On the other hand one can prove that the series
\[ \sum_{q=1}^{\infty} \frac{a_q}{q} \sigma_q(n) \]
converges for \( s > 0 \) and that its sum tends to \( f(n) \) as \( s \) tends to zero through positive values. So, if the series (2) converges for some \( n \), its sum must be \( f(n) \).

Added in proof. It has been proved by W. Schwarz that the series (2) actually converges for every \( n \) (Acta Arith. 27 (1975), pp. 269-278).

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A counterexample to a conjecture on multinomial degree

by

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Let \( K \) be a field. A polynomial \( p(x) \) with coefficients in \( K \) of the form \( a_0 + a_1x + \ldots + a_dx^d \) with all \( a_i \neq 0 \) is called a multinomial of length \( d \). The \( d \)-tuple \((m_1, \ldots, m_d)\) is the exponent vector of \( p(x) \). An element \( \theta \) in a field extension of \( K \) is of multinomial degree \( d \) over \( K \) if \( \theta \) satisfies a multinomial of length \( d \) and no multinomial of smaller length.

Clearly, \( \theta \) has multinomial degree 1 over \( K \) if and only if some positive power of \( \theta \) lies in \( K \).

The following conjecture is posed in [2]: If \( K \) is a field of characteristic 0 and \( \theta \) is an element of multinomial degree \( d \) over \( K \) so that there exist \( d+1 \) multinomials of length \( d \) satisfied by \( \theta \), \( p_i(x) \), \( i = 0, 1, \ldots, d \), where the corresponding exponent vectors are not proportional, then \([K(\theta^m) : K] = d \) for some positive power \( m \) of \( \theta \).

Let \( \theta \) be a root of the irreducible polynomial \( x^3 - x + 1 \) over the field of rational numbers \( \mathbb{Q} \). We show that \( \theta \) provides a counterexample to the above conjecture. We observe that an element of odd degree \( m \) over \( \mathbb{Q} \) has multinomial degree 1 if and only if its minimal polynomial over \( \mathbb{Q} \) has the form \( x^m - a \). For a proof see [1]. Hence \( \theta \) has degree 3 and multinomial degree 2 over \( \mathbb{Q} \). Moreover, every positive power of \( \theta \) has degree 3 = \([\mathbb{Q}(\theta^m) : \mathbb{Q}]\).

Multiplying \( x^3 - x + 1 \) by appropriately chosen polynomials of degree 2 and 4 we obtain the following additional multinomials of length 2 satisfied by \( \theta \):

\[ x^3 + x^4 + 1 = (x^3 - x + 1)(x^2 + x + 1), \]
\[ x^3 - 2x^2 - 1 = (x^3 - x + 1)(x^2 - x^2 - x - 1), \]
\[ x^3 + 2x^2 + 1 = (x^3 - x + 1)(x^2 + x^2 + x + 1). \]

Thus \( \theta \) satisfies four multinomials of length 2 with exponent vectors \((1, 3), (4, 5), (5, 7), \) and \((4, 7)\), respectively. Hence \( \theta \) does provide the desired counterexample.
Elementary methods in the theory of $L$-functions, II
On the greatest real zero of a real $L$-function

by

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1. As it is well-known, the $L$-zeros play an important role in the distribution of primes in arithmetic progressions and hence many great problems of the analytical number theory depend on the zeros of $L$-functions.

After the investigations of Gronwall [7] and Titchmarsh [19] zetafree regions were given for $L$-functions belonging to complex characters. Page [12] proved in 1934 the following theorem:

For a real zero $1 - \delta$ of an $L$-function belonging to a real primitive character modulo $D$

\begin{equation}
\delta \geq \frac{1}{\sqrt{D} \log^2 D},
\end{equation}

(1.1)

(1.1) is an easy consequence of the lower bound

\begin{equation}
L(1) \geq \frac{\pi}{\sqrt{D}},
\end{equation}

(1.2)

which we can get from Dirichlet's class number-formula and of the fact

\begin{equation}
L'(\sigma) = O(\log^2 D) \quad \text{for} \quad 1 - \frac{1}{\log D} < \sigma \leq 1,
\end{equation}

(1.3)

which we can prove easily by partial summation.

Thus by the mean value theorem of differential calculus there is a $\xi$, $0 \leq \xi \leq \delta$,

\begin{equation}
\frac{L(1)}{\delta} = L'(1 - \xi) = O(\log^2 D).
\end{equation}

(1.4)

In 1935 Siegel [16] proved

\begin{equation}
L(1) \geq C(\varepsilon) D^{-\varepsilon}
\end{equation}

(1.5)

for an arbitrary $\varepsilon > 0$,

where $C(\varepsilon)$ is an ineffective constant depending on $\varepsilon$. 