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Elementary methods in the theory of L-functions, IV
The Heilbronn phenomenon

by

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1. Gauss [8] raised two problems concerning the class number $h(-D)$ of the imaginary quadratic field belonging to the fundamental discriminant $-D<0$.

1. Determine all the negative fundamental discriminants with class number one.

2. Is it true, that $h(-D)\to\infty$ if $D\to\infty$? (Gauss conjectured the truth of this assertion.)

About one hundred years later, in 1913, Gronwall [10] proved a conditional result concerning the 2nd problem (although this result was only implicite contained in his work [10]):

If the $L(s,\chi_D)$ function belonging to the real primitive character $\chi(n)=\left(\frac{-D}{n}\right)$ has no zero in the interval

$$\left[1-\frac{a}{\log D},1\right] \tag{1.1}$$

then

$$h(-D)>\frac{b(a)\sqrt{D}}{\log D\sqrt{\log\log D}}. \tag{1.2}$$

5 years later, Hecke (see Landau [18]) proved even more, namely that from (1.1) follows

$$h(-D)>\frac{b'(a)\sqrt{D}}{\log D}, \tag{1.3}$$

where a is a constant and $b(a)$, resp. $b'(a)$, are constants depending only on a.

The fact, that zerofree regions (resp. intervals) of some L-functions have influence on the class number of imaginary quadratic fields is easily understandable if we consider the class-number formula of Dirichlet

$$(1.4)\qquad h(-D)=\frac{\sqrt{D}}{\pi}L(1,\chi_D)\quad\text{where}\quad \chi_D(n)=\left(\frac{-D}{n}\right)\ \text{and}\ D>4.$$

The results of Gronwall and Hecke showed the amenableness of Gauss's problems to the tools of the analytical number theory.

In 1933 Deuring [6] showed that if the Riemann hypothesis is not true then

$$h(-D)\geqslant 2\quad\text{for}\quad D>D_0.$$

In 1934 Mordell [23] succeeded to prove under the same assumption that

$$h(-D)\to\infty\quad\text{if}\quad D\to\infty.$$

The 2nd problem was solved in 1934 by Heilbronn [12], who deduced

$$h(-D)\to\infty\quad\text{if}\ D\to\infty$$

from the assumption that the general Riemann hypothesis is not true.

Heilbronn's theorem together with Hecke's theorem gives without any assumption

$$h(-D)\to\infty\quad\text{for}\quad D\to\infty.$$

A year later Siegel [31] proved the inequality

$$(1.5)\qquad h(-D)>D^{1/2-\varepsilon}\quad\text{for}\quad D>D_0(\varepsilon)$$

for an arbitrary $\varepsilon>0$, and with a constant $D_0(\varepsilon)$ depending only on ε.

However the constant $D_0(\varepsilon)$ in Siegel's theorem was ineffective (for the non-trivial case $\varepsilon\leqslant 1/2$), i.e. the proof gave no possibility to determine $D_0(\varepsilon)$ for a given ε. So the curious situation was that the 2nd problem of Gauss — which seemed to be obviously more difficult (and seemed to contain the first problem) — was solved affirmatively, without giving even a theoretical possibility to determine all the imaginary quadratic fields with class number one.

Another interesting fact is that Hecke's theorem is an essential part of Siegel's theorem (and it remained also essential and unavoidable in all the later proofs of Siegel's theorem, given by Heilbronn [13], Estermann [7], Chowla [5], Linnik [20], Tatuzawa [34], Rodosski [29], Knapowski [16], Pintz [25], Goldfeld [9], although the assumption of Hecke's theorem, i.e. the non-vanishing of $L(s,\chi_D)$ in the interval

$$\left[1-\frac{a}{\log D},1\right]$$

is not yet proved.

The first essential result, concerning the first problem of Gauss was achieved in 1934 by Heilbronn and Linfoot [14] showing that except for the known values of

$$-D=-3,-4,-7,-8,-11,-19,-43,-67,-163$$

there is at most a tenth negative fundamental discriminant with class number one. (Other proofs for this fact were given later by K. Iseki [15] and R. Ayoub [1].) An analogue statement concerning the determination of the imaginary quadratic fields with given class number was proved by Landau [19] in 1935. Modifying the proof of Heilbronn [12], Landau showed that if

$$h(-D)=h$$

then the inequality

$$(1.6)\qquad D\leqslant D(h)=Ch^8\log^6(3h)$$

(where C is an absolute effective constant) holds with the possible exception of at most one negative fundamental discriminant. Tatuzawa [34] proved in 1950 the theorem of Landau, i.e. (1.6) with

$$(1.7)\qquad D(h)=Ch^2\log^2(13h).$$

Towards the effectivization of Siegel's theorem Tatuzawa [34] proved, that if $h(-D)\leqslant D^{1/2-\varepsilon}$ then the inequality

$$(1.8)\qquad D\leqslant D_0'(\varepsilon)=\max(e^{12},e^{1/\varepsilon})$$

holds with the possible exception of at most one negative fundamental discriminant. Finally, more than 30 years after Siegel's result, in 1966–67, Baker [2] and Stark [32] independently proved, that there is no tenth imaginary quadratic field with class number one. A few years ago the methods of Baker and Stark led to the solution of class number two problem. (See Baker [3], Stark [33], Baker and Stark [4], and Montgomery and Weinberger [22].)

The most important consequence of Siegel's theorem, showed by Walfisz [35] in 1936, is that an L function belonging to the real character χ mod D has no zero in the interval

$$(1.9)\qquad [1-c(\varepsilon)D^{-\varepsilon},1].$$

This improves the error term in the formula, that gives the number of primes less than or equal to a given x in an arithmetic progression, and so Siegel's inequality plays a fundamental role in the analytical number theory.

The interesting fact, discovered by Deuring [6], and Heilbronn [12] is that the non-trivial zeros of $\zeta(s)$ and $L(s,\chi)$ (where χ is an arbitrary real or complex character), have influence on the real zeros of other real L-functions.

This fact was used also by Linnik [21] in his work concerning the least prime in an arithmetic progression. Analyzing this phenomenon, called by him Deuring-Heilbronn phenomenon, he succeeded to prove the following very general theorem

THEOREM (Linnik). *If an L-function belonging to a real non-principal character* mod D *has a real zero* $1-\delta$ *with*

$$\delta \leqslant A_1/\log D \tag{1.10}$$

then all the L-functions belonging to characters mod D, *have no zero in the domain*

$$\sigma \geqslant 1-\frac{A_2}{\log D(|t|+1)}\log\frac{eA_1}{\delta\log D(|t|+1)}, \quad \delta\log D(|t|+1)\leqslant A_1 \tag{1.11}$$

(A_1 *and* A_2 *are absolute constants.*)

(For simpler proofs of Linnik's theorem see Rodosskii [30], Knapowski [17], Haneke [11].)

2. In [28] we investigated the influence of zeros of $\zeta(s)$ on the exceptional zeros, i.e. the Deuring phenomenon. Now we turn our attention to the non-trivial zeros of arbitrary L-functions belonging to non-principal characters, and demonstrate their influence on the exceptional zeros. Our method will be different from the method applied in [28]. We modify the method, applied in [25] to prove Siegel's theorem in an elementary way (which is based on some ideas of Linnik [20]) to obtain some results concerning the Heilbronn phenomenon.

In our theorems we shall assume without any further reference that D (but not k) is greater than a given effective constant D_0 (computable from the proofs).

First we state

THEOREM 1. *Let us assume that an L-function belonging to a non-principal (real or complex) character* χ_k mod k *has an* $s_0 = 1-\gamma+it$ *zero with* $\gamma < 0.05$. *Then for an arbitrary real non-principal character* χ_D mod D *(for which* $\chi_k\chi_D$ *is also non-principal) the inequality*

$$L(1,\chi_D) > \frac{1}{140\,U^{6\gamma}\log^3 U} \tag{2.1}$$

holds, where $U = k|s_0|D$.

From Theorem 1 follows a weakened form of Heilbronn's theorem, namely if there is an L-function, which has a zero in the half-plane $\sigma > 0.95$ (1), then $h(-D) \gg \sqrt[5]{D}$.

(1) With little more extra trouble we could also prove that a zero in the half-plane $\sigma > 3/4$ implies $h(-D)\to\infty$.

Theorem 1 combined with the theorem of Hecke also gives Siegel's theorem.

From Theorem 1 one can easily derive a weakened form of Linnik's theorem [21] on the exceptional zeros, namely

THEOREM 2. *If an L function belonging to a non-principal character* χ_k mod k *has an* $s_0 = 1-\gamma+it$ *zero with* $\gamma < 0.05$, *and an other L-function belonging to the real non-principal character* χ_D *(for which* $\chi_k\chi_D$ *is also non-principal)* mod D *has an* $1-\delta$ *real exceptional zero, then the inequality*

$$\delta > \frac{1}{140\,U^{6\gamma}\log^5 U} \tag{2.2}$$

holds, where $U = k|s_0|D$.

This is equivalent to the inequality

$$\gamma > \frac{1}{6\log U}\log\frac{1}{140\,\delta\log^5 U}. \tag{2.3}$$

This gives Linnik's theorem in the case

$$\delta = O\left(\frac{1}{\log^{5+\varepsilon} D}\right) \quad (\varepsilon > 0). \tag{2.4}$$

Theorem 2 gives for real zeros of real L-functions the following result (proved in another way by Rodosskii [29]).

THEOREM 3. *For an arbitrary* ε, $0<\varepsilon<0.05$, *there is at most one* D, *and at most one primitive real character* χ_D mod D, *such that* $L(s,\chi_D)$ *vanishes somewhere in the interval*

$$\left[1-\min\left(\varepsilon, \frac{1}{140\cdot 32\log^5 D\cdot D^{12\varepsilon}}\right), 1\right]. \tag{2.5}$$

If we combine a trivial consequence of Theorem 3 with Hecke's theorem, we have Tatuzawa's theorems.

THEOREM 4. *For an arbitrary* $\varepsilon>0$ *there is a* $D_0(\varepsilon)$ *effective constant depending only on* ε, *with the following property: if* χ_D *is a real primitive character* mod D, *and* $D > D_0(\varepsilon)$ *then*

$$L(s,\chi_D)\neq 0 \quad \text{for} \quad s\in[1-D^{-\varepsilon}, 1] \tag{2.6}$$

and the inequality

$$L(1,\chi_D) > D^{-\varepsilon} \tag{2.7}$$

holds, with the possible exception of at most one D, *and at most one primitive character* χ_D mod D.

Applying the class-number formula of Dirichlet (1.4) (in case of (2.9) we must yet use a theorem of Landau [18]), we get from (2.7)

THEOREM 5. *For an arbitrary $\varepsilon > 0$ there is a $D_1(\varepsilon)$ effective constant with the following property: If $D \geqslant D_1(\varepsilon)$ (and $-D < 0$ is a fundamental discriminant) then the inequality*

$$h(-D) > D^{1/2-\varepsilon} \tag{2.8}$$

holds, with the possible exception of at most one negative fundamental discriminant.

If $-D$ is a negative fundamental discriminant, h an arbitrary natural number, $D > Ch^2\log^2(3h)$ (C is an absolute effective constant), then the inequality

$$h(-D) > h \tag{2.9}$$

holds, with the possible exception of at most one negative fundamental discriminant.

3. For the proof of Theorem 1 we define the following sets of natural numbers:

$$A_\nu = \{n;\ p|n,\ p\ \text{prime} \to \chi_D(p) = \nu\} \quad (\nu = -1, 0, 1), \tag{3.1}$$

$$C = \{c;\ c = ab,\ a \in A_1,\ b \in A_0\}. \tag{3.2}$$

Then an arbitrary natural number n can be written in the form

$$n = cm = abm; \quad c \in C,\ a \in A_1,\ b \in A_0,\ m \in A_{-1}. \tag{3.3}$$

As for an arbitrary θ multiplicative number theoretical function, the function

$$g_\theta(n) = \sum_{d|n} \theta(d) \tag{3.4}$$

is multiplicative, so if $\lambda(n)$ denotes Liouville's λ-function, then

$$g_\lambda(n) = \sum_{d|n} \lambda(d) = \begin{cases} 1, & \text{if } n = l^2, \\ 0, & \text{if } n \neq l^2, \end{cases} \tag{3.5}$$

and

$$g_D(n) = \sum_{d|n} \chi_D(d) = \prod_{p^\alpha \| n} \bigl(1 + \chi(p) + \ldots + \chi^\alpha(p)\bigr) \geqslant 0 \tag{3.6}$$

are multiplicative.

We shall make use of the relation

$$g_\lambda(a) = \sum_{a_l|a} \lambda(a_l) = \sum_{a_l|a} 2^{\nu(a_l)} \lambda(a_l)\, d\left(\frac{a}{a_l}\right) \tag{3.7}$$

(where $\nu(n)$ denotes the number of distinct prime divisors of n, and $d(n)$ denotes the number of divisors of n). Further we know that for

$$a \in A_1, \quad b \in A_0, \quad m \in A_{-1} \tag{3.8}$$

we have from (3.6)

$$g_D(a) = d(a), \quad g_D(b) = 1, \quad g_D(m) = g_\lambda(m). \tag{3.9}$$

Hence with (3.7) as $g_\lambda(n)$ and $g_D(n)$ are multiplicative, for $n = abm = cm$, $a \in A_1$, $b \in A_0$, $m \in A_{-1}$, we get

$$\begin{aligned} g_\lambda(n) &= g_\lambda(a) g_\lambda(b) g_\lambda(m) \\ &= \sum_{a_l|a} 2^{\nu(a_l)} \lambda(a_l) g_D\left(\frac{a}{a_l}\right) \sum_{b_l|b} \lambda(b_l) g_D\left(\frac{b}{b_l}\right) \cdot g_D(m) \\ &= \sum_{\substack{c_l|c,\ c_l = a_l b_l \\ a_l \in A_1,\ b_l \in A_0}} 2^{\nu(a_l)} \lambda(c_l) g_D\left(\frac{n}{c_l}\right). \end{aligned} \tag{3.10}$$

On the other hand, from (3.6) for $c \in C$, $c = ab$, $a \in A_1$, $b \in A_0$, we have

$$2^{\nu(a)} \leqslant g_D(c) \leqslant d(c). \tag{3.11}$$

Thus, considering (3.6), (3.10), (3.11) and the notations in Theorem 1, we have

$$\begin{aligned} 0.11 &\leqslant 1 - \sum_{l=2}^{\infty} \frac{1}{l^{2(1-\gamma)}} \leqslant \Bigl| \sum_{\substack{n \leqslant U^6 \\ n = l^2}} \frac{\chi_k(n)}{n^{s_0}} \Bigr| \\ &= \Bigl| \sum_{n \leqslant U^6} \frac{\chi_k(n)}{n^{s_0}} g_\lambda(n) \Bigr| = \Bigl| \sum_{n \leqslant U^6} \frac{\chi_k(n)}{n^{s_0}} \sum_{\substack{c \in C,\ c|n \\ c = ab,\ a \in A_1,\ b \in A_0}} 2^{\nu(a)} \lambda(c) g_D\left(\frac{n}{c}\right) \Bigr| \\ &= \Bigl| \sum_{\substack{c \leqslant U^6,\ c \in C \\ c = ab,\ a \in A_1,\ b \in A_0}} \frac{2^{\nu(a)} \lambda(c) \chi_k(c)}{c^{s_0}} \sum_{r \leqslant U^6/c} \frac{\chi_k(r)}{r^{s_0}} g_D(r) \Bigr| \\ &\leqslant \underbrace{\sum_{n \leqslant U^3} \frac{d(n)}{n^{1-\gamma}} \Bigl| \sum_{r \leqslant U^6/n} \frac{\chi_k(r)}{r^{s_0}} g_D(r) \Bigr|}_{\Sigma_1} + \underbrace{\sum_{U^3 < n \leqslant U^6} \frac{g_D(n)}{n^{1-\gamma}} \sum_{r \leqslant U^6/n} \frac{d(r)}{r^{1-\gamma}}}_{\Sigma_2}. \\ &= \Sigma_1 + \Sigma_2 \end{aligned} \tag{3.12}$$

Here as $L(s_0, \chi_k) = 0$ estimating the finite partial sums of $L(s_0, \chi_k\chi_D)$ and $L(s_0, \chi_k)$ by partial summation for $y \geqslant U^3$ we get

$$\begin{aligned} \Bigl| \sum_{r \leqslant y} \frac{\chi_k(r)}{r^{s_0}} \sum_{d|r} \chi_D(d) &= \Bigl| \sum_{d \leqslant y} \frac{\chi_k(d)\chi_D(d)}{d^{s_0}} \sum_{l \leqslant y/d} \frac{\chi_k(l)}{l^{s_0}} \Bigr| \\ &\leqslant \Bigl| \sum_{d \leqslant \sqrt{y}} \frac{\chi_k(d)\chi_D(d)}{d^{s_0}} \sum_{l = y/d} \frac{\chi_k(l)}{l^{s_0}} \Bigr| + \Bigl| \sum_{l \leqslant \sqrt{y}} \frac{\chi_k(l)}{l^{s_0}} \sum_{\sqrt{y} < d \leqslant y/l} \frac{\chi_k(d)\chi_D(d)}{d^{s_0}} \Bigr| \end{aligned} \tag{3.13}$$

$$\leqslant \sum_{d\leqslant\sqrt{y}} \frac{1}{d^{1-\gamma}}\cdot\frac{2k|s_0|}{(y/d)^{1-\gamma}} + \sum_{l\leqslant\sqrt{y}} \frac{1}{l^{1-\gamma}}\cdot\frac{2kD|s_0|}{(\sqrt{y})^{1-\gamma}}$$

$$\leqslant \frac{2k|s_0|\sqrt{y}}{y^{1-\gamma}} + \frac{2kD|s_0|\log y\cdot y^{\gamma}}{\sqrt{y}} < \frac{1}{100y^{\gamma}\log^2 y}.$$

Now (3.12) and (3.13) give the inequality

$$\Sigma_1 \leqslant \sum_{n\leqslant U^3} \frac{d(n)n^{\gamma}}{n}\cdot\frac{1}{100\left(\frac{U^6}{n}\right)^{\gamma}\log^2 U^3} \tag{3.14}$$

$$< \frac{1}{100\log^2 U^3}\sum_{n\leqslant U^3}\frac{d(n)}{n} < \frac{1}{100}.$$

Combining (3.11) with (3.14) we have

$$\Sigma_2 > \tfrac{1}{10}. \tag{3.15}$$

On the other hand using Lemma 1 of [27] with the values $x_1 = U^3$, $x_2 = U^6$ subtracting the two equalities we get by $U \geqslant D$:

$$\sum_{U^3\leqslant n\leqslant U^6} \frac{g_D(n)}{n} = \log U^3 L(1, \chi_D) + O\left(\sqrt{\frac{\sqrt{D}\log D\log U^3}{U^3}}\right) \tag{3.16}$$

$$= 3\log U\big(L(1, \chi_D) + O(U^{-5/4})\big).$$

This implies together with (3.15)

$$\frac{1}{10} < \Sigma_2 < U^{6\gamma}\sum_{U^3<n\leqslant U^6}\frac{g_D(n)}{n}\sum_{r\leqslant U^6/n}\frac{d(r)}{r} \tag{3.17}$$

$$< U^{6\gamma}3\log U\big(L(1, \chi_D) + O(U^{-5/4})\big)\big(\tfrac{1}{2}+o(1)\big)\log^2 U^3$$

$$< \big(13.5 + o(1)\big)\log^3 U\cdot L(1, \chi_D) + o(1)$$

which proves Theorem 1.

4. Theorem 2 follows from Theorem 1, if we recall that for the greatest real zero $1-\delta$ of $L(s, \chi_D)$ (where χ_D is a real non-principal character mod D), by the arguments of Page [24]

$$\frac{L(1)}{\delta} \leqslant \log^2 D \leqslant \log^2 U. \tag{4.1}$$

Theorem 2 implies Theorem 3, so let us assume that k is the minimal modulus for which an L-function belonging to a real primitive character mod k has a real zero $1-\gamma$ in the interval (2.5). Then for an arbitrary modulus $D \geqslant k$ and for the real primitive character χ_D mod D ($\chi_D \neq \chi_k$), for the greatest real zero $1-\delta$ of $L(s, \chi_D)$ holds by Theorem 2:

$$\delta > \frac{1}{140\,U^{6\gamma}\log^5 U} = \frac{1}{140\,[kD(1-\gamma)]^{6\gamma}\log^5[kD(1-\gamma)]} \tag{4.2}$$

$$> \frac{1}{140\cdot 32\log^5 D\cdot D^{12\gamma}} \geqslant \frac{1}{140\cdot 32\log^5 D\cdot D^{12\varepsilon}}.$$

Applying Theorem 3 with $\varepsilon/20$ instead of ε, and considering that for a $D > D_0(\varepsilon)$ (effective constant)

$$\min\left(\frac{\varepsilon}{20}, \frac{1}{140\cdot 32\log^5 D\cdot D^{12\varepsilon/20}}\right) > \frac{5}{D^{\varepsilon}} \tag{4.3}$$

we get (2.6) with a constant 5 instead of 1. But we proved in [26] (Hecke's theorem), if a real L-function has no zero in the interval

$$[1-a, 1] \quad \text{for} \quad a \leqslant \frac{1}{20\log D}$$

then

$$L(1, \chi_D) > 0.23\,a > a/5$$

and so (2.6) (in this modified form) implies (2.7).

(2.7) is apart from a factor π equivalent with (2.8) (which we can naturally eliminate using (2.7) with $\varepsilon/2$ instead of ε).

If in Theorem 5 we choose C sufficiently large, and regard the fundamental discriminant $-k$ with the minimal absolute value, for which the inequalities

$$k \geqslant Ch^2\log^2(3h), \qquad h(-k) \leqslant h \tag{4.4}$$

hold, then for $\chi_k(n) = (-k/n)$ we have

$$h(-k) < \frac{c_1(C)\sqrt{k}}{\log k}, \quad \text{i.e.} \quad L(1, \chi_k) < \frac{c_1(C)\pi}{\log k}. \tag{4.5}$$

This implies by the theorem of Hecke [18] (in this form see [26]), that $L(s, \chi_k)$ has a $1-\gamma$ real zero for which

$$\gamma < \frac{5c_1(C)\pi}{\log k} \tag{4.6}$$

(where $c_1(C)$ is sufficiently small if C was chosen sufficiently large).

But in this case, according to the theorem of Landau [18], for an arbitrary primitive character χ_D ($\neq \chi_k$) mod D $L(s, \chi_D)$ has no zero in interval $\left[1-\frac{C_2}{\log kD}, 1\right]$ and thus it does not vanish in the interval $\left[1-\frac{C_2}{2\log D}, 1\right]$ (where C_2 is an absolute constant), and so by the mentioned theorem of Hecke

$$h(-D) > \frac{2C_2\sqrt{D}}{5\pi\log D} \geqslant \frac{2C_2\sqrt{k}}{5\pi\log k} > h \tag{4.7}$$

(if C was choosen large enough).

On the other hand in the case $D > k^2$ we can use Theorem 1, which gives the inequality

$$L(1, \chi_D) > \frac{1}{140\cdot 8\cdot \log^3 D\cdot D^{12\gamma}} > \frac{\pi}{\sqrt[4]{D}} \tag{4.8}$$

(for $D > D_0$ absolute constant), i.e. by the class number formula of Dirichlet we have

$$h(-D) > \sqrt[4]{D} \geqslant \sqrt{k} \geqslant \sqrt{Ch^2\log^2(3h)} > h. \tag{4.9}$$

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