On the other hand one can prove that the series

$$\sum_{1}^{\infty} \frac{a_q}{q^s} c_q(n)$$

converges for $s > 0$ and that its sum tends to $f(n)$ as s tends to zero through positive values. So, if the series (2) converges for some n, its sum must be $f(n)$.

Added in proof. It has been proved by W. Schwarz that the series (2) actually converges for every n (Acta Arith. 27 (1975), pp. 269–279).

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A counterexample to a conjecture on multinomial degree

by

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Let K be a field. A polynomial $p(x)$ with coefficients in K of the form $a_0 + a_1 x^{m_1} + \ldots + a_d x^{m_d}$ with all $a_i \neq 0$ is called a *multinomial of length* d. The d-tuple $(m_1, \ldots, m_d)$ is the exponent vector of $p(x)$. An element θ in a field extension of K is of multinomial degree d over K if θ satisfies a multinomial of length d and no multinomial of smaller length. Clearly, θ has multinomial degree 1 over K if and only if some positive power of θ lies in K.

The following conjecture is posed in [2]: If K is a field of characteristic 0 and θ is an element of multinomial degree d over K so that there exist $d+1$ multinomials of length d satisfied by θ, $p_i(x)$, $i = 0, 1, \ldots, d$, where the corresponding exponent vectors are not proportional, then $[K(\theta^m): K] = d$ for some positive power m of θ.

Let θ be a root of the irreducible polynomial $x^3 - x + 1$ over the field of rational numbers Q. We show that θ provides a counterexample to the above conjecture. We observe that an element of odd degree m over Q has multinomial degree 1 if and only if its minimal polynomial over Q has the form $x^m - a$. For a proof see [1]. Hence θ has degree 3 and multinomial degree 2 over Q. Moreover, every positive power of θ has degree $3 = [Q(\theta^m): Q]$.

Multiplying $x^3 - x + 1$ by appropriately chosen polynomials of degree 2 and 4 we obtain the following additional multinomials of length 2 satisfied by θ:

$$x^5 + x^4 + 1 = (x^3 - x + 1)(x^2 + x + 1),$$
$$x^7 - 2x^5 - 1 = (x^3 - x + 1)(x^4 - x^2 - x - 1),$$
$$x^7 + 2x^4 + 1 = (x^3 - x + 1)(x^4 + x^2 + x + 1).$$

Thus θ satisfies four multinomials of length 2 with exponent vectors $(1, 3)$, $(4, 5)$, $(5, 7)$, and $(4, 7)$, respectively. Hence θ does provide the desired counterexample.

References

[1] Lawrence Risman, *On the order and degree of solutions to pure equations*, Proc. Amer. Math. Soc., 55 (1976), pp. 261–266.
[2] M. Schacher and E. G. Straus, *Some applications of a non-Archimedean analogue of Descartes' rule of signs*, Acta Arith. 25 (1974), pp. 353–357.

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Elementary methods in the theory of L-functions, II
On the greatest real zero of a real L-function

by

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1. As it is well-known, the L-zeros play an important role in the distribution of primes in arithmetic progressions and hence many great problems of the analytical number theory depend on the zeros of L-functions.

After the investigations of Gronwall [7] and Titchmarsh [19] zerofree regions were given for L-functions belonging to complex characters. Page [12] proved in 1934 the following theorem:

For a real zero $1-\delta$ of an L-function belonging to a real primitive character modulo D

$$\delta \geqslant \frac{1}{\sqrt{D}\log^2 D}. \tag{1.1}$$

(1.1) is an easy consequence of the lower bound

$$L(1) \geqslant \frac{\pi}{\sqrt{D}}, \tag{1.2}$$

which we can get from Dirichlet's class number-formula and of the fact

$$L'(\sigma) = O(\log^2 D) \quad \text{for} \quad 1-\frac{1}{\log D} \leqslant \sigma \leqslant 1, \tag{1.3}$$

which we can prove easily by partial summation.

Thus by the mean value theorem of differential calculus there is a ξ, $0 \leqslant \xi \leqslant \delta$,

$$\frac{L(1)}{\delta} = L'(1-\xi) = O(\log^2 D). \tag{1.4}$$

In 1935 Siegel [16] proved

$$L(1) \geqslant C(\varepsilon) D^{-\varepsilon} \quad \text{for an arbitrary } \varepsilon > 0, \tag{1.5}$$

where $C(\varepsilon)$ is an ineffective constant depending on ε.