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Elementary methods in the theory of L-functions, II
On the greatest real zero of a real L-function

by

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1. As it is well-known, the L-zeros play an important role in the distribution of primes in arithmetic progressions and hence many great problems of the analytical number theory depend on the zeros of L-functions.

After the investigations of Gronwall [7] and Titchmarsh [19] zerofree regions were given for L-functions belonging to complex characters. Page [12] proved in 1934 the following theorem:

For a real zero $1-\delta$ of an L-function belonging to a real primitive character modulo D

$$\delta \geqslant \frac{1}{\sqrt{D}\log^2 D}. \tag{1.1}$$

(1.1) is an easy consequence of the lower bound

$$L(1) \geqslant \frac{\pi}{\sqrt{D}}, \tag{1.2}$$

which we can get from Dirichlet's class number-formula and of the fact

$$L'(\sigma) = O(\log^2 D) \quad \text{for} \quad 1-\frac{1}{\log D} \leqslant \sigma \leqslant 1, \tag{1.3}$$

which we can prove easily by partial summation.

Thus by the mean value theorem of differential calculus there is a ξ, $0 \leqslant \xi \leqslant \delta$,

$$\frac{L(1)}{\delta} = L'(1-\xi) = O(\log^2 D). \tag{1.4}$$

In 1935 Siegel [16] proved

$$L(1) \geqslant C(\varepsilon) D^{-\varepsilon} \quad \text{for an arbitrary } \varepsilon > 0, \tag{1.5}$$

where $C(\varepsilon)$ is an ineffective constant depending on ε.

Siegel's lower bound (1.5) together with (1.3) gives, as Walfisz [21] observed, a better estimate for the $1-\delta$ real zero, namely with a $C'(\varepsilon)$ ineffective constant depending on ε

(1.6) $$\delta > C'(\varepsilon) D^{-\varepsilon}.$$

The real zeros $1-\delta$ with $\delta \leqslant 1/\log D$ we shall call Siegel-zeros.

As the ineffectiveness of the Siegel-zeros causes a lot of difficulties in the analytical number theory, it is natural to try to improve (1.1) in an effective way. To improve effectively (1.2) is very difficult. After the results of Baker [1], [2], [3], and Stark [17], [18], [3] we know only that for the class number $h(-D)$ of the imaginary quadratic field $Q(\sqrt{-D})$

(1.7) $$h(-D) \geqslant 3, \quad \text{if} \quad D > D_0,$$

where D_0 is an effective absolute constant.

Thus

(1.8) $$L(1) \geqslant \frac{3\pi}{\sqrt{D}}, \quad \text{if} \quad D \geqslant D_0.$$

(In the case $\chi(n) = \left(\frac{D}{n}\right)$, $D > 0$, the class-number formula gives

(1.9) $$L(1) \gg \frac{\log D}{\sqrt{D}}.)$$

To improve (1.3) resp. (1.4) in the full generality would be also very difficult, if possible. But Davenport [5] proved in 1966, that in the case of

(1.10) $$L(1) = O\left(\frac{\log^3 D}{\sqrt{D}}\right)$$

one has

(1.11) $$\frac{L(1)}{\delta} = O(\log\log D).$$

This result gives together with (1.2)

(1.12) $$\delta \gg \frac{1}{\sqrt{D}\log\log D}$$

in any case, because if (1.10) does not hold, then (1.12) follows directly from (1.4)

Besides, we must mention a paper of Haneke [9], from the year 1973,

where he asserts in the case of (1.10)

(1.13) $$\frac{L(1)}{\delta} = O(1),$$

i.e.

(1.14) $$\delta \gg \frac{1}{\sqrt{D}}.$$

But his very complicated proof is incorrect, since in the last of his 23 Lemmas, estimating the sum

$$\sum_{p \leqslant D^2} \frac{(1+\chi(p))}{p}$$

he neglects those primes for which

$$\chi(p) = 0, \quad \text{i.e.} \quad p|D.$$

If we correct this, Haneke's paper gives only (1.11) and (1.12), i.e. the results of Davenport (proved in another way).

On the other hand Bateman and Grosswald mention in a paper [4] written in 1962 an unpublished result of J. B. Rosser, according to which

(1.15) $$\delta > \frac{6}{\pi\sqrt{D}}.$$

In 1963 Grosswald [8] mentioned the joint unpublished result of himself and Bateman, which was essentially (1.15), and without formulating any theorem, he referred in connection with the question to a mimeographed work of J. B. Rosser [15], that perhaps contains the proof of (1.15)([1]). Davenport and Haneke in their papers [5] and [9] also did not mention these results.

In this paper we shall prove the following theorems. All the results of this paper will be effective.

THEOREM 1. *Let χ be a real non-principal character modulo D, for which*

(1.16) $$L(1) = o\left(\frac{1}{\log D}\right).$$

Then for the Siegel-zero $1-\delta$ of $L(s)$ (which exists by the theorem of Hecke [11]) *with the notation* $g(n) = \sum_{d|n} \chi(d)$

(1.17) $$L'(1) \sim \frac{L(1)}{\delta} \sim \sum_{n \leqslant D^2} \frac{g(n)}{n} \geqslant \prod_{p|D}\left(1+\frac{1}{p}\right)\left(\frac{\pi^2}{6} - o(1)\right).$$

([1]) In a letter of 8th July 1974 to Prof. J. B. Rosser I inquired after his results, but I did not get any answer until now (6. 3. 1975).

(The sign $\sim$ replaces a factor $1+o(1)$.)

Theorem 1 improves a result of Fluch [6], proved by Estermann's method, that if

$$L'(1) \leqslant 1$$

then

$$L(1) \geqslant \frac{1}{\log D}.$$

It also improves a result of Haneke [9], according to which if (1.16) holds, then

$$\frac{L(1)}{\delta} \geqslant 1. \tag{1.18}$$

This assertion was proved by us in part I of this series [13] with

$$\frac{L(1)}{\delta} \geqslant 0.23 \tag{1.19}$$

and now Theorem 1 gives

$$\frac{L(1)}{\delta} \geqslant \prod_{p|D}\left(1+\frac{1}{p}\right)\left(\frac{\pi^2}{6}-o(1)\right) \tag{1.20}$$

which is, as we shall see from Theorem 2, in some sense the best possible result. (It shows that not only the proof of (1.13) in Haneke's paper, but also the assertion (1.13) is false.)

Using Theorem 1 we can prove

THEOREM 2. *If $-D<0$ is a fundamental discriminant for which the inequality*

$$h(-D) \leqslant \frac{\log D}{2\log\log D} \tag{1.21}$$

holds, then for the greatest real zero $1-\delta$ of $L(s,\chi)$, where $\chi(n)=\left(\frac{-D}{n}\right)$, the relation

$$\frac{L(1)}{\delta} \sim \frac{\pi^2}{6}\prod_{p|D}\left(1+\frac{1}{p}\right), \tag{1.22}$$

i.e.

$$\delta \sim \frac{6L(1)}{\pi^2\prod_{p|D}\left(1+\frac{1}{p}\right)} = \frac{6h(-D)}{\pi\prod_{p|D}\left(1+\frac{1}{p}\right)\sqrt{D}} \tag{1.23}$$

holds.

As in the most critical case (1.21) we almost exactly know the place of the zero, using the deep inequality (1.7) it is easy to prove the following effective improvement of Davenport's inequality (1.12) (and also of J. B. Rosser's unpublished (?) result (1.15)).

THEOREM 3. *For the greatest real zero $1-\delta$ of an L-function belonging to a real primitive character modulo D, the inequality*

$$\delta \geqslant \frac{12-o(1)}{\pi\sqrt{D}} \tag{1.24}$$

holds [2].

As an L-function belonging to an imprimitive character modulo D has the same zero as an other L-function belonging to a primitive character with a modulus $D^* < D$, Theorem 3 remains true for imprimitive characters too.

We shall prove one more theorem as follows:

THEOREM 4. *If for the real non-principal character $\chi(\mathrm{mod}\, D)$*

$$L(1,\chi) \leqslant \frac{1}{\log^2 D}, \tag{1.25}$$

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(2) Professor Schinzel, as referee of the paper, kindly made the following remark thereby improving the result of Theorem 3:

Theorem 3 is true in the following sharper form: For the greatest real zero $1-\delta$ of an L-function belonging to a real (primitive) character modulo D,

$$\delta \geqslant \left(\frac{16}{\pi}-\varepsilon\right)\frac{1}{\sqrt{D}} \quad \text{if} \quad D > D_0(\varepsilon) \tag{1}$$

where $D_0(\varepsilon)$ is effectively computable.

As one can easily see the proof of Theorem 3 is valid for the sharpened inequality (1) except for the negative fundamental discriminants $-D$ of the form

$$-D = -4pq,\ -8pq,\ -15r,\ -21r \quad (p, q, r \text{ primes}) \tag{2}$$

with

$$h(-D) = 4 \tag{3}$$

and

$$\left(1+\frac{1}{p}\right)\left(1+\frac{1}{q}\right) > 1+\frac{\varepsilon\pi}{16} \quad (p<q). \tag{4}$$

However, an effective bound for D satisfying (2), (3), (4), follows from Theorem 2 of the paper of A. Baker–A. Schinzel *On the least integers represented by genera of binary quadratic forms*, Acta Arith. 18 (1971), pp. 137–144. Indeed, the four genera of quadratic forms with discriminant $-D$ satisfying (2) and (3) represent numbers

$$1, 2, p, 2p; \quad 1, 3, 5, 15 \quad \text{or} \quad 1, 3, 7, 21,$$

respectively and by (4) we have $p < 32/\pi\varepsilon < D^{3/8}$ for $D > D_1(\varepsilon)$, so that the assumptions of Theorem 2, l.c. are fulfilled.

then

$$\exp\Big(\sum_{p\leqslant D^2}\frac{1+\chi(p)}{p}\Big)\ll\left(\frac{\log D\log\log D}{\log\dfrac{1}{5L(1,\chi)\log D}}\right)^2 \quad (^3). \tag{1.26}$$

The theorem asserts that any upper bound for $L(1,\chi)$ which is sharper than (1.25) implies a non-trivial upper bound for

$$\sum_{p\leqslant D^2}\frac{1+\chi(p)}{p},$$

i.e. shows that for the most primes $p\leqslant D^2$, $\chi(p)=-1$.

From (1.26) with some modification easily follows a result of W. Haneke [10], proved by him in a rather complicated and deep analitical way (using e.g. the Heilbronn–Deuring phenomenon).

THEOREM (Haneke). *If an L-function belonging to a real character $\chi(\mathrm{mod}\,D)$ has an $1-\delta$ Siegel-zero with*

$$\delta=o\Big(\frac{1}{\log^2 D}\Big) \tag{1.27}$$

then, for all

$$X\geqslant\exp\left\{\left(\frac{c_2\log D\log\log D}{\log\dfrac{1}{\delta\log D}}\right)^2\right\}, \tag{1.28}$$

the inequality

$$\sum_{p\leqslant X}\frac{1+\chi(p)}{p}\log^2 p\ll\left(\frac{\log D\log\log D}{\log\dfrac{1}{\delta\log D}}\right)^2+\delta(\log^3 X+\log^3 D) \quad (^4) \tag{1.29}$$

holds.

This theorem was proved by Haneke to improve a theorem of P. Turán [20] in the case of (1.27).

We shall use Theorem 4 to prove

THEOREM 5. *If for the real non-principal character $\chi(\mathrm{mod}\,D)$*

$$L(1,\chi)\leqslant\frac{1}{\log^2 D}, \tag{1.30}$$

(3) p always denotes throughout this paper a prime.

(4) The exponent 2 on the right side is missing by a misprint in the original paper.

then for the Siegel-zero $1-\delta$ of $L(s)$ (which exists by the theorem of Hecke [11])

$$\frac{L(1)}{\delta}\ll\left(\frac{\log D\log\log D}{\log\dfrac{1}{5L(1)\log D}}\right)^2. \tag{1.31}$$

This theorem shows that any sharper upper bound for $L(1,\chi)$ than (1.30) implies an improvement of Page's result (1.4), [12], i.e. of the almost trivial upper bound

$$\frac{L(1)}{\delta}=O(\log^2 D).$$

If e.g. $L(1)\leqslant D^{-\varepsilon}$, so from (1.31) we have

$$\frac{L(1)}{\delta}\ll\Big(\frac{1}{\varepsilon}\log\log D\Big)^2.$$

2. Now we turn to the proofs. In our paper χ will denote a real non-principal character modulo D, $L(s)$ the corresponding L-series, $g(n)=\sum_{d|n}\chi(d)$.

Proof of Theorem 1. First we prove the easy

LEMMA 1. *For an arbitrary $x\geqslant 4\sqrt{D}\log^2 D$*

$$\sum_{n\leqslant x}\frac{g(n)}{n}=L'(1)+(\log x+c)L(1)+5\vartheta\sqrt{\frac{\sqrt{D}\log D\log x}{x}}, \tag{2.1}$$

where c denotes Euler's constant, ϑ denotes a real number with an absolute value not exceeding 1, possibly different in various appearances throughout this paper.

Proof. We shall use that

$$\sum_{m\leqslant u}\frac{1}{m}=\log u+c+\vartheta\frac{1}{u}.$$

Let z denote a number — to be chosen later — for which $1\leqslant z\leqslant x$. Then

$$\sum_{n\leqslant x}\frac{g(n)}{n}=\sum_{d\leqslant x}\frac{\chi(d)}{d}\sum_{m\leqslant x/d}\frac{1}{m}=\sum_{d\leqslant z}{}_1+\sum_{z<d\leqslant x}{}_2, \tag{2.1'}$$

$$\sum{}_1=\sum_{d\leqslant z}\frac{\chi(d)}{d}\Big\{\log\frac{x}{d}+c+\frac{d\vartheta}{x}\Big\} \tag{2.2}$$

$$=(\log x+c)\sum_{d\leqslant z}\frac{\chi(d)}{d}-\sum_{d\leqslant z}\frac{\chi(d)\log d}{d}+\frac{z\vartheta}{x}.$$

Making use of Pólya's inequality

$$\Big|\sum_{d=a}^{b}\chi_D(d)\Big| \leqslant \tfrac{5}{3}\sqrt{D}\log D,$$

considering that

$$\frac{1}{d},\quad \frac{\log d}{d},\quad \frac{1}{d}\sum_{m\leqslant x/d}\frac{1}{m}$$

are monotonically decreasing in d, and applying Abel's inequality, we get

$$\Big|\sum_{d\leqslant z}\frac{\chi(d)}{d}\Big| \leqslant \frac{5\sqrt{D}\log D}{3z},$$

$$\Big|\sum_{d\leqslant z}\frac{\chi(d)\log d}{d}\Big| \leqslant \frac{5\sqrt{D}\log D\log z}{3z},$$

$$\Big|\sum\nolimits_2\Big| = \Big|\sum_{z<d\leqslant x}\frac{\chi(d)}{d}\sum_{m\leqslant x/d}\frac{1}{m}\Big| \leqslant \frac{5\sqrt{D}\log D\log x}{3z}.$$

Now set $z = \sqrt{\sqrt{D}\log D\log x\cdot x}$ $(\leqslant x)$. Then

$$\sum_{n\leqslant x}\frac{g(n)}{n} = \sum\nolimits_1 + \sum\nolimits_2 = (\log x + c)L(1) + \frac{5\vartheta\sqrt{D}\log D(\log x + c)}{3z} + L'(1) +$$

$$+\frac{5\vartheta\sqrt{D}\log D\log z}{3z} + \frac{5\vartheta\sqrt{D}\log D\log x}{3z} + \frac{\vartheta z}{x}$$

$$= L'(1) + (\log x + c)L(1) + 5\vartheta\sqrt{\frac{\sqrt{D}\log D\log x}{x}}. \ \blacksquare$$

If

$$L(1) = o\Big(\frac{1}{\log D}\Big)$$

then from Lemma 1 we get

$$\sum_{n\leqslant D^2}\frac{g(n)}{n} = L'(1) + o(1). \tag{2.3}$$

Next we shall use the following observation. Let

$$A_j = \{u\leqslant D^2;\ p|u\to\chi(p) = j\}\quad \text{for}\quad j = -1, 0, 1, \tag{2.4}$$

$$R = \{r\leqslant D^2;\ r = b\cdot m,\ b\in A_0,\ m\in A_{-1}\}, \tag{2.5}$$

$$\varrho(m) = \begin{cases}1, & \text{if}\quad m = l^2,\\ 0, & \text{if}\quad m\neq l^2.\end{cases} \tag{2.6}$$

Then an arbitrary $n\leqslant D^2$ can be written as $n = ar = abm$, where $r\in R$, $a\in A_1$, $b\in A_0$, $m\in A_{-1}$. This gives

$$g(n) = \prod_{p^{\alpha}\|n}\big(1+\chi(p)+\ldots+\chi^{\alpha}(p)\big) = g(a)g(r) = g(a)g(b)g(m) \tag{2.7}$$

$$= d(a)h(m)\quad (\geqslant 0).$$

Hence

$$\sum_{n\leqslant D^2}\frac{g(n)}{n} \geqslant \sum_{\substack{ql^2\leqslant D^2\\ \mu(q)\neq 0\\ q|D}}\frac{g(ql^2)}{ql^2} \geqslant \sum_{\substack{ql^2\leqslant D^2\\ \mu(q)\neq 0\\ q|D}}\frac{1}{ql^2} \geqslant \sum_{\substack{\mu(q)\neq 0\\ q|D}}\frac{1}{q}\sum_{l^2\leqslant D}\frac{1}{l^2} \tag{2.8}$$

$$= \prod_{p|D}\Big(1+\frac{1}{p}\Big)\Big(\frac{\pi^2}{6} - o(1)\Big).$$

Next we assert

Lemma 2. *For an arbitrary τ with $0<\tau<1$ there exists a c_τ, $0<c_\tau<1$, such that for all*

$$x \geqslant \frac{3\sqrt{D}\log D}{\tau}$$

the relation

$$\sum_{n\leqslant x}\frac{g(n)}{n^{1-\tau}} = \Big(c_\tau - \frac{1}{\tau}\Big)L(1-\tau) + \frac{1}{\tau}x^{\tau}L(1) + 2x^{\tau}\vartheta\sqrt{\frac{3\sqrt{D}\log D}{\tau x}} \tag{2.9}$$

holds.

This is Lemma 0 of [14] with the only change that in the proof the trivial estimate

$$\Big|\sum_{d=a}^{b}\chi(d)\Big| < D$$

must be replaced by Pólya's inequality

$$\Big|\sum_{d=a}^{b}\chi(d)\Big| \leqslant \tfrac{5}{3}\sqrt{D}\log D.$$

Making use of the inequality $L(1)\geqslant \pi/\sqrt{D}$ mentioned above, setting $\tau = \delta$ and $x = D^2$ and multiplying (2.9) by $x^{-\delta}$ we get

$$\sum_{n\leqslant D^2}\frac{g(n)}{n}\Big(\frac{n}{x}\Big)^{\delta} = \frac{L(1)}{\delta} + 2\vartheta\sqrt{\frac{3\sqrt{D}\log D}{\delta D^2}} \tag{2.10}$$

$$= \frac{L(1)}{\delta}\Big(1 + \frac{\vartheta\sqrt{12\,\delta\log D\cdot D^{-1/2}}}{L(1)\sqrt{D}}\Big) \sim \frac{L(1)}{\delta}.$$

Since by the theorem of Hecke [11] we have

$$\delta = o\left(\frac{1}{\log D}\right),$$

it follows

(2.11) $$1 \geqslant \left(\frac{n}{x}\right)^{\delta} \geqslant \frac{1}{x^{\delta}} = \frac{1}{D^{2\delta}} = \frac{1}{e^{2\delta \log D}} = \frac{1}{e^{o(1)}} = 1 - o(1).$$

Thus as $g(n) \geqslant 0$, from (2.10) and (2.11) we get

(2.12) $$\frac{L(1)}{\delta} \sim \sum_{n \leqslant D^2} \frac{g(n)}{n}\left(\frac{n}{x}\right)^{\delta} = \sum_{n \leqslant D^2} \frac{g(n)}{n}\big(1 - o(1)\big) \sim \sum_{n \leqslant D^2} \frac{g(n)}{n}.$$

The assertion of Theorem 1 follows from (2.3), (2.8) and (2.12).

3. Now for the proof of Theorem 2 it is sufficient to prove that if (1.21) is fulfilled then

(3.1) $$\sum_{n \leqslant D^2} \frac{g(n)}{n} \sim \prod_{p \mid D}\left(1 + \frac{1}{p}\right)\frac{\pi^2}{6}.$$

It is easy to show (see Davenport [5], Hilfssatz 1, and Haneke [9], Hilfssatz 14) that

(3.2) $$\sum_{\substack{n \leqslant \sqrt{D}/2 \\ p \mid n \to \chi(p) = 1}} 1 \leqslant h(-D).$$

Hence if $\chi(p) = 1$, then

$$p^{h(-D)+1} \geqslant \frac{\sqrt{D}}{2},$$

i.e.,

(3.3) $$p \geqslant \left(\frac{\sqrt{D}}{2}\right)^{\frac{1}{h(-D)+1}} \geqslant \frac{1}{2}\exp\left(\frac{\log D}{2h(-D)+2}\right).$$

(3.2) implies also that the number of primes, for which $p \leqslant \sqrt{D}/2$ and $\chi(p) = 1$ does not exceed $h(-D)$. Thus the assumption

$$h(-D) \leqslant \frac{\log D}{2\log\log D}$$

gives together with (3.3)

(3.4) $$\sum_{\substack{\chi(p)=1 \\ p \leqslant \sqrt{D}/2}} \frac{1}{p} \leqslant 2h(-D)e^{-\frac{\log D}{2h(-D)+2}}$$
$$\leqslant \frac{\log D}{2\log\log D}\cdot 4e^{-\log\log D} = \frac{2}{\log\log D} = o(1).$$

On the other hand, applying Lemma 1 with $x = D/4$ and $x = D^4$, we get

(3.5)
$$\Bigg(\sum_{\substack{\sqrt{D}/2 < p \leqslant D^2 \\ \chi(p)=1}} \frac{1}{p}\Bigg)^2 \leqslant \sum_{D/4 < n \leqslant D^4} \frac{g(n)}{n} = L(1)\log(4D^3) + O\left(\sqrt{\frac{\sqrt{D}\log^2 D}{D}}\right) = o(1).$$

Thus from (3.4) and (3.5) follows

(3.6) $$\sum_{\substack{p \leqslant D^2 \\ \chi(p)=1}} \frac{1}{p} = o(1).$$

Applying the definitions (2.4), (2.5), (2.6), from (3.6), we get

(3.7) $$1 \leqslant \sum_{\substack{a \leqslant D^2 \\ a \in A_1}} \frac{g(a)}{a} \leqslant \prod_{\substack{p \leqslant D^2 \\ \chi(p)=1}}\left(1 + \frac{2}{p} + \frac{3}{p^2} + \ldots\right)$$
$$= \prod_{\substack{p \leqslant D^2 \\ \chi(p)=1}} \frac{1}{\left(1 - \frac{1}{p}\right)^2} = \exp\Bigg(2\sum_{\substack{p \leqslant D^2 \\ \chi(p)=1}} \log\frac{1}{1 - \frac{1}{p}}\Bigg)$$
$$\leqslant \exp\Bigg(2\sum_{\substack{p \leqslant D^2 \\ \chi(p)=1}} \frac{2}{p}\Bigg) = e^{o(1)} = 1 + o(1).$$

Now, making use of $g(n) \geqslant 0$, from (3.7), we get

(3.8) $$\sum_{n \leqslant D^2} \frac{g(n)}{n} = \sum_{\substack{r \leqslant D^2 \\ r \in R}} \frac{g(r)}{r} \sum_{\substack{a \leqslant D^2/r \\ a \in A_1}} \frac{g(a)}{a}$$
$$= \sum_{\substack{r \leqslant D^2 \\ r \in R}} \frac{g(r)}{r}\big(1 + o(1)\big) \sim \sum_{\substack{r \leqslant D^2 \\ r \in R}} \frac{g(r)}{r}.$$

As for an $r = ql^2 \in R$ ($\mu(q) \neq 0$) by (2.7)

$$g(r) = \begin{cases} 1, & \text{if } q \mid D, \\ 0, & \text{if } q \nmid D. \end{cases}$$

Considering that if $r \notin R$ then by (3.3)

$$r \geqslant \frac{1}{2}\exp\left(\frac{\log D}{2h(-D)+2}\right) \geqslant \frac{1}{4}\log D \to \infty,$$

we get

$$(3.9)\qquad \sum_{\substack{r\leqslant D^2\\ r\in R}}\frac{g(r)}{r}=\sum_{\substack{q|D\\ \mu(q)\neq 0}}\frac{1}{q}\sum_{\substack{r\leqslant D/\sqrt{q}\\ r\in R}}\frac{1}{r^2}$$

$$=\sum_{\substack{q|D\\ \mu(q)\neq 0}}\frac{1}{q}\left(\frac{\pi^2}{6}-o(1)\right)\sim\prod_{p|D}\left(1+\frac{1}{p}\right)\frac{\pi^2}{6}.$$

(3.8) and (3.9) prove (3.1) and so we finished the proof of Theorem 2.

If we take into account the error term our proof gives the result

$$\delta=\frac{6h(-D)}{\pi\prod_{p|D}\left(1+\frac{1}{p}\right)\sqrt{D}}\left\{1+O\left(h(-D)\exp\left(\frac{\log D}{2h(-D)+2}\right)\right)\right\}.$$

4. Next we turn to the proof of Theorem 3. If

$$L(1)\geqslant\frac{\log^3 D}{\sqrt{D}}$$

then (1.24) follows from Page's result (1.4) [12]

$$\frac{L(1)}{\delta}=O(\log^2 D).$$

If

$$\frac{(\log\log D)^3}{\sqrt{D}}\leqslant L(1)\leqslant\frac{\log^3 D}{\sqrt{D}}$$

then (1.24) follows either from Davenport's result (1.12) [5]

$$\frac{L(1)}{\delta}=O(\log\log D)$$

or from our Theorem 5 (which of course we shall prove independently of Theorem 3) which in this case gives

$$\frac{L(1)}{\delta}=O\big((\log\log D)^2\big).$$

If

$$L(1)=O\left(\frac{(\log\log D)^3}{\sqrt{D}}\right)$$

then $\chi(n)=(-D/n)$, where $D>0$, so by Theorem 2

$$\delta\sim\frac{6h(-D)}{\pi\prod_{p|D}\left(1+\frac{1}{p}\right)\sqrt{D}}.$$

Thus for the proof of Theorem 3 it is sufficient to prove

$$(4.1)\qquad f(D)=\frac{h(-D)}{\prod_{p|D}\left(1+\frac{1}{p}\right)}\geqslant 2-o(1).$$

Let $\nu(D)$ denote the number of different prime divisors of D. If $\nu(D)\leqslant 2$, then the results of Baker [1]–[3], and Stark [17], [18], [3] give that for $D\geqslant D_0$ (effective constant)

$$h(-D)\geqslant 3$$

so (4.1) holds since in case of $\nu(D)\leqslant 2$ for the fundamental discriminant $-D<0$

$$\prod_{p|D}\left(1+\frac{1}{p}\right)\leqslant\frac{3}{2}\big(1+o(1)\big).$$

If $\nu(D)\geqslant 3$, making use of

$$h(-D)\geqslant 2^{\nu(D)-1},$$

we get

$$f(D)\geqslant\frac{1}{2\prod_{p|D}\left(\frac{1+1/p}{2}\right)}.$$

Thus for proving (4.1) it is sufficient to prove

$$(4.2)\qquad K(D)=\prod_{p|D}\left(\frac{1+1/p}{2}\right)\leqslant\frac{1}{4}\big(1+o(1)\big).$$

If $\nu(D)=3$, then as $-D$ is a fundamental discriminant

$$K(D)\leqslant\frac{\frac{3}{2}\cdot\frac{4}{3}\big(1+o(1)\big)}{2^3}=\tfrac{1}{4}\big(1+o(1)\big).$$

If $\nu(D)\geqslant 4$, then

$$K(D)\leqslant\frac{\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}}{2^4}=\tfrac{6}{35}<\tfrac{1}{4}.$$

This proves Theorem 3.

If we do not use the result $h(-D)\geqslant 3$ (for $D>D_0$) we get $f(D)\geqslant 1$, and so

$$\delta\geqslant\frac{6-o(1)}{\pi\sqrt{D}}.$$

5. Proof of Theorem 4. At first we prove the

LEMMA 3. *For a non-principal real character* χ $(\operatorname{mod} D)$ $(\geqslant 1500)$ *for any* A, $2 \leqslant A \leqslant D^2$, *and with the notation*

$$\sum_{\substack{A<p\leqslant D^2\\ \chi(p)\neq -1}} \frac{1}{p} = a \tag{5.1}$$

the inequality

$$L(1) > \frac{1}{5\log D}\left(\frac{a\log A}{8\log D}\right)^{\frac{8\log D}{\log A}} \tag{5.2}$$

holds.

Proof. Let us consider all the intervals of the form

$$(A^{2^{i-1}}, A^{2^i}]$$

for $i = 1, 2, \ldots, m$, where m is defined by

$$C = A^{2^{m-1}} < D^4 \leqslant A^{2^m} = C^2.$$

As here $A^{2^m} = C^2 < D^8$, we have

$$2^m \frac{8\log D}{\log A}.$$

Then there exists an $i \leqslant m$ for which

$$\sum_{\substack{A^{2^{i-1}}<p\leqslant A^{2^i}\\ \chi(p)\neq -1}} \frac{1}{p} \geqslant \frac{a}{m}.$$

If we raise the two sides of this inequality to the power 2^{m-1} then on the left side we shall get numbers of the form n^{-1}

$$n = \prod_{j=1}^{2^{m-1}} p_j \quad (A^{2^{i-1}} < p \leqslant A^{2^i};\ \chi(p) \neq -1)$$

with a multiplicity $\leqslant 2^{m-1}!$, so we have (since $a < m$)

$$\sum_{\substack{C<n\leqslant C^2\\ p|n\to\chi(p)\neq -1}} \frac{1}{n} \geqslant \frac{1}{2^{m-1}!}\left(\frac{a}{m}\right)^{2^{m-1}} > \left(\frac{a}{2^m}\right)^{2^m} > \left(\frac{a\log A}{8\log D}\right)^{8\frac{\log D}{\log A}}.$$

Considering that if for all p prime factors of n $\chi(p) \neq -1$, then $g(n) \geqslant 1$, and that for an arbitrary m, $g(m) \geqslant 0$, we get

$$\sum_{C<n\leqslant C^2} \frac{g(n)}{n} > \left(\frac{a\log A}{8\log D}\right)^{8\frac{\log D}{\log A}}. \tag{5.3}$$

But our Lemma 1 asserts for $x \geqslant \sqrt{D}\log^2 D$

$$\sum_{n\leqslant x} \frac{g(n)}{n} = L'(1) + (\log x + c)L(1) + 5\vartheta\sqrt{\frac{\sqrt{D}\log D\log x}{x}}.$$

Applying this with $x = C$, and $x = C^2$, we have as $D^2 \leqslant C < D^4$, $\sqrt{D}\cdot L(1) \geqslant \pi$,

$$\sum_{C<n\leqslant C^2} \frac{g(n)}{n} = \log C\cdot L(1) + 10\vartheta\sqrt{\frac{\sqrt{D}\log D\log D^4}{D^2}} \tag{5.4}$$

$$\leqslant 4\log D\cdot L(1) + \frac{20\log D}{D^{3/4}} < 5\log D\cdot L(1).$$

(5.4) and (5.3) together gives (5.2).

To prove Theorem 4 we need a result of Mertens, that for $x \geqslant 3$

$$\sum_{p\leqslant x} \frac{1}{p} \leqslant \log\log x + B, \tag{5.5}$$

where B is an absolute constant.

Let A be defined by

$$\log A = \frac{8\log D\log\log D}{\log\dfrac{1}{5\log D\cdot L(1)}} \quad (\geqslant 8) \tag{5.6}$$

and let us assume that the assertion of Theorem 4 (i.e. (1.26)) is not true but

$$\exp\left(\sum_{p\leqslant D^2} \frac{1+\chi(p)}{p}\right) \geqslant e^{2B+2}\log^2 A. \tag{5.7}$$

Then

$$\sum_{p\leqslant D^2} \frac{1+\chi(p)}{p} \geqslant 2\log\log A + 2B + 2. \tag{5.8}$$

From (5.8) and (5.5) it follows that $A \leqslant D^2$, and

$$\sum_{A<p\leqslant D^2} \frac{1+\chi(p)}{p} \geqslant 2 \tag{5.9}$$

and so

$$a = \sum_{\substack{A<p\leqslant D^2\\ \chi(p)\neq -1}} \frac{1}{p} \geqslant 1 \geqslant \frac{8}{\log A}. \tag{5.10}$$

Now using Lemma 3 from (5.2) we have

$$5\log D\cdot L(1) > \left(\frac{1}{\log D}\right)^{\frac{8\log D}{\log A}}, \tag{5.11}$$

$$\frac{8\log D}{\log A}\log\log D > \log\frac{1}{5\log D\cdot L(1)} \tag{5.12}$$

which contradicts to (5.6) and so the Theorem 4 is proved.

6. Proof of Theorem 5. As for an arbitrary u, v, $g(uv)\leqslant g(u)d(v)$, and

$$g(q) = \prod_{p|q}(1+\chi(p)) \quad \text{if} \quad \mu(q)\neq 0,$$

considering

$$\sum_{m=1}^{\infty}\frac{d(m^2)}{m^2} = \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^2\prod_p\left(1+\frac{1}{p^2}\right) < \zeta^3(2) = O(1)$$

we have

$$\sum_{n\leqslant D^2}\frac{g(n)}{n}\leqslant \sum_{\substack{q\leqslant D^2\\ \mu(q)\neq 0}}\frac{g(q)}{q}\sum_{m^2\leqslant D^2/q}\frac{d(m^2)}{m^2} \ll \sum_{\substack{q\leqslant D^2\\ \mu(q)\neq 0}}\frac{g(q)}{q} \tag{6.1}$$

$$\leqslant \prod_{p\leqslant D^2}\left(1+\frac{1+\chi(p)}{p}\right) < \exp\left(\sum_{p\leqslant D^2}\frac{1+\chi(p)}{p}\right).$$

Now Theorem 1 and Theorem 4 give that in case of

$$L(1)\leqslant\frac{1}{\log^2 D}$$

for the Siegel-zero $1-\delta$ of $L(s)$ (which exists by the theorem of Hecke [11]) (1.31) holds, i.e.,

$$\frac{L(1)}{\delta}\sim L'(1)\sim\sum_{n\leqslant D^2}\frac{g(n)}{n} \ll \left(\frac{\log D\log\log D}{\log\frac{1}{5L(1)\log D}}\right)^2.$$

Added in proof. Professor W. Haneke informed me in the mean time that his proof for (1.14) can be completed in a relatively easy way (see p. 275).

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