEOREM. *If C is an L-cone of index m in L, where L is a lattice in R^n, then C is a union of at most $N = n^{m-1}$ non-overlapping basic L-cones (i.e., basic L-cones such that any two distinct ones have disjoint interiors).*

If L is a sublattice of Z^n, of index r, then E^+ is an L-cone since

$$E^+ = c(r\boldsymbol{e}_1, \ldots, r\boldsymbol{e}_n),$$

where $e_1, \ldots, e_n$ are the unit points on the coordinate axes. Thus we immediately have the following corollary, which is Schmidt's theorem (with non-overlapping cones).

COROLLARY. *If L is a sublattice of Z^n then the non-negative orthant E^+ is the union of a finite number of non-overlapping basic L-cones.*

The theorem can be refined and extended in various ways at the price of extra complication. In particular, the bound $N = n^{m-1}$ is very crude. The proof uses induction on the index m rather than, as in Schmidt's approach, on the dimension n. With some extra attention to detail, we could arrange that the basic L-cones of the theorem form a *complex* of cones, i.e., that any two of the basic L-cones meet in a common (lower-dimensional) face.

2. Two lemmas. First we give two simple lemmas on which the proof of the theorem will be based. Lemma 1 specifies a method of subdivision of a cone C which is essentially barycentric subdivision with a specified point $\boldsymbol{a}$ as barycentre. Lemma 2 will be used in choosing the point $\boldsymbol{a}$.

LEMMA 1. *Let $C = c(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$, where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are a basis of R^n, and suppose that*

$$\boldsymbol{a} = \sum_{i=1}^{n} a_i \boldsymbol{a}_i \in C, \quad \boldsymbol{a} \neq \boldsymbol{0}.$$

Let

$$I = \{i \mid a_i > 0\},$$

$$C_i = c(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{a}, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_n) \quad (i \in I). \tag{1}$$

Then

$$C = \bigcup_{i \in I} C_i$$

and the cones C_i are non-overlapping.

Proof. For $i \in I$, we have $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{a}, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_n$ are a basis of R^n, so that the cones C_i are well-defined. Since $\boldsymbol{a}$ is non-zero, the set I is non-empty. If

$$\boldsymbol{b} = \sum_{i=1}^{n} \beta_i \boldsymbol{a}_i \in C,$$

then for $j \in I$, since $a_j \neq 0$, we have

$$\boldsymbol{b} = \frac{\beta_j}{a_j} \boldsymbol{a} + \sum_{\substack{i \in I \\ i \neq j}} \left(\beta_i - \frac{\beta_j}{a_j} a_i\right) \boldsymbol{a}_i + \sum_{i \notin I} \beta_i \boldsymbol{a}_i. \tag{2}$$

Since I is non-empty, we may choose $j \in I$ so that

$$\frac{\beta_j}{a_j} \leqslant \frac{\beta_i}{a_i} \quad \forall i \in I,$$

and then the coefficients in (2) are all non-negative, and so $\boldsymbol{b} \in C_j$. Thus

$$C = \bigcup_{i \in I} C_i.$$

Finally, if $\boldsymbol{b} \in \operatorname{int} C_j$, then

$$\frac{\beta_j}{a_j} < \frac{\beta_i}{a_i} \quad \forall i \in I, \quad i \neq j.$$

Hence the $\operatorname{int} C_j$ are pairwise disjoint, and this completes the proof of the lemma.

LEMMA 2. *Suppose that $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are points of Z^n, and that*

$$m = |\det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)| > 1.$$

Then there is a primitive point $\boldsymbol{a}$ in Z^n such that

$$\boldsymbol{a} = \frac{1}{m} \sum_{i=1}^{n} \mu_i \boldsymbol{a}_i, \tag{3}$$

where

$$0 \leqslant \mu_i \leqslant m-1, \quad \mu_i \in Z \quad (i = 1, \ldots, n). \tag{4}$$

Proof. Let M be the sublattice of Z^n with basis $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, so that $(Z^n : M) = m$. As every point of Z^n is uniquely expressible in the form (3) with $\mu_1, \ldots, \mu_n$ integral, it follows that the points of Z^n of this form such that (4) hold forms a complete set of representatives for the m cosets of M in Z^n. Since $m > 1$, we may choose a non-zero representative, and division by a positive integer, if necessary, then yields a primitive point $\boldsymbol{a}$ as required.

3. Proof of the theorem. Under any non-singular linear transformation φ of R^n, subdivisions of an L-cone C into basic L-cones correspond to subdivisions of the $\varphi(L)$-cone $\varphi(C)$ into basic $\varphi(L)$-cones. Hence, taking φ so that $\varphi(L) = Z^n$, we may suppose, without loss of generality, that $L = Z^n$, and C is a Z^n-cone. Then

$$C = c(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n), \tag{5}$$

where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are primitive points in Z^n.

The theorem is trivially true for cones of index 1. So to prove it by induction on the index we now assume that it holds for cones of index less than m and consider a cone C as in (5) above, of index m, where $m > 1$, so that

$$|\det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)| = m > 1.$$

By Lemma 2, there exists a primitive point $\boldsymbol{a}$ in Z^n such that (3) and (4) hold, and by Lemma 1 with this $\boldsymbol{a}$ we have

$$C = \bigcup_{i \in I} C_i,$$

where

$$I = \{i \mid \mu_i > 0\}$$

and the C_i are the non-overlapping Z^n-cones defined by (1).

For each i, the index of C_i with respect to Z^n is

$$|\det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{a}, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_n)|,$$

and, by (3) and (4), this is

$$\frac{\mu_i}{m} |\det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)| = \mu_i \leqslant m-1.$$

Hence by our inductive hypothesis there is a subdivision

$$C_i = \bigcup_{i \in J_i} C_{ij}$$

of each C_i into at most n^{m-2} non-overlapping basic Z^n-cones. Since the C_i are also non-overlapping it follows that the C_{ij}, $i \in I$, $j \in J_i$ are non-overlapping. Hence

$$C = \bigcup_{\substack{i \in I \\ j \in J_i}} C_{ij}$$

is a subdivision of C into at most n^{m-1} non-overlapping basic Z^n-cones, and the theorem follows by induction.

References

[1] W. M. Schmidt, *A problem of Schinzel on lattice points*, Acta Arith. 15 (1969), pp. 199–203.

Received on 7. 5. 1975 (708)

ACTA ARITHMETICA
XXXI (1976)

On products of special linear forms with algebraic coefficients

by

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Dedicated to Professor Dr. Theodor Schneider on his 65th birthday

1. Introduction. In a recent paper [6] I generalized the results of W. M. Schmidt [8] on real linear forms with algebraic coefficients to include the p-adic case. By means of the results of [6] we shall derive in this paper theorems on rational diophantine approximation, considering special linear forms.

Suppose n is a natural number, $t_0, \ldots, t_{n+1}$ are non-negative integers and $p_{10}, \ldots, p_{t_0 0}, \ldots, p_{1,n+1}, \ldots, p_{t_{n+1},n+1}$ form a fixed system of primes distinct in pairs. Let further $\alpha_{10}, \ldots, \alpha_{n0}$ be real algebraic numbers and let $\alpha_{1\tau}, \ldots, \alpha_{n\tau}$ be $p_{\tau 0}$-adic algebraic numbers $(1 \leqslant \tau \leqslant t_0)$; writing $\|\mathfrak{s}\| = \max\{|s_1|, \ldots, |s_{n+1}|\}$ for any $\mathfrak{s} = (s_1, \ldots, s_{n+1}) \in \boldsymbol{Z}^{n+1}$ we obtain

THEOREM 1.1. *Let $\varepsilon > 0$ be any real number. Then the inequality*

$$0 < |s_1\alpha_{10} + \ldots + s_n\alpha_{10} + s_{n+1}| \prod_{\tau=1}^{t_0} |s_1\alpha_{1\tau} + \ldots + s_n\alpha_{n\tau} + s_{n+1}|_{p_{\tau 0}} \times \prod_{i=1}^{n+1} \Big(\prod_{\tau=1}^{t_i} |s_i|_{p_{\tau i}} \Big) |s_1| \cdot \ldots \cdot |s_n| \leqslant \|\mathfrak{s}\|^{-\varepsilon} \tag{1.1}$$

is satisfied by at most a finite number of $\mathfrak{s} = (s_1, \ldots, s_{n+1}) \in \boldsymbol{Z}^{n+1} \setminus \{0\}$.

COROLLARY 1.1. *Let in addition to the hypotheses of Theorem 1.1 $\varkappa_1, \ldots, \varkappa_{n+1}$ be real numbers with*

$$0 \leqslant \varkappa_i \leqslant 1 \quad (1 \leqslant i \leqslant n+1). \tag{1.2}$$

Let $s_1, \ldots, s_{n+1}$ be restricted to integers of the form

$$s_i = s_i^* p_{1i}^{\varrho_{1i}} \cdot \ldots \cdot p_{t_i i}^{\varrho_{t_i i}} \quad (1 \leqslant i \leqslant n+1), \tag{1.3}$$

where $\varrho_{1i}, \ldots, \varrho_{t_i i}$ are non-negative integers and s_i^* are integers satisfying

$$0 < |s_i^*| \leqslant c |s_i|^{\varkappa_i} \quad (1 \leqslant i \leqslant n+1), \tag{1.4}$$