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On a theorem of G. Halász and P. Turán

by

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1. Denote by $N(\alpha, T)$ the number of roots of the Riemann zeta function $\zeta(s) = \zeta(\sigma+it)$ for $(\frac{1}{2}\leqslant)\ \alpha\leqslant\sigma\leqslant 1$, $0<t\leqslant T$. G. Halász and P. Turán [2] proved the following theorem:

For a sufficiently small positive constant A, for all α-values such that $1-A\leqslant\alpha\leqslant 1$ and for $T>c$, the inequality

$$N(\alpha, T) < T^{(1-\alpha)^{3/2}\log^3\frac{1}{1-\alpha}} \tag{1.1}$$

holds (1).

Denote by $L(s,\chi,q)$ a Dirichlet's L-function belonging to the character $\chi \bmod q$, $q\leqslant k$ and by $N(\alpha, T, \chi, q)$ the number of roots of $L(s,\chi,q)$ in the region

$$(\tfrac{1}{2}\leqslant)\ \alpha\leqslant\sigma\leqslant 1,\quad 0<t\leqslant T.$$

The purpose of this note is to prove the following

THEOREM. *For $T>c$, c — numerical constant, and for*

$$1-0.284e^{-14}\leqslant\alpha\leqslant 1 \tag{1.2}$$

the inequality

$$\sum_{q\leqslant k}\sideset{}{^*}\sum_{\chi \bmod q} N(\alpha, T, \chi, q) < \exp\left[26M_1(k,T)(1-\alpha)^{3/2}\log^3\frac{0.284}{1-\alpha}\right] \tag{1.3}$$

holds, where

$$M_1(k,T) = \max(\log^{3/2}k, \log T) \tag{1.4}$$

and the inner sum in (1.3) *is taken over all the primitive characters.*

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(1) The somewhat stronger inequality

$$N(\alpha, T) \ll T^{167(1-\alpha)^{3/2}}\log^{17}T$$

was found later by H. L. Montgomery. See his *Topics in Multiplicative Number Theory*, Lecture Notes in Math. 227, p. 102.

The coefficient 26 in the exponent on the right-hand side of (1.3) can easily be replaced by the number 1 but in this case the interval (1.2) has to be a bit shorter. This theorem extends obviously the method of G. Halász and P. Turán to Dedekind zeta-functions belonging to the cyclotomic fields with explicit bounds for the constants of the field, and to the quadratic fields as well (see [1], pp. 372–390).

2. In the proof of the theorem (1.2)–(1.4) we shall need the following lemmas:

LEMMA 1 (Turán's second main theorem, see [8] and [10]). *For arbitrary positive m, integer $n \leqslant N^*$ and complex numbers $\omega_1, \ldots, \omega_n$ there is an integer ν_0 satisfying the condition*

$$m \leqslant \nu_0 \leqslant m+N^* \tag{2.1}$$

that

$$\Big|\sum_{j=1}^{n} \omega_j^{\nu_0}\Big| \geqslant \left(\frac{N^*}{8e(m+N^*)}\right)^{N^*} |\omega_k|^{\nu_0}, \tag{2.2}$$

where ω_k stands for any of the ω_j's.

Denote by $\zeta(s, \omega)$ the well known Hurwitz-function. H.-E. Richert proved the following inequality (see [7]):

LEMMA 2. *For $\frac{1}{2} \leqslant \sigma \leqslant 1$, $t \geqslant 2$, $0 < \omega \leqslant 1$ we have*

$$\zeta(s, \omega) = \omega^{-s} + O(t^{100(1-\sigma)^{3/2}} \log^{2/3} t). \tag{2.3}$$

L. Schoenfeld could replace the constant 100 in the exponent by 39 (see [2], p. 132).

LEMMA 3. *Denoting*

$$M(k, T) = \max[\log k, (\log T)^{2/3} (\log\log T)^{1/3}] \tag{2.4}$$

we have $L(s, \chi, q) \neq 0$ in the region

$$\sigma \geqslant 1 - \frac{B}{M(k, T)}, \quad 4 \leqslant t \leqslant T, \tag{2.5}$$

where B is a positive numerical constant and $q \leqslant k$.

Proof. Using (2.3) we get for

$$\sigma \geqslant 1 - \frac{(\log\log t)^{2/3}}{(\log t)^{2/3}}, \quad t \geqslant 4,$$

the estimate

$$L(s, \chi, q) = O\left(\exp\left\{\left(\frac{\log\log t}{\log t}\right)^{2/3} \log q + 39\tfrac{2}{3} \log\log t\right\}\right). \tag{2.6}$$

From this it follows

$$L(s, \chi, q) \neq 0 \quad \text{for} \quad \sigma \geqslant 1 - \frac{B}{M(q, t)}$$

(see [6], p. 297 and [9]). But $M(q, t) \leqslant M(k, T)$ and our lemma follows.

LEMMA 4. *Denoting by $N_\chi(T)$ the number of roots of $L(s, \chi, q)$ in the region $|t| \leqslant T$, $0 < \sigma \leqslant 1$, $T \geqslant 2$, we have*

$$N_\chi(T) = O(T \log qT), \tag{2.7}$$

$$N_\chi(T+1) - N_\chi(T) = O(\log qT), \tag{2.8}$$

where the constants implied by the O-notation are independent of q (see [6], pp. 200–221).

LEMMA 5 (see [2], p. 130, and [5]). *Let $G(z)$ be regular for $|z| \leqslant R$, $G(0) \neq 0$ and let the inequality*

$$\left|\frac{G(z)}{G(0)}\right| \leqslant U \tag{2.9}$$

holds. Then if $0 < r < R$ and the zeros of $G(z)$ in the disc $|z| \leqslant r$ *are $z_1, z_2, \ldots$ then for all non-negative integers μ we have*

$$\left|\frac{1}{\mu!}\frac{G'}{G}(0)^{(\mu)} + (-1)^\mu \sum_{|z_j| \leqslant r} \frac{1}{z_j^{\mu+1}}\right| \leqslant \frac{2(\mu+1)\log U}{r^{\mu+1}}\left(1 + \frac{1}{\log(R/r)}\right). \tag{2.10}$$

The proof of the theorem (1.2)–(1.4) is based essentially on ideas contained in both papers of G. Halász and P. Turán ([2] and [3]).

3. Proof of the theorem. Let θ be such that

$$\exp(-14) \geqslant 1-\theta \geqslant \frac{B}{0.284\, M(k, T)} \tag{3.1}$$

where $M(k, T)$ is determined by (2.1) and the constant B is taken from (2.5). With such a θ we introduce the abbreviations

$$\sigma_0 = 1 + \theta(1-\theta), \tag{3.2}$$

$$\lambda = (1-\theta)^{3/2} \log^3 \frac{1}{1-\theta}, \tag{3.3}$$

$$s_0' = 2\sigma_0 - 1 + it_0, \quad r_0' = 2\sigma_0 - 1 - \theta, \quad t_0 \geqslant 3, \tag{3.4}$$

$$s_0'' = 2\sigma_0 - 1, \quad r_0'' = \left(1 - \frac{1}{\mu}\right) 2\theta(1-\theta), \quad \mu \geqslant 6. \tag{3.5}$$

As in [2] we get the estimates([2])

(3.6) $$|L^{(\mu)}(s_0', \chi, q)| \leqslant c\frac{\mu!\, q^{1-\theta} M^{5/3}(q, t_0)\, t_0^{39(1-\theta)^{3/2}}}{(r_0')^{\mu}},$$

(3.7) $$|L^{(\mu)}(s_0'', \chi_0, q)| \leqslant \frac{3(\mu+1)!}{(s_0''-1)^{\mu+1}},$$

where χ_0 denotes the principal character, and using in the disc $|s-s_0''| \leqslant r_0'$ the estimate

$$\zeta(s, \omega) = \omega^{-s} + \frac{1}{s-1} + O(1),$$

valid for $\sigma \geqslant \frac{1}{2}$, $|t| \leqslant 11$, where the constant implied by the O-notation is independent of ω and s (see [6], p. 115), we get for $\chi \neq \chi_0$ owing to (3.1),

(3.8) $$|L^{(\mu)}(s_0'', \chi, q)| \leqslant c\frac{\mu!\, M(k, T)\, k^{1-\theta}}{(r_0')^{\mu}}.$$

It is easy to notice that the function $x^{3/2}\log^2\frac{1}{x}$ is monotonically increasing in the interval $(0, e^{-4/3})$. Hence, owing to (3.1) and (3.3), we have

(3.9) $$\frac{\lambda}{\log\frac{1}{1-\theta}} = (1-\theta)^{3/2}\log^2\frac{1}{1-\theta} \geqslant \frac{\gamma\log M_1(k, T)}{M_1(k, T)},$$

where $M_1(k, T)$ is determined by (1.4) and γ is a sufficiently large numerical constant, if only $T > c$.

4. Let us consider on the segment

(4.1) $$I:\quad \sigma = \sigma_0,\ \tfrac{1}{2}T \leqslant t \leqslant T$$

with a fixed natural ν the set $H = H(\nu, k)$ (consisting of finitely many closed intervals) on which the inequality

(4.2) $$\left|\frac{L'}{L}(s, \chi^*, q)^{(\nu)}\right| \geqslant \frac{\nu!}{(1-\theta)^{\nu}} e^{-\lambda M_1(k,T)}$$

holds, for some L-function belonging to primitive character $\chi^* \bmod q$, $q \leqslant k$ (not necessary the same one in all points of H). Then denoting the measure of the set H by $|H|$ we assert the following

LEMMA. *For λ determined by* (3.3) *and θ determined by* (3.1) *and for*

(4.3) $$\frac{\lambda M_1(k, T)}{\log(\frac{1}{2}+\theta)}\left(1+\frac{20}{\log\frac{1}{1-\theta}}\right) \leqslant \nu+1 \leqslant \frac{\lambda M_1(k, T)}{\log(\frac{1}{2}+\theta)}\left(1+\frac{20.54}{\log\frac{1}{1-\theta}}\right)$$

([2]) Here and later c means unspecified positive numerical constant not necessarily the same in different occurrences.

the inequality

(4.4) $$|H| \leqslant M_1^9(k, T)\exp\{(2+10^{-5})\lambda M_1(k, T)\}$$

holds, if only T is sufficiently large.

Proof (compare [2], pp. 126–128). Let τ_1 be the smallest t-value in H and $\tau_1, \ldots, \tau_l$ being defined, let τ_{l+1} be the smallest t-value in H satisfying

$$\tau_{l+1} \geqslant \tau_l + 6$$

(if there is any). If $\tau_1, \ldots, \tau_M$ are all these points then

$$H \subset \bigcup_{l=1}^{M} [\tau_l, \tau_l+6]$$

and hence

(4.5) $$|H| \leqslant 6M.$$

Denote

$$s_j = \sigma_0 + i\tau_j, \quad j = 1, \ldots, M$$

and

$$F(s, q, \chi) = \frac{L'}{L}(s, \chi, q)^{(\nu)}.$$

Hence, owing to (4.2), we have

(4.6) $$\frac{M\nu!}{(1-\theta)^{\nu}} e^{-\lambda M_1(k,T)} \leqslant \sum_{j=1}^{M} |F(s_j, \chi_j^*, q_j)|,$$

where χ_j^* is one of these primitive characters $\bmod q_j$, $q_j \leqslant k$, for which the inequality (4.2) in the point s_j holds. Putting

$$\eta_j = \exp\left(-i\arg F(s_j, \chi_j^*, q_j)\right)$$

we have $|\eta_j| = 1$ and $|F(s_j, \chi_j^*, q_j)| = F(s_j, \chi_j^*, q_j)\eta_j$. But

$$F(s_j, \chi_j^*, q_j) = (-1)^{\nu+1}\sum_{n=2}^{\infty}\frac{\Lambda(n)\chi_j^*(n)\log^{\nu} n}{n^{s_j}}$$

and from (4.6) it follows

(4.7) $$\frac{M\nu!}{(1-\theta)^{\nu}} e^{-\lambda M_1(k,T)} \leqslant \sum_{n=2}^{\infty}\frac{1}{n^{1/2}\log n}\cdot\frac{\log^{\nu+2} n}{n^{\sigma_0-1/2}}\left|\sum_{j=1}^{M}\eta_j n^{-i\tau_j}\chi_j^*(n)\right|.$$

Applying Cauchy's inequality we get

$$(4.8)\qquad \frac{M^2\nu!^2}{(1-\theta)^{2\nu}}e^{-2\lambda M_1(k,T)}\leqslant c\sum_{n=2}^{\infty}\sum_{j_1,j_2=1}^{M}\frac{\log^{2\nu+4}n\,\chi^*_{j_1}(n)\overline{\chi^*_{j_2}(n)}}{n^{2\sigma_0-1+i(\tau_{j_1}-\tau_{j_2})}}\eta_{j_1}\overline{\eta}_{j_2}$$

$$=c\left|\sum_{j_1,j_2=1}^{M}\eta_{j_1}\overline{\eta}_{j_2}\sum_{n=2}^{\infty}\frac{\log^{2\nu+4}n\,\chi^*_{j_1}(n)\overline{\chi^*_{j_2}(n)}}{n^{2\sigma_0-1+i(\tau_{j_1}-\tau_{j_2})}}\right|$$

$$\leqslant c\left(\sum_{j=1}^{M}\left|\sum_{\substack{n=2\\(n,q_j)=1}}^{\infty}\frac{\log^{2\nu+4}n}{n^{2\sigma_0-1}}\right|+\sum_{\substack{j_1\neq j_2\\ j_1,j_2=1}}^{M}\left|\sum_{n=2}^{\infty}\frac{\log^{2\nu+4}n\,\chi_{j_1j_2}(n)}{n^{2\sigma_0-1+i(\tau_{j_1}-\tau_{j_2})}}\right|\right)$$

$$\leqslant cM\Big(\max_{1\leqslant q\leqslant k}|L^{(2\nu+4)}(2\sigma_0-1,\chi_0,q)|+$$

$$+M\max_{\substack{2\leqslant q\leqslant k^2\\ 6\leqslant t'-t''\leqslant T/2\\ \chi\neq\chi_0}}\left|L^{(2\nu+4)}\big(2\sigma_0-1+i(t'-t''),\chi,q\big)\right|\Big).$$

Using the estimates (3.6) and (3.7) we have

$$(4.9)\qquad \frac{M\nu!^2}{(1-\theta)^{2\nu}}e^{-2\lambda M_1(k,T)}$$

$$\leqslant\frac{c(2\nu+5)!}{(2\sigma_0-2)^{2\nu+5}}+\frac{c_1M(2\nu+4)!\,k^{2(1-\theta)}M^{5/3}(k^2,T)\,T^{39(1-\theta)^{3/2}}}{(2\sigma_0-1-\theta)^{2\nu+4}}.$$

Applying (4.3) and then (3.9) we get

$$(4.10)\qquad (\tfrac12+\theta)^{2\nu+2}\geqslant M_1^{\nu}(k,T)\exp\left\{2\lambda M_1(k,T)\left(1+\frac{19.5}{\log\frac{1}{1-\theta}}\right)\right\}.$$

Since

$$2\sigma_0-1-\theta=2(1-\theta)(\tfrac12+\theta)$$

and for θ from (3.1) and T sufficiently large

$$2(1-\theta)\log k+39(1-\theta)^{3/2}\log T-39\lambda\frac{M_1(k,T)}{\log\frac{1}{1-\theta}}\leqslant 0$$

then by the use of the Stirling formula, the second term in (4.9) is less than

$$\frac12\,\frac{M\nu!^2e^{-2\lambda M_1(k,T)}}{(1-\theta)^{2\nu}}.$$

Hence for T sufficiently large

$$(4.11)\qquad M\leqslant c\frac{(2\nu+5)!\,e^{2\lambda M_1(k,T)}}{2^{2\nu+5}\nu!^2(1-\theta)^5\theta^{2\nu+5}}.$$

Owing to (4.3) and (3.1)

$$(4.12)\qquad \theta^{-2\nu-2}\leqslant\exp\left\{2\lambda M_1(k,T)\left(1+\frac{20.54}{\log\frac{1}{1-\theta}}\right)\frac{1-\theta}{\theta\log(\frac12+\theta)}\right\}$$

$$\leqslant\exp\{10^{-5}\lambda M_1(k,T)\}.$$

Therefore from (4.11) by the use of the Stirling formula for T sufficiently large we have

$$M\leqslant\tfrac16 M_1^9(k,T)\exp\{(2+10^{-5})\lambda M_1(k,T)\}$$

which ends the proof of the lemma.

What we shall use of this lemma is a simple corollary. Let us consider on I the set H^* of s-values for which the inequality

$$(4.13)\qquad \left|\frac{L'}{L}(s,\chi^*,q)^{(\nu)}\right|<\frac{\nu!\,e^{-\lambda M_1(k,T)}}{(1-\theta)^{\nu}}$$

holds for all ν-values permitted by (4.3) and for all $L(s,\chi^*,q)$-functions, $q\leqslant k$. Its complementary set $\bar H^*$ with respect to $[\frac12T,T]$ is certainly covered by the union of the above $H=H(\nu,k)$ sets

$$\bar H^*\subset\bigcup_{\nu}H(\nu,k).$$

Hence, owing to (4.3), we have from (4.4) the

COROLLARY. *Dropping from I a suitable set $\bar H^*$ (consisting of finitely many closed intervals) of measure*

$$(4.14)\qquad \leqslant M_1^{10}(k,T)\exp\{(2+10^{-5})\lambda M_1(k,T)\}$$

in the remaining points of I the inequality (4.13) holds simultaneously for all ν-values permitted by (4.3) and for all $L(s,\chi^*,q)$-functions, $q\leqslant k$, if only $T>c$.

5. Let us consider the horizontal strips l_j defined by

$$(5.1)\qquad \frac T2+\frac{j}{M_1^3(k,T)}\leqslant t\leqslant\frac T2+\frac{j+1}{M_1^3(k,T)},$$

$$j=0,1,\ldots,[\tfrac12TM_1^3(k,T)].$$

We call a strip l_j a "good" one if its intersection with I contains at least one point of the set H^*, otherwise we call it a "bad" one. The number of "bad" strips is, owing to (4.14),

$$(5.2)\qquad \leqslant\frac{|\bar H^*|}{1/M_1^3(k,T)}\leqslant M_1^{13}(k,T)\exp\{(2+10^{-5})\lambda M_1(k,T)\}.$$

Let us fix any "bad" strip l_j'' and let $z_j'' = \sigma_0 + it''$ be a fixed point in it. Let us count the number of these $L(s, \chi^*, q)$-functions, $q \leqslant k$, for which the inequality

$$\left|\frac{L'}{L}(z_j'', \chi^*, q)^{(\nu)}\right| \geqslant \frac{\nu!\, e^{-\lambda M_1(k,T)}}{(1-\theta)^\nu} \tag{5.3}$$

for a fixed ν from (4.3) holds (compare [3], pp. 408–410). Let

$$\chi_j^* \bmod q_j, \quad q_j \leqslant k, \quad j = 1, 2, \ldots, N$$

be all primitive characters, for which the inequality (5.3) holds. Analogously to (4.8) we get

$$\frac{N\nu!^2 e^{-2\lambda M_1(k,T)}}{(1-\theta)^{2\nu}} \leqslant c\Big(\max_{1\leqslant q\leqslant k} |L^{(2\nu+4)}(2\sigma_0-1, \chi_0, q)| + N \max_{\substack{2\leqslant q\leqslant k^2\\ \chi\neq\chi_0}} |L^{(2\nu+4)}(2\sigma_0-1, \chi, q)|\Big). \tag{5.4}$$

Applying (3.8) and (3.9) for the first and the second term respectively we get

$$\frac{N\nu!^2 e^{-2\lambda M_1(k,T)}}{(1-\theta)^{2\nu}} \leqslant c\left(\frac{(2\nu+5)!}{(2\sigma_0-2)^{2\nu+5}} + N\frac{(2\nu+4)!\, M^2(k,T)\, k^{2(1-\theta)}}{(2\sigma_0-1-\theta)^{2\nu+4}}\right). \tag{5.5}$$

By the same reasoning as in lemma (4.3)–(4.4), we get the following statement:

The number of "bad" $L(s, \chi^*, q)$-functions for which the inequality (5.3) holds, for some ν-values from (4.3) in a fixed z_j'' point belonging to a "bad" strip l_j'' cannot exceed

$$M_1^{10}(k, T)\exp\{(2+10^{-5})\lambda M_1(k, T)\} \tag{5.6}$$

if only T is sufficiently large and θ satisfy (3.1). Owing to Lemma 4 each "bad" strip contains at most

$$c \log qt < cM_1(k, T)$$

zeros of $L(s, \chi^*, q)$. Hence from (5.2) and (5.6) the number of roots of "bad" $L(s, \chi^*, q)$-functions in all "bad" strips of the rectangle

$$\exp(-14) \geqslant 1-\theta \geqslant \frac{B}{0.284 M(k, T)}, \quad T/2 \leqslant t \leqslant T, \ T > c \tag{5.7}$$

cannot exceed

$$M_1^{24}(k, T)\exp\{2(2+10^{-5})\lambda M_1(k, T)\}. \tag{5.8}$$

Let

$$z^* = \sigma_0 + it^*$$

be a point of H^* in any fixed "good" strip or a point z_j'' in any fixed "bad" strip l_j''. Hence for all ν-values from the interval (4.3) we have

$$\left|\frac{L'}{L}(z^*, \chi^*, q)^{(\nu)}\right| < \frac{\nu!\, e^{-\lambda M_1(k,T)}}{(1-\theta)^\nu}, \tag{5.9}$$

where $L(s, \chi^*, q)$ is any L-function in the first case and a "good" one in the second case. We shall apply Lemma 5 taking

$$G(z) = L(z+z^*, \chi^*, q), \quad R = 3.2(1-\theta), \quad r = 3.2\, e^{-11/24}(1-\theta), \quad \mu = \nu.$$

This gives

$$\left|\frac{1}{\nu!}\frac{L'}{L}(z^*, \chi^*, q)^{(\nu)} - \sum_{|z^*-\varrho|\leqslant r} \frac{1}{(z^*-\varrho)^{\nu+1}}\right| \leqslant \frac{70(\nu+1)\log U}{11 r^{\nu+1}}, \tag{5.10}$$

where

$$U = \max_{|z|\leqslant R}\left|\frac{L(z+z^*, \chi^*, q)}{L(z^*, \chi^*, q)}\right|.$$

Owing to (3.2) and (3.1) we have

$$\frac{1}{L(z^*, \chi^*, q)} = O\left(\frac{1}{\sigma_0-1}\right) = O\big(M(k, T)\big).$$

Using Richert's estimate (2.3) we get in the disc $|s-z^*| \leqslant R$

$$L(s, \chi^*, q) = O\big(M(k, T)\, k^{(3.2-\theta)(1-\theta)}\, T^{39(3.2-\theta)^{3/2}(1-\theta)^{3/2}} \log^{2/3} T\big).$$

Therefore we can put

$$U = M^3(k, T)\, k^{(3.2-\theta)(1-\theta)}\, T^{39(3.2-\theta)^{3/2}(1-\theta)^{3/2}}. \tag{5.11}$$

Owing to (5.9) we can write the formula (5.10) in the form

$$\left|\sum_{|z^*-\varrho|\leqslant r}\left(\frac{1-\theta}{z^*-\varrho}\right)^{\nu+1}\right| \leqslant \frac{70(\nu+1)\log U}{11(3.2\, e^{-11/24})^{\nu+1}} + (1-\theta)\, e^{-\lambda M_1(k,T)}. \tag{5.12}$$

From (4.3), (3.9) and (5.11) it is easily seen that for $T > c$ and for all ν-values permitted by (4.3) we have

$$\left|\sum_{|z^*-\varrho|\leqslant r}\left(\frac{1-\theta}{z^*-\varrho}\right)^{\nu+1}\right| < \frac{1}{2}\, e^{-\lambda M_1(k,T)}. \tag{5.13}$$

6. In order to estimate the sum of (5.13) from below we shall apply Lemma 1. We choose

$$m = \frac{\lambda M_1(k,T)}{\log(\frac{1}{2}+\theta)}\left(1+\frac{20}{\log\frac{1}{1-\theta}}\right). \tag{6.1}$$

In order to estimate the number of terms in the sum (5.13) we shall apply Jensen's inequality which gives for the number of zeros of the regular function $f(s)$ in the disc $|s-s_0|\leqslant r<R$ the bound

$$\frac{1}{\log(R/r)}\max_{|s-s_0|\leqslant R}\log\left|\frac{f(s)}{f(s_0)}\right|.$$

Owing to (5.11), by the use of (3.9), for sufficiently large T we have

$$\frac{24}{11}\log U < \frac{\lambda M_1(k,T)}{\log(\frac{1}{2}+\theta)\log\frac{1}{1-\theta}}\left(\frac{96\log M(k,T)\log(\frac{1}{2}+\theta)}{11\gamma\log M_1(k,T)}+\right.$$
$$\left.+\frac{87\log k\log(\frac{1}{2}+\theta)}{\sqrt{1-\theta}M_1(k,T)\log^2\frac{1}{1-\theta}}+\frac{936(3.2-\theta)^{3/2}\log(\frac{1}{2}+\theta)}{11M_1(k,T)\log^2\frac{1}{1-\theta}}\log T\right).$$

It is easily to notice that the first two terms on the right-hand side are, owing to (3.1), arbitrary small if only T is sufficiently large. The third term in the brackets is $\leqslant 0.537$. Hence we can choose

$$N^* = \frac{0.54\lambda M_1(k,T)}{\log(\frac{1}{2}+\theta)\log\frac{1}{1-\theta}}. \tag{6.2}$$

Moreover in any strip l_j we have, owing to (4.3),

$$\left(\frac{2-\theta-\sigma_\varrho}{z^*-\varrho}\right)^{\nu+1}\geqslant\left(1+\frac{c}{M_1^2(k,T)}\right)^{-\nu-1}>\frac{1}{2}$$

if only T is sufficiently large. Hence from Lemma 1 it follows that there exist such a ν_0 satisfying (4.3) for which

$$\left|\sum_{|z^*-\varrho|\leqslant r}\left(\frac{1-\theta}{z^*-\varrho}\right)^{\nu_0+1}\right|\geqslant\frac{1}{2}\exp\left\{-0.698\lambda M_1(k,T)-(\nu_0+1)\log\frac{2-\theta-\sigma_{\varrho^*}}{1-\theta}\right\}. \tag{6.3}$$

where $\varrho^*=\sigma_{\varrho^*}+it_{\varrho^*}$ denotes the zero which has in our strip the greatest real part σ_{ϱ^*}.

Using the right-hand side of (4.3) and comparing (6.3) with (5.13) we get for θ satisfying (3.1)

$$\frac{1-\sigma_{\varrho^*}}{1-\theta}>0.284. \tag{6.4}$$

This proves, that the $\varrho=\sigma_\varrho+it_\varrho$ zeros of any $L(s,\chi^*,q)$-function in "good" strips or of "good" one in "bad" strips satisfy the inequality

$$\sigma_\varrho<1-0.284(1-\theta).$$

Owing to (5.7), (5.8) we have then for

$$\frac{B}{M(k,T)}\leqslant 1-\alpha\leqslant 0.284\exp(-14) \tag{6.5}$$

the estimate

$$\sum_{q\leqslant k}\sum_{\chi \bmod q}^{*}\big(N(\alpha,T,\chi,q)-N(\alpha,T/2,\chi,q)\big) < M_1^{24}(k,T)e^{2(2+10^{-5})\lambda M_1(k,T)}. \tag{6.6}$$

Analogously as in proof of (5.6) one can show

$$\sum_{q\leqslant k}\sum_{\chi \bmod q}^{*}N(\alpha,c,q) = O\Big(\log^{20}k\exp\big((2+10^{-5})\lambda\log^{3/2}k\big)\Big) \tag{6.7}$$

where c is a constant and α belongs to the interval (1.2). Replacing in (6.6) T by

$$\frac{T}{2},\ldots,\frac{T}{2^i}\geqslant c\geqslant\frac{T}{2^{i+1}}$$

we get after summation and application of (6.7)

$$\sum_{q\leqslant k}\sum_{\chi \bmod q}^{*}N(\alpha,T,\chi,q)<M_1^{25}(k,T)\exp\{2(2+10^{-5})\lambda M_1(k,T)\}. \tag{6.8}$$

Applying now (3.9) we get our theorem.

References

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On the problem of divisors

by

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1. Introduction. Let k be an integer greater than 2. Let $\tau_k(n)$ be the number of the solutions of the equation $n = m_1 m_2 \ldots m_k$ in integers $m_i \geqslant 1$. We are concerned with the estimate of the number

$$D_k(x) = \sum_{n \leqslant x} \tau_k(n).$$

Let $M_k(x)$ be the residue of $\zeta^k(s)x^s/s$ at $s = 1$, where $\zeta(s)$ is the Riemann zeta-function. It is well known that $D_k(x) \sim M_k(x)$ and if we put $\Delta_k(x) = D_k(x) - M_k(x)$,

$$\Delta_k(x) \ll x^{1-1/k}(\log x)^{k-2} \quad \text{for} \quad k = 2, 3, 4, \ldots$$

(Cf. 12.1.4 of [5].) It was shown by Hardy and Littlewood that

$$\Delta_k(x) \ll x^{\frac{k-1}{k+2}+\varepsilon} \quad \text{for each } k \geqslant 4.$$

(Cf. 12.3 of [5].)

Generally if we put $\zeta(\frac{1}{2}+it) \ll |t|^{\lambda}$, then their method gives

$$\Delta_k(x) \ll x^{\frac{2(k-4)\lambda+1}{2(k-4)\lambda+2}+\varepsilon} \quad \text{for each } k \geqslant 4. \tag{1}$$

It is well known that we can take $\lambda = 173/1067$ which is due to Kolesnik [3]. In 1971 Karatsuba [1] showed

$$\Delta_k(x) \ll x^{1-Ck^{-2/3}+\varepsilon} \quad \text{for each } k \geqslant 2,$$

where C is some positive absolute constant. In this paper we shall improve these results for k in $10 \leqslant k \leqslant k_0$, where k_0 is some positive constant which depends on C above. Our proof depends on only the well known properties of $\zeta(s)$. To state our result we shall introduce some notations. Let b be an integer greater than 3. Let $j(b)$ be determined by

$$(j-1)2^{j-2}+1 < b \leqslant j2^{j-1}+1.$$