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On Ramanujan expansions of certain arithmetical functions

by

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Dedicated to Theodor Schneider on his 65th birthday

1. Introduction. Let us associate to each real or complex-valued arithmetical function f the arithmetical function $f' = \mu * f$ (defined by $f'(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$).

The following results are known[1].

(a) If we have

$$\sum_{n=1}^{\infty} \frac{|f'(n)|}{n} < \infty, \tag{1}$$

then f is limit-periodic (B).

Moreover the Fourier-series of f is, after a suitable grouping of its terms, the Ramanujan series

$$\sum_{q=1}^{\infty} a_q c_q(n) \tag{2}$$

where $c_q(n)$ is Ramanujan's sum

$$\sum_{\substack{1 \leqslant h \leqslant q \\ (h,q)=1}} \exp\left(2\pi i \frac{hn}{q}\right)$$

and the coefficient a_q is given by the formula

$$a_q = \sum_{m=1}^{\infty} \frac{f'(mq)}{mq}. \tag{3}$$

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(1) See Wintner, *Eratosthenian averages* (Waverly Press, 1943), paragraphs 26, 27, 33, 35.

(b) If we have not only (1) but the stronger condition

$$\sum_{n=1}^{\infty} d(n)\frac{|f'(n)|}{n} < \infty, \tag{4}$$

where $d(n)$ is the number of divisors of n, then for each n the Ramanujan-series (2) is absolutely convergent and equal to $f(n)$.

The proof of (a) is very simple: By Möbius inversion formula we have for every n

$$f(n) = \sum_{d|n} f'(d).$$

For each positive integer k define an arithmetical function f_k by

$$f_k(n) = \sum_{\substack{d|n \\ d\leqslant k}} f'(d).$$

Then f_k is obviously periodic with period $k!$. Moreover we have for each n

$$|f(n)-f_k(n)| = \Big|\sum_{\substack{d|n \\ d>k}} f'(d)\Big| \leqslant \sum_{\substack{d|n \\ d>k}} |f'(d)|^{(2)}.$$

It follows that we have for $x \geqslant k+1$

$$\sum_{n\leqslant x} |f(n)-f_k(n)| \leqslant \sum_{k<d\leqslant x} |f'(d)|\left[\frac{x}{d}\right],$$

and therefore

$$\frac{1}{x}\sum_{n\leqslant x} |f(n)-f_k(n)| \leqslant \sum_{d=k+1}^{\infty} \frac{|f'(d)|}{d}.$$

Since the right-hand side tends to zero as k tends to infinity, this implies that f is limit-periodic (B).

Moreover its Fourier-coefficients are the limits as k tends to infinity of the Fourier-coefficients of f_k, which are very easily computed.

If $a = h/q$, where h and q are integers, $q>0$ and $(h,q)=1$, this gives

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x} f(n)\exp(-2\pi i a n) = \sum_{q|d}\frac{f'(d)}{d} = \sum_{m=1}^{\infty}\frac{f'(mq)}{mq}.$$

It is interesting to consider the particular case when f is a multiplicative function (3).

(2) An empty sum is assigned the value zero.

(3) Loc. cit., § 46. To avoid any misunderstanding we emphasize that in our terminology a function f is said to be *multiplicative* if we have not only $f(mn) = f(m)f(n)$ whenever $(m,n)=1$ but also $f(1)=1$ (we exclude the function which is identically zero).

Then f' is also multiplicative and therefore (1) is equivalent to

$$\sum_{p,r}\frac{|f'(p^r)|}{p^r} < \infty,$$

where in the summation p runs through the set of all primes and r runs through the set of all positive integers.

Since $f'(p^r) = f(p^r)-f(p^{r-1})$ this reads

$$\sum_{p,r}\frac{|f(p^r)-f(p^{r-1})|}{p^r} < \infty. \tag{5}$$

Similarly (4) is equivalent to

$$\sum_{p,r}(r+1)\frac{|f(p^r)-f(p^{r-1})|}{p^r} < \infty$$

(or the same with r instead of $r+1$).

Our main purpose here is to prove that (b) *still holds if* (4) *is replaced by the weaker condition*

$$\sum_{n=1}^{\infty} 2^{\omega(n)}\frac{|f'(n)|}{n} < \infty, \tag{6}$$

where $\omega(n)$ is the number of distinct prime divisors of n.

The proof is very simple and it is surprising that Wintner did not obtain this result.

In the case when f is multiplicative (6) is equivalent to (5).

Thus, *if f is a multiplicative function satisfying* (5), *then we can assert not only that f is limit-periodic (B) but also that for each n its Ramanujan-series is absolutely convergent and equal to $f(n)$*(4).

We will also prove for that case a formula which expresses the coefficient a_q by means of an infinite product:

Let

$$\varrho_p(q) = \begin{cases} 0 & \text{if} \quad p\nmid q, \\ a & \text{if} \quad p^a \| q,\ a\geqslant 1. \end{cases}$$

Then we have

$$a_q = \prod_p\Big(\sum_{r=\varrho_p(q)}^{\infty}\frac{f'(p^r)}{p^r}\Big), \tag{7}$$

where the series and the infinite product are absolutely convergent.

(4) This has already been proved in special cases by E. Cohen (Bull. Amer. Math. Soc. 67 (1961), pp. 145–147).

If $f'(p^r)$ is replaced by its value, (7) becomes

$$a_q = \left(\prod_{p|q}\left(\sum_{r=e_p(q)}^{\infty}\frac{f(p^r)-f(p^{r-1})}{p^r}\right)\right)\left(\prod_{p\nmid q}\left(1+\sum_{r=1}^{\infty}\frac{f(p^r)-f(p^{r-1})}{p^r}\right)\right).$$

We will conclude by some remarks on Ramanujan series of multiplicative functions.

2. Proof of the main result. We suppose that the arithmetical function f satisfies (6) (which obviously implies (1)). We will prove that for every positive integer n the series (2), where the coefficients a_q are given by (3), is absolutely convergent and equal to $f(n)$.

We consider a fixed n.

2.1. We first show that it suffices to prove that the double series

$$\sum_{m,q\geqslant 1}\frac{f'(mq)}{mq}c_q(n) \tag{8}$$

is absolutely convergent.

Suppose this has been proved. Then, since

$$\sum_{m=1}^{\infty}\frac{f'(mq)}{mq}c_q(n) = a_q c_q(n),$$

we see that (2) is absolutely convergent and its sum is equal to the sum of (8). The latter is equal to

$$\sum_{k=1}^{\infty} w_k,\quad \text{where}\quad w_k = \sum_{mq=k}\frac{f'(mq)}{mq}c_q(n) = \frac{f'(k)}{k}\sum_{mq=k}c_q(n).$$

But we have

$$\sum_{mq=k}c_q(n) = \sum_{h=1}^{k}\exp\left(2\pi i\frac{hn}{k}\right) = \begin{cases}k & \text{if } k|n,\\ 0 & \text{otherwise.}\end{cases}$$

In fact,

$$\sum_{h=1}^{k}\exp\left(2\pi i\frac{hn}{k}\right) = \sum_{m|k}\left(\sum_{\substack{1\leqslant h\leqslant k\\(h,k)=m}}\exp\left(2\pi i\frac{hn}{k}\right)\right)$$

and, since "$m|k$ and $(h,k)=m$" is equivalent to "$k=mq$ and $h=mh'$ with $(h',q)=1$", this is equal to

$$\sum_{mq=k}\left(\sum_{\substack{1\leqslant h\leqslant q\\(h,q)=1}}\exp\left(2\pi i\frac{hn}{q}\right)\right) = \sum_{mq=k}c_q(n).$$

Thus

$$w_k = \begin{cases}f'(k) & \text{if } k|n,\\ 0 & \text{otherwise,}\end{cases}$$

and therefore

$$\sum_{k=1}^{\infty} w_k = \sum_{k|n} f'(k) = f(n).$$

2.2. Now the absolute convergence of (8) is equivalent to

$$\sum_{k=1}^{\infty} W_k < \infty,\quad \text{where}\quad W_k = \sum_{mq=k}\frac{|f'(mq)|}{mq}|c_q(n)| = \frac{|f'(k)|}{k}\sum_{q|k}|c_q(n)|.$$

That this is implied by (6) follows from the following

LEMMA. *For every positive integer k,*

$$\sum_{q|k}|c_q(n)| \leqslant n\cdot 2^{\omega(k)}.$$

2.2.1. Proof. Define arithmetic functions g_n and h_n by

$$g_n(q) = c_q(n) \quad\text{and}\quad h_n(k) = \sum_{q|k}|g_n(q)|.$$

We have to prove that $h_n(k)\leqslant n\cdot 2^{\omega(k)}$.

It is well known that the function g_n is multiplicative and that

$$g_n(q) = c_q(n) = \sum_{d|(q,n)} d\mu\left(\frac{q}{d}\right)\text{(5)}. \tag{9}$$

Since g_n is multiplicative, h_n is also multiplicative. So it is completely determined by its values for the powers of primes, i.e. by the numbers

$$h_n(p^r) = \sum_{j=0}^{r}|g_n(p^j)|.$$

Now, if $p\nmid n$, (9) gives $g_n(p^r) = \mu(p^r)$, so that

$$h_n(p^r) = 2 \quad \text{for every } r\geqslant 1.$$

If $p^a\|n$, $a\geqslant 1$, then (9) gives

$$g_n(p^r) = \sum_{j=0}^{\min(r,a)} p^j\mu(p^{r-j}) = \begin{cases}p^r-p^{r-1} & \text{if } 1\leqslant r\leqslant a,\\ -p^a & \text{if } r=a+1,\\ 0 & \text{if } r>a+1,\end{cases}$$

(5) See, for instance, Hardy and Wright, *An Introduction to the Theory of Numbers*, theorems 67 and 271.

and it follows that

$$h_n(p^r) = \begin{cases} p^r & \text{if} \quad 1 \leqslant r \leqslant \alpha, \\ 2p^\alpha & \text{if} \quad r > \alpha, \end{cases}$$

so that $0 \leqslant h_n(p^r) \leqslant 2p^\alpha$ for every $r \geqslant 1$.

Thus we always have $0 \leqslant h_n(p^r) \leqslant 2p^{\varrho_p(n)}$ for every $r \geqslant 1$, where ϱ_p is defined as in formula (7).

This gives

$$h_n(k) = \prod_{p|k} h_n(p^{\varrho_p(k)}) \leqslant 2^{\omega(k)} \prod_{p|(k,n)} p^{\varrho_p(n)} \leqslant 2^{\omega(k)} \prod_{p|n} p^{\varrho_p(n)} = n\cdot 2^{\omega(k)}.$$

2.2.2. Remark. It is clear that

$$\sum_{q|k} |c_q(n)| = h_n(k) = n\cdot 2^{\omega(k)} \quad \text{when } k \text{ is a multiple of } n' = n\prod_{p|n} p.$$

So our lemma is best possible.

Wintner used the crude estimate

$$\sum_{q|k} |c_q(n)| \leqslant \sigma(n)\, d(k),$$

which follows from the fact that (9) implies $|c_q(n)| \leqslant \sigma(n)$. He was probably not aware of the multiplicative property of Ramanujan's sum.

3. Proof of formula (7). We now suppose that f is a multiplicative function satisfying (5), and therefore (1), and that a_q is given by formula (3). We will prove that we have (7).

3.1. (5) obviously implies that for each prime p the series

$$\sum_{r=\varrho_p(q)}^{\infty} \frac{f'(p^r)}{p^r}$$

is absolutely convergent and that the infinite product

$$\text{(P)} \qquad \prod_p \Big(\sum_{r=\varrho_p(q)}^{\infty} \frac{f'(p^r)}{p^r} \Big)$$

is absolutely convergent.

We also see that (7) is trivial for $q = 1$.

3.2. We now suppose that $q > 1$. One of the following circumstances occurs:

(i) There is a prime p dividing q for which $f'(p^r) = 0$ whenever

$$r \geqslant \varrho_p(q) \text{ (so that } \sum_{r=\varrho_p(q)}^{\infty} f'(p^r)/p^r = 0);$$

(ii) For each prime p dividing q there is some $r \geqslant \varrho_p(q)$ for which $f'(p^r) \neq 0$.

In case (i) we have $f'(mq) = 0$ for every m (for $\varrho_p(mq) \geqslant \varrho_p(q)$) and therefore $a_q = 0$. Then (7) holds since the infinite product (P) has a zero factor.

Now consider case (ii). For each prime p dividing q, let $\varrho_p(q)+\alpha_p$ be the smallest $r \geqslant \varrho_p(q)$ for which $f'(p^r) \neq 0$.

Set $\delta = \prod_{p|q} p^{\alpha_p}$. We obviously have $f'(\delta q) \neq 0$.

On the other hand $f'(mq) = 0$ if $\delta \nmid m$, and it follows that

$$a_q = \sum_{\nu=1}^{\infty} \frac{f'(\nu\delta q)}{\nu\delta q}.$$

It is very easy to check that for every positive integer ν

$$f'(\nu\delta q) = f'(\delta q)\, g(\nu),$$

where g is the multiplicative function determined by

$$g(p^r) = \begin{cases} f'(p^r) & \text{if} \quad p\nmid q, \\ f'(p^{\varrho_p(q)+\alpha_p+r})/f'(p^{\varrho_p(q)+\alpha_p}) & \text{if} \quad p|q. \end{cases}$$

Thus we have

$$\text{(10)} \qquad a_q = \frac{f'(\delta q)}{\delta q} \sum_{\nu=1}^{\infty} \frac{g(\nu)}{\nu}.$$

(5) obviously implies

$$\sum_{\substack{p,r \\ r\geqslant 1}} \frac{|g(p^r)|}{p^r} < \infty,$$

and it follows that

$$\sum_{\nu=1}^{\infty} \frac{g(\nu)}{\nu} = \prod_p \Big(\sum_{r=0}^{\infty} \frac{g(p^r)}{p^r} \Big).$$

Now it is very easy to check that

$$\prod_p \Big(\sum_{r=0}^{\infty} \frac{g(p^r)}{p^r} \Big) = \frac{\delta q}{f'(\delta q)} \prod_p \Big(\sum_{r=\varrho_p(q)}^{\infty} \frac{f'(p^r)}{p^r} \Big),$$

so that (10) yields (7).

First, for each prime p which does not divide q,

$$\sum_{r=0}^{\infty} \frac{g(p^r)}{p^r} = \sum_{r=0}^{\infty} \frac{f'(p^r)}{p^r}.$$

On the other hand, for each p dividing q,

$$\sum_{r=0}^{\infty}\frac{g(p^r)}{p^r} = p^{\varrho_p(q)+a_p} f'(p^{\varrho_p(q)+a_p})^{-1}\sum_{r=0}^{\infty}\frac{f'(p^{\varrho_p(q)+a_p+r})}{p^{\varrho_p(q)+a_p+r}}.$$

We obviously have

$$\sum_{r=0}^{\infty}\frac{f'(p^{\varrho_p(q)+a_p+r})}{p^{\varrho_p(q)+a_p+r}} = \sum_{r=\varrho_p(q)+a_p}^{\infty}\frac{f'(p^r)}{p^r},$$

and this is equal to $\sum_{r=\varrho_p(q)}^{\infty} f'(p^r)/p^r$ since, if $a_p>0$, $f'(p^r)=0$ for $\varrho_p(q)\leqslant r < \varrho_p(q)+a_p$.

It follows that

$$\prod_{p|q}\Big(\sum_{r=0}^{\infty}\frac{g(p^r)}{p^r}\Big) = \Big(\prod_{p|q}\frac{p^{\varrho_p(q)+a_p}}{f'(p^{\varrho_p(q)+a_p})}\Big)\Big(\prod_{p|q}\Big(\sum_{r=\varrho_p(q)}^{\infty}\frac{f'(p^r)}{p^r}\Big)\Big)$$
$$= \frac{\delta q}{f'(\delta q)}\prod_{p|q}\Big(\sum_{r=\varrho_p(q)}^{\infty}\frac{f'(p^r)}{p^r}\Big).$$

4. Remarks on Ramanujan series of multiplicative functions

4.1. Let us consider a multiplicative function f satisfying (5) and the coefficients a_q given by formula (7).

4.1.1. Looking at (7) we see that $a_1\neq 0$ *if and only if*

$$\sum_{r=0}^{\infty}\frac{f'(p^r)}{p^r}\neq 0 \quad \textit{for every } p.$$

Moreover, if this holds, then a_q/a_1 is a multiplicative function of q.

In the general case, denote by E the set of those primes p for which

$$\sum_{r=0}^{\infty}\frac{f'(p^r)}{p^r} = 0.$$

E must be finite for we have

$$\sum_{r=0}^{\infty}\frac{f'(p^r)}{p^r} = 1+u_p \quad \text{where} \quad u_p=\sum_{r=1}^{\infty}\frac{f'(p^r)}{p^r},$$

and (5) implies $\sum|u_p|<\infty$.

Let $\omega=\prod_{p\in E}p$.

We now see that $a_q=0$ *if q is not a multiple of ω*, for in that case $\varrho_p(q)=0$ for some $p\in E$. But $a_\omega\neq 0$ (for $\sum_{r=1}^{\infty}\frac{f'(p^r)}{p^r}=-1$ when $p\in E$) and *$a_{m\omega}/a_\omega$ is a multiplicative function of m.*

4.1.2. *If f is strongly multiplicative* (i.e. $f(p^r)=f(p)$ for every p and every $r>1$), *then $a_q=0$ whenever q is not squarefree* (for $f'(p^r)=0$ for $r>1$).

4.2. It is interesting to consider specially multiplicative functions satisfying

(11) $$|f(n)|\leqslant 1 \quad \text{for every } n.$$

4.2.1. For such a function (5) is obviously equivalent to

(12) $$\sum\frac{|f(p)-1|}{p}<\infty.$$

So, *if f is a multiplicative function satisfying* (11) *and* (12), *then f is limit periodic* (B) *and, for each n, the Ramanujan series* (2), *where the coefficients a_q are given by* (7), *is absolutely convergent and equal to $f(n)$.*

It is easy to see that $\sum_{r=0}^{\infty}\frac{f'(p^r)}{p^r}\neq 0$ for $p>2$ and that $\sum_{r=0}^{\infty}\frac{f'(2^r)}{2^r}$ cannot be zero unless we have

$$f(2^r)=-1 \quad \text{for every } r\geqslant 1.$$

In fact, we have for each p

$$\sum_{r=0}^{\infty}\frac{f'(p^r)}{p^r} = 1+\sum_{r=1}^{\infty}\frac{f(p^r)-f(p^{r-1})}{p^r} = \Big(1-\frac{1}{p}\Big)\Big(1+\sum_{r=1}^{\infty}\frac{f(p^r)}{p^r}\Big).$$

Since

$$\Big|\sum_{r=1}^{\infty}\frac{f(p^r)}{p^r}\Big|\leqslant\sum_{r=1}^{\infty}\frac{1}{p^r}=\frac{1}{p-1},$$

this cannot be zero if $p>2$.

Besides we have

$$\mathrm{Re}\Big(1+\sum_{r=1}^{\infty}\frac{f(2^r)}{2^r}\Big) = \sum_{r=1}^{\infty}\frac{1+\mathrm{Re}f(2^r)}{2^r}.$$

Since all terms of the last series are non-negative, this cannot be zero unless they are all zero. But, since $|f(2^r)|\leqslant 1$, $\mathrm{Re}f(2^r)=-1$ implies $f(2^r)=-1$.

In the case when $f(2^r) = -1$ for every $r > 1$, we actually have

$$\sum_{r=0}^{\infty} \frac{f'(2^r)}{2^r} = 0.$$

The results of § 4.1.1 yield the following conclusions:

If $f(2^r) \neq -1$ for some $r > 1$, then $a_1 \neq 0$ and a_q/a_1 is a multiplicative function of q.

If $f(2^r) = -1$ for every $r \geqslant 1$, then $a_q = 0$ for every odd q, but $a_2 \neq 0$ and a_{2m}/a_2 is a multiplicative function of m.

In the latter case we can indeed say a little more, namely that $a_q = 0$ *if* $q \not\equiv 2 \pmod 4$, (so that a_{2m}/a_2 is zero when m is even).

This follows at once from the fact that we have

$$\sum_{r=a}^{\infty} \frac{f'(2^r)}{2^r} = 0 \quad \text{if} \quad a = 0 \text{ or } a > 1,$$

for $f'(2) = -2$ and $f'(2^r) = 0$ when $r > 1$.

We may add that, since f is bounded, it is actually limit-periodic (B^λ) for every $\lambda \geqslant 1$. Therefore we have *Parseval's equality*, which gives

$$(13) \qquad \sum_{q=1}^{\infty} \varphi(q)|a_q|^2 = \prod \left(1 - \frac{1}{p}\right)\left(1 + \sum_{r=1}^{\infty} \frac{|f(p^r)|^2}{p^r}\right).$$

In fact the series $\sum \frac{1 - |f(p)|^2}{p}$ is convergent, for

$$1 - |f(p)|^2 = \big(1 + |f(p)|\big)\big(1 - |f(p)|\big) \leqslant 2\,|f(p) - 1|,$$

and by a known result [6] $|f(n)|^2$ has a mean value equal to the right-hand side of (13).

The results of this paragraph contain as particular cases some results proved by W. Schwarz in a recent paper [7].

4.2.2. [8] Now consider again a multiplicative function f satisfying (11), but replace the hypothesis that we have (12) by the weaker assumption

(6) H. Delange, *Sur les fonctions arithmétiques multiplicatives*, Ann. Sci. Ecole Norm. Sup. (3), 78 (1961), pp. 273–304, th. 2, p. 275.

(7) *Ramanujan-Entwicklungen stark multiplikativer zahlentheoretischer Funktionen*, Acta Arith. 22 (1973), pp. 329–338.

(8) The results given in this paragraph are contained in results stated without proof in H. Daboussi et H. Delange, *Quelques propriétés des fonctions multiplicatives de module au plus égal à 1*, C. R. Acad. Sci. Paris, 278 (1974), série A, pp. 657–660.

that *the series* $\sum \frac{1 - f(p)}{p}$ *is convergent* (which holds in particular if f possesses a non zero mean-value [9]).

Then we cannot apply the above general theory.

However it can be proved that in this case too f *is limit-periodic* (B).

The proof runs as follows. Let y be any real number $\geqslant 2$ and let f_y be the multiplicative function determined by

$$f_y(p^r) = \begin{cases} f(p^r) & \text{if} \quad p \leqslant y \\ 1 & \text{if} \quad p > y \end{cases} \quad \text{(for every prime } p \text{ and every } r \geqslant 1).$$

We have

$$f'_y(p^r) = \begin{cases} f'(p^r) & \text{if} \quad p \leqslant y, \\ 0 & \text{if} \quad p > y, \end{cases}$$

and by the above theory f_y is limit-periodic (B), and even limit-periodic (B^2) since it is bounded.

Using the equality

$$|f(n) - f_y(n)|^2 = |f(n)|^2 - \overline{f(n)} f_y(n) - f(n)\overline{f_y(n)} + |f_y(n)|^2$$

and applying the known result quoted in note (6) to each of the functions $|f|^2$, $\bar{f} f_y$, $f \bar{f}_y$ and $|f_y|^2$ we see that as x tends to infinity

$$\frac{1}{x} \sum_{n \leqslant x} |f(n) - f_y(n)|^2$$

tends to a limit which can be expressed by means of three infinite products and a finite one. This limit is seen to tend to zero as y tends to infinity.

This proves not only that f is limit-periodic (B^2) but also that its Fourier-coefficients are the limits as y tends to infinity of the Fourier-coefficients of f_y.

It follows that *the Fourier-series of f is still, after the usual grouping of its terms, the Ramanujan series* (2) *where the coefficients a_q are given by* (7). But now the infinite product in (7) is convergent but not necessarily absolutely convergent [10]. The properties of a_q given in § 4.2.1 are still valid.

It is easy to see that *the series* (2) *is absolutely convergent for no n if we have not* (12). It suffices to consider $\sum |a_p c_p(n)|$ if $a_1 \neq 0$, $\sum_{p>2} |a_{2p} c_{2p}(n)|$ if $a_1 = 0$.

(9) H. Delange, loc. cit., Th. 1, p. 274.

(10) The convergence of that product follows at once from the fact that $\sum_0^{\infty} f'(p^r)/p^r = 1 + u_p$ where $\sum |u_p|^2 < \infty$ and $\sum u_p$ converges. The convergence of the series $\sum (1 - f(p))/p$ is indeed a necessary and sufficient condition for the convergence of the infinite product.

On the other hand one can prove that the series

$$\sum_{1}^{\infty} \frac{a_q}{q^s} c_q(n)$$

converges for $s > 0$ and that its sum tends to $f(n)$ as s tends to zero through positive values. So, if the series (2) converges for some n, its sum must be $f(n)$.

Added in proof. It has been proved by W. Schwarz that the series (2) actually converges for every n (Acta Arith. 27 (1975), pp. 269–279).

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A counterexample to a conjecture on multinomial degree

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Let K be a field. A polynomial $p(x)$ with coefficients in K of the form $a_0 + a_1 x^{m_1} + \ldots + a_d x^{m_d}$ with all $a_i \neq 0$ is called a *multinomial of length d*. The d-tuple $(m_1, \ldots, m_d)$ is the exponent vector of $p(x)$. An element θ in a field extension of K is of multinomial degree d over K if θ satisfies a multinomial of length d and no multinomial of smaller length. Clearly, θ has multinomial degree 1 over K if and only if some positive power of θ lies in K.

The following conjecture is posed in [2]: If K is a field of characteristic 0 and θ is an element of multinomial degree d over K so that there exist $d+1$ multinomials of length d satisfied by θ, $p_i(x)$, $i = 0, 1, \ldots, d$, where the corresponding exponent vectors are not proportional, then $[K(\theta^m): K] = d$ for some positive power m of θ.

Let θ be a root of the irreducible polynomial $x^3 - x + 1$ over the field of rational numbers Q. We show that θ provides a counterexample to the above conjecture. We observe that an element of odd degree m over Q has multinomial degree 1 if and only if its minimal polynomial over Q has the form $x^m - a$. For a proof see [1]. Hence θ has degree 3 and multinomial degree 2 over Q. Moreover, every positive power of θ has degree $3 = [Q(\theta^m): Q]$.

Multiplying $x^3 - x + 1$ by appropriately chosen polynomials of degree 2 and 4 we obtain the following additional multinomials of length 2 satisfied by θ:

$$x^5 + x^4 + 1 = (x^3 - x + 1)(x^2 + x + 1),$$
$$x^7 - 2x^5 - 1 = (x^3 - x + 1)(x^4 - x^2 - x - 1),$$
$$x^7 + 2x^4 + 1 = (x^3 - x + 1)(x^4 + x^2 + x + 1).$$

Thus θ satisfies four multinomials of length 2 with exponent vectors $(1, 3)$, $(4, 5)$, $(5, 7)$, and $(4, 7)$, respectively. Hence θ does provide the desired counterexample.