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On the problem of divisors

by

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1. Introduction. Let k be an integer greater than 2. Let $\tau_k(n)$ be the number of the solutions of the equation $n = m_1 m_2 \dots m_k$ in integers $m_i \geqslant 1$. We are concerned with the estimate of the number

$$D_k(x) = \sum_{n \leqslant x} \tau_k(n).$$

Let $M_k(x)$ be the residue of $\zeta^k(s)x^s/s$ at $s = 1$, where $\zeta(s)$ is the Riemann zeta-function. It is well known that $D_k(x) \sim M_k(x)$ and if we put $\Delta_k(x) = D_k(x) - M_k(x)$,

$$\Delta_k(x) \ll x^{1-1/k}(\log x)^{k-2} \quad \text{for} \quad k = 2, 3, 4, \dots$$

(Cf. 12.1.4 of [5].) It was shown by Hardy and Littlewood that

$$\Delta_k(x) \ll x^{\frac{k-1}{k+2}+\varepsilon} \quad \text{for each } k \geqslant 4.$$

(Cf. 12.3 of [5].)

Generally if we put $\zeta(\frac{1}{2}+it) \ll |t|^\lambda$, then their method gives

$$\Delta_k(x) \ll x^{\frac{2(k-4)\lambda+1}{2(k-4)\lambda+2}+\varepsilon} \quad \text{for each } k \geqslant 4. \tag{1}$$

It is well known that we can take $\lambda = 173/1067$ which is due to Kolesnik [3]. In 1971 Karatsuba [1] showed

$$\Delta_k(x) \ll x^{1-Ck^{-2/3}+\varepsilon} \quad \text{for each } k \geqslant 2,$$

where C is some positive absolute constant. In this paper we shall improve these results for k in $10 \leqslant k \leqslant k_0$, where k_0 is some positive constant which depends on C above. Our proof depends on only the well known properties of $\zeta(s)$. To state our result we shall introduce some notations. Let b be an integer greater than 3. Let $j(b)$ be determined by

$$(j-1)2^{j-2}+1 < b \leqslant j2^{j-1}+1.$$

We put $\beta(b) = (j(b)+1)/(2b+2^{j(b)}-2)$. Let $l(b)$ be determined by

$$1-\frac{l-1}{2^{l-1}-2} < 1-\beta(b) \leqslant 1-\frac{l}{2^l-2}.$$

We put

$$\mu(b) = \frac{\beta(b)2^{l-1}-1}{2^{l-1}l-2^l+2}, \quad \text{where} \quad l = l(b).$$

Then our result is the following

THEOREM. *For each $k \geqslant 10$, we have*

$$\Delta_k(x) \ll x^{1-\frac{\beta(b)}{1+\mu(b)(k-2b)}+\varepsilon},$$

where b is an integer in $4 \leqslant 2b \leqslant k$, $\beta(b)$ and $\mu(b)$ are the same as above, the constant involved in $\ll$ may depend on k, and ε is an arbitrarily small positive number.

We shall prove our theorem in § 2. In § 3 we shall give some remarks about k_0 and C which we have mentioned above.

2. Proof of theorem

2.1. Lemmas. Let σ_{2b} be the lower bound of the numbers σ such that

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^{2b}dt \ll 1.$$

LEMMA 1. *Let b be an integer greater than 1, and let $j = j(b)$ be determined by $(j-1)2^{j-2}+1 < b \leqslant j2^{j-1}+1$. Then*

$$\sigma_{2b} \leqslant 1-\frac{j+1}{2b+2^j-2}.$$

(Cf. 7.10 of [5].)

LEMMA 2. *If $l \geqslant 2$, $L = 2^{l-1}$, $\sigma = 1-\dfrac{l}{2L-2}$, $t > t_0$,*

$$\zeta(\sigma+it) \ll t^{1/(2L-2)}\log t.$$

(Cf. 5.14 of [5].)

2.2. Proof of theorem. We always denote an arbitrarily small positive number by ε. Let x be half an odd integer. We start from

$$D_k(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta^k(s)\frac{x^s}{s}ds + O\left(\frac{x^c}{T(c-1)^k}\right) + O\left(\frac{x^c}{T}\right),$$

where $c = 1+\varepsilon$. (Cf. the last line of page 264 in [5].)

For integral b in $4 \leqslant 2b \leqslant k$, we put $\beta' = \beta(b)-\varepsilon$, where $\beta(b)$ is defined in the Introduction. We move the line of the above integration to the line $\sigma = 1-\beta'$. Then taking the pole of $\zeta^k(s)$ into account, we get

$$\begin{aligned}\Delta_k(x) &\ll x^{1-\beta(b)+\varepsilon}\int_{-T}^{T} |\zeta(1-\beta'+it)|^k\frac{dt}{\tau} + T^{-1}\int_{1-\beta'}^{c} |\zeta(\sigma+iT)|^k x^\sigma d\sigma + O(x^cT^{-1})\\ &= I_1+I_2+O(x^cT^{-1}),\end{aligned}$$

say, where we put $\tau = |t|+1$. Now let $l = l(b)$ satisfy

$$1-\frac{l-1}{2^{l-1}-2} < 1-\beta(b) \leqslant 1-\frac{l}{2^l-2}$$

as in the Introduction.

By the convexity argument, using Lemma 2, we get

$$\zeta(\sigma+it) \ll t^{\mu(b)\frac{c-\sigma}{c-(1-\beta')}+\varepsilon} \tag{2}$$

uniformly for σ in $1-\beta' \leqslant \sigma \leqslant c$, where $\mu(b)$ is defined in the Introduction. Now, by Lemma 1 and (2), we get

$$\begin{aligned}I_1 &\ll x^{1-\beta(b)+\varepsilon}\max_{|t|\leqslant T}|\zeta(1-\beta'+it)|^{k-2b}\int_{-T}^{T}|\zeta(1-\beta'+it)|^{2b}\frac{dt}{\tau}\\ &\ll x^{1-\beta(b)+\varepsilon}T^{\mu(b)(k-2b)+\varepsilon}.\end{aligned}$$

By (2) we get also

$$I_2 \ll x^cT^{-1}+x^{1-\beta'}T^{k\mu(b)-1+\varepsilon}.$$

Since $\mu(b) < (2b)^{-1}$, taking $T = x^{\beta(b)(1+\mu(b)(k-2b))^{-1}}$, we get

$$\Delta_k(x) \ll x^{1-\beta(b)(1+\mu(b)(k-2b))^{-1}+\varepsilon} \quad \text{for} \quad 4 \leqslant 2b \leqslant k. \tag{3}$$

Here we may remove the restriction on x which was given at the beginning of the proof. Finally we have to show that for $k \geqslant 10$ (3) gives the better estimate than (1) in the Introduction with $\lambda = 173/1067$. For this we have to show that there exists a b in $4 \leqslant 2b \leqslant k$ satisfying

$$\big(1+\mu(b)(k-2b)\big)/\beta(b) < (346/1067)\big(k+(375/173)\big). \tag{4}$$

Here we shall show that for $k \geqslant 23$ there exists a b in $4 \leqslant 2b \leqslant k$ satisfying

$$\mu(b)(k-2b) < 1 \tag{5}$$

and

$$\big(\beta(b)\big)^{-1} < (173/1067)\big(k+(375/173)\big). \tag{6}$$

Then such b satisfies (4), and we are done for $k \geqslant 23$. For $10 \leqslant k \leqslant 22$, by simple calculations, we can find a b in $4 \leqslant 2b \leqslant k$ satisfying (4). (For example, for $k = 10$, $b = 3$ or 4 gives $\Delta_{10}(x) \ll x^{77/104}$.)

Now we assume first that $k \geqslant 198$. For given $k \geqslant 198$, we choose b satisfying

(7) $$\frac{178 j(b)}{1067\beta(b)} < \left(\frac{375}{173}+k\right)\frac{173}{1067} \leqslant \frac{178 j(b+1)}{1067\beta(b+1)}.$$

Then for $k \geqslant 198$, $b \geqslant 82$ and $j(b) \geqslant 6$. Now we have first

$$\frac{1}{\beta(b)} < \frac{1067\cdot 173}{178 j(b) 1067}\left(k+\frac{375}{173}\right) \leqslant \frac{173}{1067}\left(k+\frac{375}{173}\right) \quad \text{since} \quad j(b) \geqslant 6.$$

Since

$$\frac{1}{\beta(b)} = \frac{2b+2^{j(b)}-2}{j(b)+1} \geqslant \frac{2b+(2b-2)/j(b)-2}{j(b)+1} \geqslant \frac{(2b-2)}{j(b)},$$

we have

$$2b \leqslant \big(j(b)/\beta(b)\big)+2 < (173/178)\big(k+(375/173)\big)+2 \leqslant k.$$

Next,

$$\frac{1}{\mu(b)} = \frac{2^{l(b)-1}l(b)-2^{l(b)}+2}{\beta(b)2^{l(b)-1}-1} > \frac{2^{l(b)-1}l(b)-2^{l(b)}+2}{\frac{l(b)-1}{2^{l(b)-1}-2}\cdot 2^{l(b)-1}-1}$$

$$= 2^{l(b)-1}-2 \geqslant \frac{1}{4}\frac{l(b+1)}{\beta(b+1)}-\frac{3}{2}$$

since

$$\frac{l(b+1)}{\beta(b+1)} \leqslant 2^{l(b+1)}-2 \leqslant 2^{l(b)+1}-2.$$

We have also

$$2b > \frac{j(b+1)-1}{\beta(b+1)}$$

since

$$\frac{1}{\beta(b+1)} = \frac{2(b+1)+2^{j(b+1)}-2}{j(b+1)+1} < \frac{2(b+1)+\dfrac{4(b+1)-4}{j(b+1)-1}-2}{j(b+1)+1}$$

$$= \frac{2(b+1)-2}{j(b+1)-1} = \frac{2b}{j(b+1)-1}.$$

Hence we have

$$\frac{1}{\mu(b)}+2b > \frac{(1/4)l(b+1)+j(b+1)-1}{\beta(b+1)}-\frac{3}{2} \geqslant \frac{(5/4)j(b+1)-1}{\beta(b+1)}-\frac{3}{2}$$

$$= \frac{178 j(b+1)}{173\beta(b+1)}+\frac{(153/692)j(b+1)-1}{\beta(b+1)}-\frac{3}{2} \geqslant k$$

since

$$j(b+1) \geqslant j(b) \geqslant 6.$$

Hence for given $k \geqslant 198$, b chosen by (7) satisfies both (5) and (6). For $k \leqslant 197$, by simple calculations we can find b satisfying (5) and (6) as follows for example; $b = 40$ for $110 \leqslant k \leqslant 197$, $b = 22$ for $70 \leqslant k \leqslant 109$, $b = 15$ for $52 \leqslant k \leqslant 69$, $b = 10$ for $38 \leqslant k \leqslant 51$, $b = 8$ for $32 \leqslant k \leqslant 37$, $b = 6$ for $26 \leqslant k \leqslant 31$, $b = 5$ for $23 \leqslant k \leqslant 25$. ∎

3. Concluding remarks. It might not be redundant to have a rough computation of k_0 in the Introduction. For this we shall use the following

LEMMA 3. *For $1/2 \leqslant \sigma \leqslant 1$, $t \geqslant 2$, we have*

$$\zeta(\sigma+it) \ll t^{39(1-\sigma)^{3/2}}(\log t)^{2/3}.$$

(Cf. [4] and [6].)

From this we deduce at once

$$\sigma_{2k} \leqslant \min_{2\leqslant a\leqslant k}\left(1-\left(a+78(k-a)a^{-1/2}\right)^{-1}\right),$$

where σ_{2k} was defined in § 2. Hence, in particular, we have

COROLLARY.

$$\sigma_{2k} \leqslant 1-\left(2(78k)^{2/3}-78(78k)^{1/3}+2\right)^{-1} \quad \text{for} \quad k \geqslant 1.$$

Now we take $\beta(b) = \left(2(78b)^{2/3}\right)^{-1}$ in the argument in § 2. Then as before we get

$$\Delta_k(x) \ll x^{1-\left(2(78b)^{2/3}(1+(k-2b)2^{-5/2}b^{-1})\right)^{-1}+\varepsilon} \quad \text{for} \quad 4 \leqslant 2b \leqslant k.$$

Taking $b = [k/(2^{5/2}-2)]$, we get

THEOREM. *For each $k \geqslant k_0 \geqslant 2^{24}$, we have*

$$\Delta_k(x) \ll x^{1-Ak^{-2/3}+\varepsilon},$$

where we put $A^{-1} = 2^{1/2}(2^{3/2}-1)^{1/3}(39)^{2/3}$, k_0 is some effectively computable integer and ε is an arbitrarily small positive number.

More generally, let a be a positive number satisfying

$$\zeta(\sigma+it) \ll t^{a(1-\sigma)^{3/2}}(\log t),$$

then the above argument gives $A^{-1} = 2^{1/2}(2^{3/2}-1)^{1/3}a^{2/3}$ in the theorem above. We may compare this with Karatsuba's uniform estimation [2] with respect to k and x;

$$\Delta_k(x) \ll x^{1-(2(2ak)^{2/3})^{-1}}(B\log x)^k,$$

where a is the same as above and B is some positive absolute constant. We may remark here that a slight improvement of the above theorem can be obtained by choosing $\beta(b) = \big(2(78b)^{2/3}-78(78b)^{1/3}+2\big)^{-1}$ in the above argument.

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Fonctions g-additives et formule asymptotique pour la propriété $\big(n, f(n)\big) = q$

par

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1. Introduction. Un théorème bien connu de Čebyšev dit que la probabilité que deux entiers n et m soient premiers entre eux vaut $6/\pi^2$; autrement dit:

$$\lim_{\substack{x\to\infty\\ y\to\infty}} \frac{1}{xy} \sum_{\substack{1\leqslant n\leqslant x\\ 1\leqslant m\leqslant y\\ (n,m)=1}} 1 = \frac{6}{\pi^2}.$$

On peut s'attendre à ce que ce résultat reste vrai si les entiers n et m sont tels que $n = f(m)$, où f est une fonction arithmétique à valeurs entières, pourvu que f ne conserve pas les propriétés arithmétiques de l'entier n. Plusieurs résultats de cette nature ont été obtenus notamment par G. L. Watson [14], Erdös et Lorentz [4], T. Estermann [5], R. R. Hall [8], [9], [10], et enfin par A. S. Fainleib [6].

Au contraire, si la fonction arithmétique f préserve certaines propriétés arithmétiques, on peut espérer un résultat différent. C'est ce qui a été démontré pour les fonctions multiplicatives par P. Erdös [3] et par E. J. Scourfield [13].

Dans ce qui suit nous étudions sous cet aspect une fonction g-additive. Cette notion a été introduite par A. O. Gelfond [7] et développée entre autres par J. Bésineau [1], H. Delange [2] et M. Mendès-France [11].

Si g est un entier > 1, on dit que la fonction arithmétique f est g-additive si, quel que soit $k \geqslant 0$, on a:

$$f(ag^k+b) = f(ag^k)+f(b) \quad \text{pour} \quad 0\leqslant a\leqslant g-1 \text{ et } 0\leqslant b\leqslant g^k-1.$$

On dit que la fonction arithmétique F est g-multiplicative si, quel que soit $k \geqslant 0$, on a:

$$F(ag^k+b) = F(ag^k)\cdot F(b) \quad \text{pour} \quad 0\leqslant a\leqslant g-1 \text{ et } 0\leqslant b\leqslant g^k-1.$$