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Some non-linear diophantine approximations

by

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Introduction. Throughout the paper, k denotes a positive integer, ε an arbitrary positive number, and $C(k, \varepsilon)$ a positive number depending at most on k and ε, not necessarily the same at each occurrence, similarly for $C(k)$, $C(\varepsilon)$. $\|\alpha\|$ denotes the distance between α and the nearest integer. We write $K = 2^{k-1}$.

In 1948 Heilbronn proved the following deep and important theorem [11].

Theorem 1. *For any $N \geqslant 1$ and any real θ there is an integer x satisfying*

$$1 \leqslant x \leqslant N \quad \text{and} \quad \|\theta x^2\| < C(\varepsilon) N^{-1/2+\varepsilon}.$$

Heilbronn's result is analogous to Dirichlet's theorem (Lemma 3, below) in that the degree of approximation and the constant are independent of θ, N. We can rephrase it as

$$\min_{1 \leqslant x \leqslant N} \|\theta x^2\| < C(\varepsilon) N^{-1/2+\varepsilon} \quad (N \geqslant 1,\ \theta \text{ real}).$$

The method of [11] has been applied by several authors. Thus Danicic [6] and Davenport [10] proved independently:

Theorem 2.

$$\min_{1 \leqslant x \leqslant N} \|\theta x^k\| < C(k, \varepsilon)\, N^{-1/K+\varepsilon} \quad (N \geqslant 1,\ \theta \text{ real});$$

and Davenport [10] proved

Theorem 3. *For any polynomial f of degree k without constant term,*

$$\min_{1 \leqslant x \leqslant N} \|f(x)\| < C(k, \varepsilon) N^{-1/(2K-1)+\varepsilon} \quad (N \geqslant 1).$$

Davenport's paper forms a very good introduction to Heilbronn's method and to this paper in particular.

Simultaneous diophantine approximations of this kind have also been studied. In [15], [16] Liu (improving a result of Danicic [6], [7]) proved the following

THEOREM 4.

$$\min_{1\leqslant x\leqslant N}\max(\|\theta_1x^2\|,\|\theta_2x^2\|)<C(\varepsilon)N^{-1/7+\varepsilon}$$

and if $k\geqslant 3$

$$\min_{1\leqslant x\leqslant N}\max(\|\theta_1x^k\|,\|\theta_2x^k\|)<C(k,\varepsilon)N^{-k/(3Kk+1)+\varepsilon}$$

($N\geqslant 1$, θ_1, θ_2 *real*).

Along similar lines Cook [4] proved

THEOREM 5. *If* f_1, f_2 *are quadratic polynomials without constant term,*

$$\min_{1\leqslant x\leqslant N}\max_{i=1,2}\|f_i(x)\|<C(\varepsilon)N^{-1/22+\varepsilon}\quad (N\geqslant 1).$$

Cook also proved results for r quadratic polynomials, $r=3,4,\ldots$ ([3], [4]). All the above theorems depend on Weyl's estimates for exponential sums. Using the method of Vinogradov, Cook [3] proved

THEOREM 6. *For* $k\geqslant 12$,

$$\min_{1\leqslant x\leqslant N}\|\theta x^k\|<C(\varepsilon,k)N^{-\varrho_k+\varepsilon}\quad (N\geqslant 1,\ \theta\ \textit{real})$$

where

$$\varrho_k^{-1}=4k(k-1)\log\big(12(k-1)^2\big)\sim 8k^2\log k.$$

Cook has also applied Vinogradov's estimates to obtain results for r polynomials of degree k ([3], [5]). Other results on Heilbronn's method include two beautiful papers on quadratic forms by Danicic ([7], [9]) and results on additive forms of degree k ([2], [16]); see also [14] where Liu slightly sharpens Theorem 1.

In this paper we consider simultaneous approximations to monomials of different degrees, $\min_{1\leqslant x\leqslant N}\max(\|\theta_1x^{a_1}\|,\ldots,\|\theta_rx^{a_r}\|)$ where (throughout the paper) $1\leqslant a_1<\ldots<a_r\leqslant k$ are integers. More generally, we prove an analogue of Minkowski's theorem:

THEOREM 7. *Let* $0<\eta_i<1$, $i=1,\ldots,k$, $\eta_1\ldots\eta_k\geqslant N^{-1}$. *Let* $\theta_1,\ldots,\theta_k$ *be real. There is an integer* x *such that*

$$1\leqslant x\leqslant N,\quad \|\theta_ix\|\leqslant\eta_i\quad (1\leqslant i\leqslant k).$$

See [1] for Minkowski's convex body theorem, which implies Theorem 7. We prove the following result.

THEOREM 8. *Let* $0<\eta_i<1$ $(i=1,\ldots,k)$,

$$\eta_1\ldots\eta_k\geqslant C(k,\varepsilon)N^{-1/K+\varepsilon}. \tag{1.1}$$

Let θ *be real. Then there is an integer* x *such that*

$$1\leqslant x\leqslant N,\quad \|\theta x^i\|\leqslant\eta_i\quad (i=1,\ldots,k). \tag{1.2}$$

THEOREM 9. *Let* $0<\eta_i<1$ $(i=1,2)$,

$$\eta_1^3\eta_2^2\geqslant C(\varepsilon)N^{-1+\varepsilon}. \tag{1.3}$$

Let θ_1, θ_2 *be real. Then there is an integer* x *such that*

$$1\leqslant x\leqslant N,\quad \|\theta_ix^i\|\leqslant\eta_i\quad (i=1,2). \tag{1.4}$$

Taking $\eta_1=\ldots=\eta_{k-1}=N^{-\varepsilon}$ in Theorem 8, we recover Theorem 2. If the η_i's are all equal we obtain from Theorems 8 and 9

$$\min_{1\leqslant x\leqslant N}\max_{1\leqslant i\leqslant k}\|\theta x^i\|<C(k,\varepsilon)N^{-1/Kk+\varepsilon}, \tag{1.5}$$

$$\min_{1\leqslant x\leqslant N}\max(\|\theta_1x\|,\|\theta_2x^2\|)<C(\varepsilon)N^{-1/5+\varepsilon} \tag{1.6}$$

($N>1$, θ_1, θ_2 real). These represent great improvements on Theorem 5 in particular cases. (1.6) is a special case of

THEOREM 10. *Let* $1\leqslant a_1<\ldots<a_r\leqslant k$, *then*

$$\min_{1\leqslant x\leqslant N}\max_{1\leqslant i\leqslant r}\|\theta_ix^{a_i}\|<C(k,\varepsilon)N^{-1/u_{k,r}+\varepsilon}$$

($\theta_1,\ldots,\theta_k$ *real*, $N\geqslant 1$), *where*

$$u_{k,r}=\begin{cases}2^{k-r}\big((r-1)2^r+1\big), & 1\leqslant r\leqslant 4,\\ 2^{k-4}\big(65+2(r-5)(r+4)\big), & r\geqslant 5.\end{cases}$$

We prove Theorems 8 and 9 in § 3 and Theorem 10 in § 4. In §§ 5 and 6 we use Hua's improvement [12] of Vinogradov's method to improve Theorem 6, inequality (1.5) and Theorem 10 for large k.

In the remainder of the paper, $e(z)$ denotes $e^{2\pi iz}$. Until further notice, by $F\ll G$ we mean $G>0$, $|F|<C(k,\varepsilon)G$. In proving any theorem we may assume (without making it explicit) that $\varepsilon<\varepsilon(k)$, N is an integer and $N>C(\varepsilon,k)$. b denotes an integer >0 depending on k, not necessarily the same at each occurrence.

We should like to thank our friend and teacher Dr. A. J. Jones, who initiated us into Heilbronn's method and constantly encouraged our work.

2. Heilbronn's method. We state a number of lemmas of analytic number theory.

LEMMA 1. *Let* $0<\Delta<\frac{1}{2}$ *and let* a *be a positive integer. There exists a real function* $\psi(z)$, *of period* 1, *such that*

$$\psi(z)=0\quad \textit{for}\quad \|z\|>\Delta,$$

$$\psi(z)=\sum_{m=-\infty}^{\infty}a_me(mz),\quad a_0=\Delta,\quad a_{-m}=a_m$$

and

$$|a_m|<C(a)\min(\Delta,m^{-a-1}\Delta^{-a}),\quad m\geqslant 1. \tag{2.1}$$

Thus

$$\sum_{m} |a_m| \ll C(a). \tag{2.2}$$

Proof. This is a special case of Lemma 12 of Chapter I of [17].

Lemma 2. *Let $f_m(x) = \theta x^k + \theta_{m1} x^{k-1} + \ldots + \theta_{m,k-1} x$ be a polynomial with real coefficients. Let m be an integer $\geqslant 1$ and let*

$$S(m) = \sum_{x=1}^{N} e\big(mf_m(x)\big).$$

Suppose $|\theta - aq^{-1}| \leqslant q^{-2}$ for some integers $q \geqslant 1$, a, where $(a, q) = 1$. Then for $H \geqslant 1$,

$$\sum_{m=1}^{H} |S(m)|^K \ll (HN)^{\varepsilon}\left(HN^{K-1} + N^{K-k}\left(\frac{HN^{k-1}}{q} + 1\right)(N + q\log q)\right). \tag{2.3}$$

If $k = 1$, $q \geqslant 2H$ then

$$\sum_{m=1}^{H} |S(m)| \ll q\log q. \tag{2.4}$$

Proof. By Satz 266 of [13],

$$|S(m)|^K \ll N^{K-1+\varepsilon/2} + N^{K-k+\varepsilon/2} \sum_{v=1}^{k!N^{k-1}} \min\left(N, \frac{1}{\|mv\theta\|}\right).$$

But

$$\sum_{m=1}^{H} \sum_{v=1}^{k!N^{k-1}} \min\left(N, \frac{1}{\|mv\theta\|}\right) \ll (HN)^{\varepsilon/2} \sum_{x=1}^{k!HN^{k-1}} \min\left(N, \frac{1}{\|x\theta\|}\right)$$

because $x = mv$ has $\ll (HN)^{\varepsilon/2}$ solutions m for any x. Now we can complete the proof using well-known techniques; see for example Lemma 8a of Chapter I of [17]. For the second part,

$$\sum_{m=1}^{H} |S(m)| \leqslant \sum_{m=1}^{H} \|m\theta\|^{-1} \leqslant \sum_{1 \leqslant m < q/2} \|m\theta\|^{-1}.$$

Since $(a, q) = 1$, the integers $k \equiv am \pmod q$, $0 \leqslant k < q$, are distinct and not zero $(1 \leqslant m < q)$, so for $1 \leqslant m < q/2$,

$$\|m\theta\| = \left\| maq^{-1} + \frac{\varphi}{2q} \right\| \quad (|\varphi| < 1)$$

$$= \left\| kq^{-1} + \frac{\varphi}{2q} \right\| \gg \frac{\min(k, q-k)}{q};$$

$$\sum_{1 \leqslant m < q/2} \|m\theta\|^{-1} \ll \sum_{1 \leqslant k \leqslant q/2} qk^{-1} \ll q\log q.$$

Lemma 3. *Let θ be a real number and $Q \geqslant 1$. There are integers a, q such that $(a, q) = 1$, $1 \leqslant q \leqslant Q$ and*

$$|\theta - aq^{-1}| \leqslant q^{-1}Q^{-1}.$$

Proof. This is a special case of Theorem 7.

The next lemma contains the essence of Heilbronn's method. An 'improvement on Dirichlet's theorem', (2.6), (2.7), is 'purchased' by a 'Heilbronn hypothesis', as we call the hypothesis of Lemma 4 about the absence of an integral solution.

Lemma 4. *Let $\lambda_1, \ldots, \lambda_r$ be positive real numbers and $\theta_1, \ldots, \theta_r$ any real numbers. Suppose there is no integral solution of the inequalities*

$$1 \leqslant x \leqslant N, \quad \|x^{a_j}\theta_j\| \leqslant N^{-\lambda_j} \quad (j = 1, \ldots, r).$$

Then there is a j, $1 \leqslant j \leqslant r$, such that either

$$K_j(\lambda_1 + \ldots + \lambda_j) \geqslant 1 - \varepsilon(jK_j + 3) \quad (K_j = 2^{a_j-1}) \tag{2.5}$$

or there is an integer $q \geqslant 1$, such that

$$q \ll N^{K_j(\lambda_1+\ldots+\lambda_j)+\varepsilon(jK_j+2)}, \tag{2.6}$$

$$\|q\theta_j\| \ll N^{-a_j-\lambda_j+K_j(\lambda_1+\ldots+\lambda_j)+\varepsilon(jK_j+2)}. \tag{2.7}$$

Proof. Let $\psi_i(z)$ be as in Lemma 1 with $\Delta = \frac{1}{2}N^{-\lambda_i}$, $a = [1/\varepsilon]+1$ $(i = 1, \ldots, r)$. By hypothesis,

$$\sum_{x=1}^{N} \prod_{i=1}^{r} \psi_i(x^{a_i}\theta_i) = 0,$$

or

$$2^{-r}N^{1-(\lambda_1+\ldots+\lambda_r)} + \sum{}^{*} a_{m_1} \ldots a_{m_r} T(\boldsymbol{m}^r) = 0 \tag{2.8}$$

where $\sum^*$ denotes a summation over all nonzero integral vectors

$$\boldsymbol{m}^r = (m_1, \ldots, m_r);$$

and

$$T(\boldsymbol{m}^j) = \sum_{x=1}^{N} e(m_1\theta_1 x^{a_1} + \ldots + m_j\theta_j x^{a_j}) \quad (1 \leqslant j \leqslant r).$$

From (2.8),

$$N^{1-(\lambda_1+\ldots+\lambda_r)} \ll \sum{}^{*} |a_{m_1} \ldots a_{m_r} T(\boldsymbol{m}^r)|.$$

Summing over the $\boldsymbol{m}^r$ with $|m_1| > N^{\lambda_1+\varepsilon}$ we have by (2.1), (2.2),

$$\sum |a_{m_1} \ldots a_{m_r} T(\boldsymbol{m}^r)| \ll N \sum_{m_1 > N^{\lambda_1+\varepsilon}}' N^{a\lambda_1} m_1^{-a-1} \ll N^{1-a\varepsilon} \ll 1$$

and similarly for $|m_2| > N^{\lambda_1+\varepsilon}$, etc. Thus (unless $1-(\lambda_1+\ldots+\lambda_r) < \varepsilon$, which we may ignore as it would end the proof),

$$N^{1-(\lambda_1+\ldots+\lambda_r)} \ll \sum\nolimits^{(r)} |a_{m_1} \ldots a_{m_r} T(\boldsymbol{m}^r)|,$$

where $\sum^{(r)}$ is a sum over $|m_i| \leqslant N^{\lambda_i+\varepsilon}$ $(i = 1, \ldots, r)$, $\boldsymbol{m}^r \neq 0$, and using (2.1),

$$N \ll \sum\nolimits^{(r)} |T(\boldsymbol{m}^r)|.$$

Clearly there is a j, $1 \leqslant j \leqslant r$, such that

$$N \ll \sum_{m_j \neq 0}^{(j)} |T(\boldsymbol{m}^j)|.$$

This completes the first stage of the lemma. Now, if $a_j > 1$, we use the Weyl estimate (2.3). By Hölder's inequality,

(2.9) $$N^{K_j} \ll N^{(\lambda_1+\ldots+\lambda_j+j\varepsilon)(K_j-1)} \sum_{m_j \neq 0}^{(j)} |T(\boldsymbol{m}^j)|^{K_j}$$
$$\ll N^{K_j(\lambda_1+\ldots+\lambda_j+j\varepsilon)-\lambda_j+\varepsilon}\left(N^{K_j-1+\lambda_j}+N^{K_j-a_j}\left(\frac{N^{a_j-1+\lambda_j+\varepsilon}}{q}+1\right)(N+q\log q)\right)$$

for any integer $q \geqslant 1$ such that

(2.10) $$\left|\theta_j - \frac{a}{q}\right| \leqslant \frac{1}{qQ}$$

(a integer, $(a, q) = 1$, $q \leqslant Q$). We take $Q = N^{\mu}$, where $\mu = a_j - K_j(\lambda_1+\ldots+\lambda_j)+\lambda_j - b\varepsilon$; b is a large integer depending on k; and choose such a q using Lemma 3.

We may assume that (2.5) is false. Thus

$$N^{K_j(\lambda_1+\ldots+\lambda_j+j\varepsilon)} N^{K_j-1+\varepsilon} \ll N^{K_j-\varepsilon}$$

and (2.9) yields

$$N^{a_j-K_j(\lambda_1+\ldots+\lambda_j)+\lambda_j-(jK_j+1)\varepsilon} \ll \left(\frac{N^{a_j-1+\lambda_j+\varepsilon}}{q}+1\right)(N+q\log q)$$
$$\ll \frac{N^{a_j+\lambda_j+\varepsilon}}{q}+N^{a_j-1+\lambda_j+\varepsilon}\log q+q\log q.$$

If $b = jK_j+2$, the term $q\log q$ is negligible. Since (2.5) is false, the term $N^{a_j-1+\lambda_j+\varepsilon}\log q$ is negligible, and we deduce that (2.6) holds. Finally, (2.10) implies (2.7).

If $a_j = 1$, we have instead of (2.9),

$$N \ll \sum_{m_j=1}^{N^{\lambda_1+\varepsilon}} |T(\boldsymbol{m}^1)|.$$

We let q be an integer, $1 \leqslant q \leqslant Q = N^{1-\varepsilon}$, such that (2.10) holds with $j = 1$. If $q \geqslant 2N^{\lambda_1+\varepsilon}$, Lemma 2 gives

$$N \ll q\log q$$

which is absurd. So $q \leqslant 2N^{\lambda_1+\varepsilon}$, $\|q\theta_1\| \leqslant N^{-1+\varepsilon}$ which implies (2.6) and (2.7). This completes the proof of the lemma.

3. Proofs of Theorems 8 and 9

Proof of Theorem 8. Suppose there is no solution of the simultaneous inequalities (1.2). We apply Lemma 4 with $r = k$, $a_j = j$, $N^{-\lambda_j} = \eta_j$, $\theta_j = \theta$ $(1 \leqslant j \leqslant k)$. We shall show that

(3.1) $$K_k(\lambda_1+\ldots+\lambda_k) \geqslant 1 - b\varepsilon \quad (K_k = K)$$

for a suitable b. This can be restated as

$$\eta_1 \ldots \eta_k \leqslant N^{-(1-b\varepsilon)/K}$$

so with (3.1) we will have proved the theorem.

In Lemma 5, we may assume (2.5) is false since if it held the proof would be complete. Thus for some j, $1 \leqslant j \leqslant k$, we have an integer $q \geqslant 1$ such that

$$q \ll N^{K_j(\lambda_1+\ldots+\lambda_j)+\varepsilon(jK_j+2)} \quad (K_j = 2^{j-1})$$

so that $q \leqslant N$; and

$$\|q\theta\| \ll N^{-j-\lambda_j+K_j(\lambda_1+\ldots+\lambda_j)+\varepsilon(jK_j+2)}.$$

Thus for $1 \leqslant i \leqslant k$,

$$\|q^i\theta\| \leqslant q^{i-1}\|q\theta\| \ll N^{iK_j(\lambda_1+\ldots+\lambda_j)-j-\lambda_j+b\varepsilon}$$

where $b = i(jK_j+2)$. By the 'Heilbronn hypothesis' we have for some i,

$$N^{iK_j(\lambda_1+\ldots+\lambda_j)-j-\lambda_j+(b+1)\varepsilon} \geqslant N^{-\lambda_i}$$

and

$$\frac{iK_j(\lambda_1+\ldots+\lambda_j)}{j} - \frac{\lambda_j}{j} + \frac{\lambda_i}{j} \geqslant 1-(b+1)\varepsilon.$$

If $i \geqslant j$, (3.1) holds since $K_j j^{-1} \leqslant K_k k^{-1}$ $(1 \leqslant j \leqslant k)$. If $i < j$,

$$\frac{iK_j+1}{j} \leqslant \frac{kK_j}{j} \leqslant K_k$$

and again (3.1) holds. This completes the proof.

All the proofs in this paper use the principle of this one. The Heilbronn hypothesis has already been used to purchase an improvement on Dirichlet's theorem: now we use it again to show this improvement cannot be too great, and thus $\lambda_1+\ldots+\lambda_k$ must exceed some positive bound. This deduction yields the contrapositive form of the theorem: if the Heilbronn hypothesis is satisfied, the box $[0, \eta_1] \times \ldots \times [0, \eta_k]$ has small volume.

Proof of Theorem 9. Suppose there is no solution of the simultaneous inequalities (1.4). As above we have only to show that

$$3\lambda_1 + 2\lambda_2 \geqslant 1 - b\varepsilon \tag{3.2}$$

where $\eta_i = N^{-\lambda_i}$ $(i = 1, 2)$. We apply Lemma 4 with $r = 2$, $a_1 = 1$, $a_2 = 2$. If $j = 1$, we may assume (2.5) false. There is thus an integer $q \geqslant 1$ such that

$$q \leqslant N^{\lambda_1 + 3\varepsilon}, \quad \|q\theta_1\| \leqslant N^{-1+3\varepsilon}.$$

By Theorem 1 there is an integer z, $1 \leqslant z \leqslant N^{1-\lambda_1-4\varepsilon}$, such that

$$\|z^2 q^2 \theta_2\| \leqslant N^{-\frac{1}{2}+\frac{1}{2}\lambda_1+3\varepsilon}.$$

Since $zq \leqslant N$ and $\|zq\theta_1\| \leqslant N^{-\lambda_1-\varepsilon}$, the Heilbronn hypothesis gives

$$\|z^2 q^2 \theta_2\| \geqslant N^{-\lambda_2}$$

so that

$$-\tfrac{1}{2} + \tfrac{1}{2}\lambda_1 + 4\varepsilon \geqslant -\lambda_2$$

which implies (3.2).

If $j = 2$, we may assume (2.5) false. There is thus an integer $q \geqslant 1$ such that

$$q \leqslant N^{2(\lambda_1+\lambda_2)+6\varepsilon}, \quad \|q\theta_2\| \leqslant N^{-2+2\lambda_1+\lambda_2+6\varepsilon}.$$

By Lemma 3 there is an integer $z \leqslant N^{1-2(\lambda_1+\lambda_2)-7\varepsilon}$ such that

$$\|zq\theta_1\| \leqslant N^{-1+2(\lambda_1+\lambda_2)+7\varepsilon}.$$

Since $zq \leqslant N$ and

$$\|z^2q^2\theta_2\| \leqslant z^2 q \|q\theta_2\| \leqslant N^{2-2(\lambda_1+\lambda_2)-8\varepsilon} N^{-2+2\lambda_1+\lambda_2+6\varepsilon} = N^{-\lambda_2-2\varepsilon},$$

we must have

$$\|zq\theta_1\| \geqslant N^{-\lambda_1}$$

so that

$$-1 + 2(\lambda_1 + \lambda_2) + 8\varepsilon \geqslant -\lambda_1$$

which implies (3.2). This completes the proof.

4. Proof of Theorem 10. We need a "recursive" lemma.

LEMMA 5. *Let $u_{k,r}$ $(k \geqslant 1, r = 1, \ldots, k)$ be positive integers defined recursively by*

$$u_{k,1} = 2^{k-1}, \quad u_{k,r} = \max\left(u_{k,r-1} + (r-1)2^{k-2},\ u_{k-1,r-1} + r2^{k-1}\right) \tag{4.1}$$

if $r > 1$. Then

$$u_{k,r} = \begin{cases} 2^{k-r}\left((r-1)2^r + 1\right), & r = 1, 2, 3, 4; \\ 2^{k-4}\left(65 + 2(r-5)(r+4)\right), & r \geqslant 5. \end{cases}$$

Proof. The cases $r \leqslant 4$ are left to the reader. Now

$$\begin{aligned} u_{k,5} &= \max(u_{k,4} + 4\cdot 2^{k-2},\ u_{k-1,4} + 5\cdot 2^{k-1}) \\ &= \max\left(2^{k-4}(3\cdot 2^4 + 1) + 16\cdot 2^{k-4},\ 2^{k-5}\cdot 49 + 80\cdot 2^{k-5}\right) \\ &= \max(130\cdot 2^{k-5},\ 129\cdot 2^{k-5}) = 65\cdot 2^{k-4}. \end{aligned}$$

Now suppose $h \geqslant 6$ and the result known for all $r < h$, for all k. Then

$$u_{k,h} = \max\Big(2^{k-4}\big(65 + 2(h-6)(h+3)\big) + (h-1)2^{k-2},\ 2^{k-5}\big(65 + 2(h-6)(h+3)\big) + h\cdot 2^{k-1}\Big).$$

The first of this pair is seen to be larger, and is

$$2^{k-4}\big(65 + 2(h-5)(h+4)\big).$$

This completes the proof.

Proof of Theorem 10. We call the theorem 'case (k, r)'. Case $(k, 1)$ is Theorem 2, or Lemma 3. Assume case (j, h) is known for $1 \leqslant h \leqslant j < k$ and that the cases $(k, 1), \ldots, (k, r-1)$ are known $(r > 1)$. We deduce case (k, r); this will obviously prove the theorem.

We use the definition of $u_{k,r}$ given by (4.1) wherever convenient.

Suppose there are no integer solutions of the inequalities

$$1 \leqslant x \leqslant N, \quad \max_{1 \leqslant j \leqslant r} \|x^{a_j}\theta_j\| \leqslant N^{-\lambda}$$

where $\lambda > 0$. We shall deduce that

$$\lambda \geqslant \frac{1}{u_{k,r}} - b\varepsilon \tag{4.2}$$

which will complete the proof. By Lemma 4 (with $\lambda_j = \lambda$, $1 \leqslant j \leqslant k$) there exists j, $1 \leqslant j \leqslant r$, such that either

$$jK_j\lambda \geqslant 1 - \varepsilon(jK_j + 3) \quad (K_j = 2^{a_j-1}) \tag{4.3}$$

or there is an integer $q \geqslant 1$ such that

$$q \leqslant N^{jK_j\lambda + \varepsilon(jK_j+2)}, \tag{4.4}$$

$$\|q\theta_j\| \leqslant N^{-a_j + (jK_j-1)\lambda + \varepsilon(jK_j+2)}. \tag{4.5}$$

Since $jK_j \leqslant r2^{r-1} \leqslant u_{k,r}$, we may assume (4.3) false. There are two cases to consider.

(a) $j = r$. By the case $(k-1, r-1)$ there is an integer z, $1 \leqslant z \leqslant N^{1-rK_r\lambda-\varepsilon(rK_r+3)}$ such that

$$\max_{1\leqslant i\leqslant r-1} \|z^{a_i} q^{a_i}\theta_i\| \ll N^{-(1-rK_r\lambda-\varepsilon(rK_r+3))u_{k-1,r-1}^{-1}+\varepsilon} \tag{4.6}$$

(since $1 \leqslant a_1 < \ldots < a_{r-1} \leqslant k-1$). Now $zq \leqslant N$, and by (4.4), (4.5),

$$\|z^{a_r} q^{a_r}\theta_r\| \leqslant z^{a_r} q^{a_r-1}\|q\theta_r\| \ll N^{\mu} < N^{-\lambda}$$

where

$$\mu = a_r(1-rK_r\lambda)+(a_r-1)rK_r\lambda - a_r + (rK_r-1)\lambda - a_r\varepsilon = -\lambda - a_r\varepsilon.$$

Thus, by (4.6) and the Heilbronn hypothesis, we must have

$$-\big(1-rK_r\lambda-\varepsilon(rK_r+3)\big)u_{k-1,r-1}^{-1}+2\varepsilon \geqslant -\lambda$$

or

$$\lambda(rK_r+u_{k-1,r-1}) \geqslant 1-b\varepsilon$$

which implies (4.2).

(b) $j \leqslant r-1$. By the case $(k, r-1)$ there is an integer z, $1 \leqslant z \leqslant N^{1-jK_j\lambda-\varepsilon(jK_j+3)}$ such that

$$\max_{\substack{1\leqslant i\leqslant r\\ i\neq j}} \|z^{a_i} q^{a_i}\theta_i\| \ll N^{-(1-jK_j\lambda)u_{k,r-1}^{-1}+b\varepsilon}. \tag{4.7}$$

Now $zq \leqslant N$, and by (4.4), (4.5),

$$\|z^{a_j} q^{a_j}\theta_j\| \leqslant z^{a_j} q^{a_j-1}\|q\theta_j\| \ll N^{\nu} \leqslant N^{-\lambda}$$

where

$$\nu = a_j(1-jK_j\lambda)+a_j jK_j\lambda - \lambda - a_j - a_j\varepsilon = -\lambda - a_j\varepsilon.$$

Thus, by (4.7) and the Heilbronn hypothesis, we must have

$$-(1-jK_j\lambda)u_{k,r-1}^{-1}+(b+1)\varepsilon \geqslant -\lambda$$

or

$$\lambda(jK_j+u_{k,r-1}) \geqslant 1-b\varepsilon$$

which implies (4.2). This completes the proof of Theorem 10.

5. The analogue of Lemma 4 using Hua's estimates. For the rest of the paper, $k \geqslant 2$. We begin with Hua's estimate of an exponential sum, which corresponds to Weyl's estimate (2.3), and is much sharper for large k. In the next lemma, $\ll$ depends on k, t and δ'.

LEMMA 6. *Make all the hypotheses of Lemma 2. Suppose further that p is an integer, $1 \leqslant p \leqslant N$, and that the number of solutions in integers of*

$$x_1^h+\ldots+x_{2t}^h = y_1^h+\ldots+y_{2t}^h \quad (1\leqslant h\leqslant k-1,\ 1\leqslant x_j, y_j\leqslant p)$$

is

$$\ll p^{4t-\frac{1}{2}k(k-1)+\delta'} \tag{5.1}$$

for some $\delta' > 0$, where t is a positive integer. Then

$$\sum_{m=1}^{H}|S(m)|^{4t} \ll (HN)^{\varepsilon}\Big(Hp^{4t}+p^{\delta'}N^{4t-1}\Big(\Big(\frac{HN}{q}+1\Big)(1+p^{1-k}q\log q)+H\Big)\Big). \tag{5.2}$$

In particular, the estimates (5.1) and hence (5.2) hold if

$$\delta' = \tfrac{1}{2}k(k-1)\Big(1-\frac{1}{k-1}\Big)^{l}+\varepsilon, \qquad t = \Big[\tfrac{1}{8}k(k-1)+\frac{l(k-1)}{2}\Big]+1$$

where l is any positive integer.

Proof. The second assertion of the lemma is Hua's 'mean value theorem' (Theorem 1 of [12] with s replaced by $2t$ and k replaced by $k-1 \geqslant 1$). The first assertion follows by slightly adapting the argument of Theorem 4 of [12]. Imitating the argument up to [12], (56) with no significant change (but writing N, p, k for $P, p_1, k+1$)

$$|S(m)|^{4t} \ll p^{4t}+p^{-4t}\big(N^{4t-2}p^{4t-\frac{1}{2}k(k-1)+\delta'}\big)p^{\frac{1}{2}(k-1)(k-2)} \times$$
$$\times\Big(N\sum_{Y=1}^{kN}\min\Big(p^{k-1}, \frac{1}{\|mY\theta\|}\Big)+Np^{k-1}\Big)$$

so using the fact that $x = mY$ has $\ll (NH)^{\varepsilon}$ solutions m,

$$\sum_{m=1}^{H}|S(m)|^{4t} \ll p^{4t}H+(NH)^{\varepsilon}N^{4t-1}p^{\delta'+1-k}\Big(\sum_{x=1}^{kNH}\min\Big(p^{k-1}, \frac{1}{\|x\theta\|}\Big)+Hp^{k-1}\Big)$$
$$\ll p^{4t}H+(NH)^{\varepsilon}N^{4t-1}p^{\delta'+1-k}\Big(\Big(\frac{kNH}{q}+1\Big)(p^{k-1}+q\log q)+Hp^{k-1}\Big)$$

by the same argument that leads to (2.3), namely Lemma 8a of [17]. This completes the proof of Lemma 6.

Naturally we can prove an analogue of Lemma 4 using (5.2) instead of (2.3). For simplicity we assume $\lambda_1 = \ldots = \lambda_r = \lambda$. We suppose that with every $j \leqslant r$ such that $a_j > 1$ is associated an integer $l_j \geqslant 1$ and write

$$\delta_j = \tfrac{1}{2}a_j(a_j-1)\Big(1-\frac{1}{a_j-1}\Big)^{l_j}, \qquad t_j = \Big[\tfrac{1}{8}a_j(a_j-1)+\frac{l_j(k-1)}{2}\Big]+1;$$

but $\delta_1 = 0$, $4t_1 = 1$ if $a_1 = 1$. We shall always have $\delta_j < 1$. The constant implied by $\ll$ is now allowed to depend on l_j $(1\leqslant j\leqslant r)$ as well as k, ε. Where there is no ambiguity we drop the indices from l_j, δ_j and t_j. The integer b used in awkward multiplies of ε can also depend on l_j.

Lemma 7. *Let $\theta_1, \ldots, \theta_r$ be real. Suppose there is no integer solution of the inequalities*

$$1 \leqslant x \leqslant N, \quad \|x^{a_i}\theta_i\| < N^{-\lambda}.$$

Then there exists j, $1 \leqslant j \leqslant r$, such that either

$$\lambda > \frac{1-\delta}{j(4t+k-1-\delta)} - b\varepsilon \tag{5.3}$$

or there is an integer $q \geqslant 1$ such that

$$q \ll N^{\delta+\lambda(4tj-\delta j)+b\varepsilon}, \tag{5.4}$$

$$\|q\theta_j\| \ll N^{-a_j+\delta+j\lambda(4t+a_j-1-\delta-\frac{1}{j})+b\varepsilon}. \tag{5.5}$$

In all the inequalities, $\delta = \delta_j$, $t = t_j$.

Proof. The first stage of Lemma 4 can be repeated and so for some j, $1 \leqslant j < r$,

$$N \ll \sum_{m_j \neq 0}^{(j)} |T(\boldsymbol{m}^j)|. \tag{5.6}$$

If $a_j = 1$ we either have $\lambda \geqslant 1-4\varepsilon$ (which implies (5.3)), or (5.4) and (5.5), exactly as in Lemma 4. If $a_j > 1$, we apply Hölder's inequality:

$$N^{4t} \ll N^{j(\lambda+\varepsilon)(4t-1)} \sum_{m_j \neq 0}^{(j)} |T(\boldsymbol{m}^j)|^{4t}.$$

Now let $Q = N^{\mu}$ where

$$\mu = a_j - \delta - j\lambda\left(4t + a_j - 1 - \delta - \frac{1}{j}\right) - b\varepsilon \tag{5.7}$$

and choose a, q $(1 \leqslant q \leqslant Q,\ (a, q) = 1)$ so that (2.10) holds. Then by Lemma 6,

$$N^{4t} \ll N^{(4tj-1)(\lambda+\varepsilon)}\left(p^{4t}N^{\lambda+2\varepsilon} + p^{\delta+\varepsilon}N^{4t-1}\left(\left(\frac{N^{1+\lambda+\varepsilon}}{q}+1\right)\times\right.\right.$$
$$\left.\left.\times\left(1+p^{1-a_j}q\log q\right) + N^{\lambda+\varepsilon}\right)\right). \tag{5.8}$$

In order that $p^{4t}N^{4tj\lambda+(4tj+1)\varepsilon}$ shall be negligible, we choose $p = [N^{1-j\lambda-(j+1)\varepsilon}]$; then $p \geqslant 1$, unless $\lambda \geqslant 1/j - 3\varepsilon$ which may be excluded as it implies (5.3). Thus from (5.8)

$$N^{1-(4tj-1)(\lambda+\varepsilon)}p^{-\delta-\varepsilon} \ll \frac{N^{1+\lambda+\varepsilon}}{q} + N^{\lambda+\varepsilon} + \frac{N^{1+\lambda+\varepsilon}\log q}{p^{a_j-1}} + \frac{q\log q}{p^{a_j-1}}.$$

We may assume that (5.3) is false. Thus, if b is suitably defined in (5.3), the terms $N^{\lambda+\varepsilon}$ and $N^{1+\lambda+\varepsilon}\log q \cdot p^{1-a_j}$ are negligible; and with a suitable b in (5.7), the term $q\log q \cdot p^{1-a_j}$ is negligible. Thus

$$q \ll N^{4tj\lambda+b\varepsilon}p^{\delta+\varepsilon}$$

which is (5.4); while (2.10) implies (5.5).

To complete this section we deduce from Lemma 7 the following improvement of Theorem 6.

Theorem 11. *Let θ be real. Then for $k \geqslant 3$,*

$$\min_{1 \leqslant x \leqslant N} \|\theta x^k\| < C(k)N^{-\sigma_k} \quad (N \geqslant 1)$$

where

$$\sigma_k^{-1} = \frac{2k^2\log k}{\log k - 1}(\log k^2 + \log\log k) \sim 4k^2\log k.$$

Proof. We suppose that there are no integer solutions of

$$1 \leqslant x \leqslant N, \quad \|\theta x^k\| \leqslant N^{-\lambda} \quad (\lambda > 0).$$

We apply Lemma 7 with $r = 1$, $a_1 = k$, $\theta_1 = \theta$, and

$$l = \left[\frac{\log\frac{1}{2}k(k-1) + \log\log k}{-\log\left(1 - \dfrac{1}{k-1}\right)}\right] + 1,$$

so that

$$\delta < 1/\log k,$$

and

$$4t \leqslant (k-1)(2l+\tfrac{1}{2}k) + 4 \leqslant (k-1)^2(2\log\tfrac{1}{2}k^2 + 2\log\log k) + \tfrac{1}{2}k(k+3) + 2.$$

We assume first that (5.3) is false and (5.4) and (5.5) hold. Then

$$\|q^k\theta\| \leqslant q^{k-1}\|q\theta\| \ll N^{-k+k\delta+\lambda(k(4t+1-\delta)-2)+b\varepsilon}$$

and $q \leqslant N$, so by the Heilbronn hypothesis,

$$-k + k\delta + \lambda\big(k(4t+1-\delta)-2\big) + b\varepsilon \geqslant -\lambda$$

or

$$\lambda \geqslant \frac{1-\delta}{4t+k-1} - b\varepsilon.$$

Even if (5.3) holds instead, we can always conclude that

$$\lambda \geqslant \frac{1-\delta}{4t+k-1} - b\varepsilon \geqslant \frac{1-(\log k)^{-1}}{(k-1)^2(2\log\frac{1}{2}k^2 + 2\log\log k) + \frac{1}{2}k(k+5) + 1}$$
$$\geqslant \frac{1-(\log k)^{-1}}{2k^2(\log k^2 + \log\log k)}$$

and the theorem is proved.

6. An improvement of Theorem 10 for large k. We begin with the special case when the θ_j's are equal.

THEOREM 12. *For* $k \geqslant 3$,

$$\min_{1\leqslant x\leqslant N}\ \max_{1\leqslant j\leqslant k} \|\theta x^j\| < C(k) N^{-\tau_k} \quad (\theta \textit{ real},\ N \geqslant 1)$$

where

$$\tau_k^{-1} = \frac{2k^3\log k}{\log k - 1}(\log k^2 + \log\log k).$$

Proof. Suppose there are no integer solutions of

$$1 \leqslant x \leqslant N, \qquad \max_{1\leqslant j\leqslant k} \|\theta x^j\| \leqslant N^{-\lambda}$$

where $\lambda > 0$. We apply Lemma 7 with $r = k$, $a_j = j$, $\theta_j = \theta$ $(1 \leqslant j \leqslant k)$ and

$$l_j = \left[\frac{\log\frac{1}{2}k(j-1) + \log\log k}{-\log\left(1 - \dfrac{1}{j-1}\right)}\right] + 1$$

so that t_j satisfies the same estimate as before, and

$$\delta_j = \frac{1}{2}j(j-1)\left(1 - \frac{1}{j-1}\right)^{l_j} < \frac{j}{k\log k}.$$

Suppose first that (5.3) is false and (5.4) and (5.5) hold. Then for $1 \leqslant i \leqslant k$,

$$\|q^i\theta\| \leqslant q^{k-1}\|q\theta\| \ll N^{-j+k\delta+\lambda((4t-\delta)jk+j^2-j)}$$

and $q \leqslant N$. Thus, by the Heilbronn hypothesis,

$$-j + k\delta + \lambda\big((4t-\delta)jk + j^2 - j\big) + \varepsilon \geqslant -\lambda,$$

or

$$\lambda > \frac{1 - kj^{-1}\delta}{4tk - \delta k + j - 1} - \varepsilon.$$

Even if (5.3) holds instead, we can always conclude that

$$\lambda > \frac{1 - kj^{-1}\delta}{k(4t + k - 1)} \geqslant \frac{1 - (\log k)^{-1}}{k(4t + k - 1)} \geqslant \tau_k$$

and the theorem is proved.

When the θ_j's are general real numbers, we discard some of the minor savings of Theorem 10 in order to simplify the calculation.

THEOREM 13. *For* $k \geqslant 3$, $\theta_1, \ldots, \theta_r$ *real*,

(6.1) $$\min_{1\leqslant x\leqslant N}\ \max_{1\leqslant i\leqslant r} \|x^{a_i}\theta_i\| \leqslant C(k) N^{-1/w_{k,r}}$$

where

$$w_{k,r} = e^{2/\log k}\big(\sigma_k^{-1} + (r+2)(r-1)k^2(\log k^3 + \log\log k)\big).$$

Proof. Define $v_{k,1} = \sigma_k^{-1}$ in the notation of Theorem 11. Let $r \geqslant 2$ and let us assume that for some $v_{k,r-1} > 0$ we have

(6.2) $$\min_{1\leqslant x\leqslant M}\ \max_{1\leqslant i\leqslant r-1} \|x^{b_i}\varphi_i\| \leqslant C(k) M^{-1/v_{k,r-1}}$$

($\varphi_1, \ldots, \varphi_r$ real, $M \geqslant 1$, $1 \leqslant b_1 < \ldots < b_{r-1} \leqslant k$ integers).

Now suppose there is no integral solution of

$$1 \leqslant x \leqslant N, \qquad \max_{1\leqslant i\leqslant r} \|x^{a_i}\theta_i\| \leqslant N^{-\lambda}.$$

We apply Lemma 7 with

$$l_j = \left[\frac{\log\left(\frac{1}{2}ka_j(a_j-1)\log k\right)}{-\log\left(1 - \dfrac{1}{a_j-1}\right)}\right] + 1 \leqslant (k-1)(\log\tfrac{1}{2}k^3 + \log\log k) + 1$$

so that

$$\delta_j < \frac{1}{k\log k},$$

$$4t_j \leqslant 2(k-1)^2\log\tfrac{1}{2}k^3 + 2k^2\log\log k + \tfrac{1}{2}k(k+3) + 2 \qquad (j = 1, \ldots, r).$$

Assume first that (5.3) is false, and (5.4) and (5.5) hold. Let (for a suitable b)

(6.3) $$\mu = 1 - \delta - \lambda(4t - \delta + 1)j - b\varepsilon;$$

then $\mu \geqslant 0$ and by (6.2) there is an integer z, $1 \leqslant z \leqslant N^\mu$, such that

$$\max_{\substack{1\leqslant i\leqslant r\\ i\neq j}} \|z^{a_i}q^{a_i}\theta\| \ll N^{-\mu/v_{k,r-1}};$$

and

$$\|z^{a_j}q^{a_j}\theta_j\| \leqslant z^{a_j}q^{a_j-1}\|q\theta_j\| \ll N^\nu$$

where

$$\nu = \mu a_j + (a_j-1)\big(\delta + \lambda(4tj - \delta j)\big) - a_j + \delta + j\lambda\left(4t + a_j - \frac{1}{j} - \delta\right) + b\varepsilon \leqslant -\lambda - \varepsilon$$

if b is suitably chosen in (6.3). Now $qz \leqslant N$, so by the Heilbronn hypothesis,

$$\mu/v_{k,r-1} \leqslant \lambda + \varepsilon$$

or

$$\lambda \geqslant \frac{1-\delta}{(4t-\delta+1)j + v_{k,r-1}} - b\varepsilon.$$

Even if (5.3) holds instead we can always conclude that

$$\lambda \geqslant \frac{1-\delta}{(4t+k-1)j+v_{k,r-1}};$$

and so

(6.4) $$\min_{1\leqslant x\leqslant N} \max_{1\leqslant i\leqslant r} \|x^{a_i}\theta_i\| \leqslant C(k)N^{-1/v_{k,r}}$$

where

$$v_{k,r} = \big(v_{k,r-1}+(4t+k-1)r\big)\left(1+\frac{2}{k\log k}\right).$$

Thus (6.4) is true for $r = 1, \ldots, k$. Now

$$v_{k,r} \leqslant v_r, \quad \text{where} \quad v_1 = \sigma_k^{-1},$$
$$v_r = (v_{r-1}+ar)(1+\beta) \quad (r \geqslant 2)$$

$(a = 2k^2(\log k^3 + \log\log k)$, $\beta = 2/(k\log k)$. By induction,

$$v_r = v_1(1+\beta)^{r-1} + \sum_{j=2}^{r} ja(1+\beta)^{r+1-j} < (1+\beta)^k\big(v_1 + a\big(\tfrac{1}{2}r(r+1)-1\big)\big)$$

$$< \exp\left(\frac{2}{\log k}\right)\big(\sigma_k^{-1} + (r+2)(r-1)k^2(\log k^3 + \log\log k)\big).$$

Since $v_{k,r} \leqslant v_r \leqslant w_{k,r}$, the theorem is proved.

COROLLARY.

$$\min_{1\leqslant x\leqslant N} \max_{1\leqslant i\leqslant k} \|x^i\theta_i\| < C(k)N^{-(1-\varepsilon_k)/3k^4\log k}$$

$(N \geqslant 1$, θ_i real) where $\varepsilon_k \to 0$ as $k \to \infty$.

Finally we observe that in Theorem 9, $\eta_1^3\eta_2^2$ can be replaced by $\eta_1\eta_2^4$. To see this suppose $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_1 + 4\lambda_2 \leqslant 1-6\varepsilon$. There is an integer q,

$$1 \leqslant q \leqslant N^{1-2\lambda_2-3\varepsilon}, \quad \|q\theta_1\| \leqslant N^{-1+2\lambda_2+3\varepsilon},$$

and an integer z,

$$1 \leqslant z \leqslant N^{2\lambda_2+3\varepsilon}, \quad \|z^2q^2\theta_2\| \leqslant C(\varepsilon)N^{-\lambda_2-\varepsilon} \leqslant N^{-\lambda_2}.$$

Now

$$x = zq \leqslant N, \quad \|x^2\theta_2\| \leqslant N^{-\lambda_2} \quad \text{and} \quad \|x\theta_1\| \leqslant z\|q\theta_1\| \leqslant N^{-1+4\lambda_2+6\varepsilon} \leqslant N^{-\lambda_1}.$$

This method was pointed out to us by Dr. R. J. Cook.

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