tive Obengrenze für

$$\max\left(|Nm(x)|, |Nm(y)|, p_1^{h_1 f_1 w_1}, \ldots, p_s^{h_s f_s w_s}\right).$$

Schließlich merken wir an, daß der Satz aus § 2 in unseren Untersuchungen als eine Hilfsbetrachtung benutzt wurde. Aber sie ist als solches interessant weil die Analyse der bestimmten Klasse der Diophantschen Gleichungen auf die Analyse der ganzwertigen Lösungen der Gleichung der Form (8) zurückgeführt wird.

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Eingegangen 17. 2. 1974 (649)

ACTA ARITHMETICA
XXXI (1976)

On the frequency of small fractional parts in certain real sequences, IV

by

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1. Introduction. For real t, let $\|t\|$ denote the distance between t and the nearest integer, so that $0 \leqslant \|t\| \leqslant 1/2$, and let $x, \alpha_1, \alpha_2, \ldots$ be real numbers which may without loss of generality be taken in the interval $[0, 1)$. Let $a_1, a_2, \ldots$ be an increasing sequence of positive integers, let $f(1), f(2), \ldots$ be numbers in $[0, 1/2]$ for which $\sum f(k) = \infty$, and put $F(n) = 2\sum_1^n f(k)$. Finally, define $N_n(x)$ to be the number of positive integers $k \leqslant n$ for which

$$\|a_k x - \alpha_k\| < f(k). \tag{1}$$

The problem addressed in this sequence of papers is that of finding conditions under which

$$N_n(x) \sim F(n) \quad \text{as} \quad n \to \infty, \tag{2}$$

this being the rate of growth of $N_n(x)$ one would expect on probabilistic grounds for "most" x [6]. Relation (2) does not hold for almost all (a.a.) x, for some pairs of sequences $\{a_k\}$, $\{f(k)\}$ as described above [2], but a number of conditions on $\{a_k\}$ and $\{f(k)\}$ together are known which guarantee (2) for a.a. x. (See Part III [6] and its bibliography, and also see [7], [8], for example. For a complete list of the literature in this area, see my *Reviews in Number Theory*, vol. 3, Section J24, American Mathematical Society, 1974.)

In the first part of the present paper it is shown that if $F(n) \gg n^a$ $(1/2 < a < 1)$ then (2) holds for a.a. x no matter what $\{a_k\}$ may be.

In the second part the case $a_k = k$, $\alpha_k = 0$, $f(k) = c/k$ (c a positive constant) is studied, so that $N_n(x)$ now counts the number of solutions p, q of the inequality

$$|qx - p| < c/q, \quad q \leqslant n \tag{3}$$

and one would expect $N_n(x) \sim 2c\log n$. This is true for almost all but not for all x, and either of the two quantities $N_n(x)$ and $\log n$ may be of smaller order of magnitude than the other, for suitable x. (E.g., x and c may be chosen so that $N_n(x) = 0$ for all n, and again so that $\log n = o(N_n)$.) It is shown, however, that if $N_n^0(x)$ counts the *primitive* solutions of (3) — those with $q \leqslant n$ and $(p, q) = 1$ — then $N_n^0(x) \ll \log n$ for all x without exception. It would be interesting to obtain more general theorems of this sort; the present proof, using continued fraction theory, is clearly restricted in scope.

2. A metric theorem

THEOREM 1. *Let $\{f(k)\}$ be an arbitrary positive sequence such that*

$$F(n) = 2\sum_{1}^{n} f(k) \gg n^{a} \quad \text{for some fixed } a \in (1/2, 1].$$

Then for each $\varepsilon > 0$, and for arbitrary $\{a_k\}$ and $\{\alpha_k\}$,

$$N_n(x) = F(n) + O(n^{1/2}\log^{3/2+\varepsilon} n) \quad \text{for a.a. } x. \tag{4}$$

In particular, if $\{f(k)\}$ is non-increasing and $f(k) \gg k^{a-1}$, then (4) holds.

Remark. In the special cases $a_k = p(k)$ (p a polynomial) and $a_k = a^k$ ($a > 1$ an integer), Schmidt [8] and Philipp [7] have proved strengthened versions of this theorem in which all restrictions on $f(k)$ are dropped and the error term in (4) is replaced by $O\big(F^{1/2}(n)\log^{3/2+\varepsilon}F(n)\big)$, which is generally smaller than that in (4). But it seems unlikely that much more than (4) is true for arbitrary $\{a_k\}$. Specifically, I conjecture that even the relation $N_n(x) \sim F(n)$ does not hold for a.a. x, for $\alpha_k = 0$, $f(k) = k^{a-1}$ ($a \in [0, 1/2)$) and suitable $\{a_k\}$.

Proof. Let $\psi(t, \gamma) = 1$ for $\|t\| < \gamma$ and $= 0$ for other t, and put

$$I_{kl} = \int_0^1 \big\{\psi\big(a_k x - \alpha_k, f(k)\big) - 2f(k)\big\}\big\{\psi\big(a_l x - \alpha_l, f(l)\big) - 2f(l)\big\}\,dx,$$

so that

$$\sum_{k,l=m+1}^{n} I_{kl} = \int_0^1 \big\{\big(N_n(x) - F(n)\big) - \big(N_m(x) - F(m)\big)\big\}^2\,dx.$$

It was shown in [6] that

$$I_{kl} = \frac{2}{\pi^2}\frac{d_{kl}^2}{a_k a_l}\sum_{j=1}^{\infty} j^{-2}\sin 2\pi j\frac{a_l f(k)}{d_{kl}}\sin 2\pi j\frac{a_k f(l)}{d_{kl}}\cos\frac{a_l\alpha_k - a_k\alpha_l}{d_{kl}}$$

with $d_{kl} = (a_k, a_l)$. Thus if $[\cdot,\cdot]$ is the LCM function,

$$I_{kl} \leqslant \frac{2}{\pi^2}\frac{d_{kl}^2}{a_k a_l}\sum_{j=1}^{\infty} j^{-2} = \frac{1}{3}\frac{(a_k, a_l)}{[a_k, a_l]},$$

a relation first proved (in a special case) by Koksma [5]. Hence

$$\sum_{k,l=m+1}^{n} I_{kl} < \sum_{k,l=m+1}^{n}\frac{(a_k, a_l)}{[a_k, a_l]} < cn(\log\log n)^2, \tag{5}$$

the second inequality being a theorem of Gál [3] which is valid for all increasing sequences $\{a_k\}$ of integers, for some absolute constant c. It has been shown by Gál and Koksma ([4], Theorem 3) that if $u_1(x), u_2(x), \ldots$ are in $L^2(0, 1)$ and

$$\int_0^1 \{u_{m+1}(x) + \ldots + u_n(x)\}^2\,dx = O\big(\Psi(n)\big)$$

uniformly in m, where $\Psi(n)/n$ is a non-decreasing function, then for every $\varepsilon > 0$ one has

$$|u_1(x) + \ldots + u_n(x)| = o\big(\Psi^{1/2}(n)\log^{3/2+\varepsilon} n\big)$$

for almost all $x \in [0, 1]$. This result, in combination with (5), gives the theorem.

3. The inequality $|sx - r| < c/s$. It was asserted in the introduction that the total number of solutions with $s \leqslant n$ may be of either smaller or larger order of magnitude than $\log n$. In fact, if x has bounded partial quotients, the inequality will have no solutions at all for c sufficiently small. On the other hand by continued fraction theory, using the notation (6) below,

$$\left|x - \frac{p_k}{q_k}\right| = \frac{1}{q_k^2\{x_{k+1} + q_{k-1}/q_k\}} < \frac{1}{a_{k+1}q_k^2},$$

so

$$\left|x - \frac{mp_k}{mq_k}\right| < \frac{m^2}{a_{k+1}(mq_k)^2},$$

so we have imprimitive solutions of the inequality $|sx - r| < c/s$ for $s = mq_k$, $m = 2, \ldots, [\sqrt{ca_{k+1}}]$. Indeed, these values of s can form an arbitrarily long arithmetic progression with difference q_k, and each new a_{k+1} can be chosen so large in comparison with q_k that the total number of solutions with $s \leqslant [\sqrt{ca_{k+1}}]q_k = n_{k+1}$ is substantially $\sqrt{ca_{k+1}}$. Thus, the ratio $N_{n_k}(x)/n_k$ can be made to go to zero arbitrarily slowly as a function of n_k, for suitable x and suitable values of n_k.

In fact, the partial quotients do not have to increase at all rapidly to produce an overabundance of solutions. Adams [1] has shown that if $x = e = \{2; \overline{1, 2n, 1}\}_{n=1}^{\infty}$, then $N_n(e)$ grows like $c_1(\log n/\log\log n)^{3/2}$.

THEOREM 2. *For real x and $c > 0$, let $N_n^0(x, c)$ be the number of pairs of integers p, q with $(p, q) = 1$ and $0 < q \leqslant n$ for which $|qx - p| < c/q$. Let $Z_n(x)$ be the number of denominators q_k of convergents to x for which $q_k \leqslant n$. Then for each real x,*

$$N_n^0(x, c) \leqslant CZ_n(x) + C',$$

where $C = 1 + 4c(1 + c\log(c+1))$ *and* $C' = c\log(c+1)$.

COROLLARY. *For all real x, $N_n^0(x, c) \ll \log n$.*

Remarks. For $c \leqslant 1/2$ the theorem is a weak version of the well-known fact that all rational numbers p/q counted by $N^0(x, 1/2)$ are convergents. The corollary follows from the fact that for all x, $q_k > a^k$ for some absolute constant $a > 1$. In the paper mentioned above, Adams showed that $N^0(e, 1) \sim c_2 Z_n(e) \sim c_3(\log n/\log\log n)$, and used this to obtain the above estimate for $N_n(e)$.

Proof. The theorem is trivial for x rational, so suppose x is irrational. We will make use of the following standard relations concerning the continued fraction expansion $x = \{a_0; a_1, a_2, \ldots\}$ and its convergents p_k/q_k and complete quotients x_k:

$$(6)\qquad \frac{p_{k+2}}{q_{k+2}} = \frac{p_{k+1}a_{k+2} + p_k}{q_{k+1}a_{k+2} + q_k}, \quad x = \frac{p_{k+1}x_{k+2} + p_k}{q_{k+1}x_{k+2} + q_k}, \quad a_{k+1} = [x_{k+1}],$$

$$q_k p_{k-1} - q_{k-1}p_k = (-1)^k, \quad \frac{p_0}{q_0} < \frac{p_2}{q_2} < \ldots < x < \ldots < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

To be definite, let us consider the approximations on one side of x, say the p/q for which $0 < x - p/q < c/q^2$; the other case is treated similarly.

As usual, define the quasi-convergents between p_k/q_k and p_{k+2}/q_{k+2} by

$$\frac{p_{ka}}{q_{ka}} = \frac{p_k + ap_{k+1}}{q_k + aq_{k+1}}, \quad 0 \leqslant a < a_{k+2},$$

so that for k even we have the following ordering for the "left" quasi-convergents:

$$\frac{p_k}{q_k} = \frac{p_{k0}}{q_{k0}} < \frac{p_{k1}}{q_{k1}} < \ldots < \frac{p_{k,a_{k+2}-1}}{q_{k,a_{k+2}-1}} < \frac{p_{k+2,0}}{q_{k+2,0}} = \frac{p_{k+2}}{q_{k+2}} < \ldots$$

A simple computation gives

$$(7)\qquad q_{ka}p_{k,a+1} - q_{k,a+1}p_{ka} = 1 \quad (k \text{ even}, 0 \leqslant a < a_{k+2}).$$

Among the pairs of integers $r = p$ and $s = q > 0$ for which

$$(8)\qquad 0 < x - \frac{r}{s} < \frac{c}{s^2},$$

only finitely many have ratio p/q less than $[x] = p_0/q_0$; for then $0 < 1/q \leqslant p_0 - p/q \leqslant x - p/q < c/q^2$, which implies $q < c$ and $p_0 q - c/q < p < p_0 q$, so the number of such pairs is $< \sum_{q<c} c/q < c\log(c+1)$. Each of the remaining p/q is either equal to a left quasi-convergent or lies strictly between two successive quasi-convergents. By a standard property of Farey sequences, (7) implies that a fraction between two successive quasi-convergents has larger denominator than either. Hence if

$$\frac{p_{ka}}{q_{ka}} < \frac{p}{q} \leqslant \frac{p_{k,a+1}}{q_{k,a+1}} < x$$

and p/q satisfies (8), so also does $p_{k,a+1}/q_{k,a+1}$. The proof will therefore be complete when the following assertions have been proved:

I. For each k, at most $4c$ quasi-convergents p_{ka}/q_{ka} with $0 < a < a_{k+2}-1$ satisfy (8).

II. At most $c\log(c+1)$ rational numbers strictly between two successive quasi-convergents satisfy (8).

Proof of I. From the expression for x occurring in (6), form

$$(q_k x - p_k) + a(q_{k+1}x - p_{k+1}),$$

obtaining

$$q_{ka}x - p_{ka} = \frac{x_{k+2} - a}{q_k + q_{k+1}x_{k+2}} \quad \text{for} \quad 0 < a < x_{k+2}.$$

Thus if $q_{ka}x - p_{ka} = \beta/q_{ka}$, then

$$\beta = \frac{(x_{k+2} - a)(q_k + aq_{k+1})}{q_k + x_{k+2}q_{k+1}} = \frac{(x_{k+2} - a)a}{x_{k+2}} \cdot \frac{q_{k+1} + q_k/a}{q_{k+1} + q_k/x_{k+2}} > \frac{(x_{k+2} - a)a}{x_{k+2}}$$

$$\geqslant \begin{cases} \frac{1}{2}a & \text{if} \quad 0 < a < \frac{1}{2}x_{k+2} \\ \frac{1}{2}(x_{k+2} - a) & \text{if} \quad \frac{1}{2}x_{k+2} \leqslant a < x_{k+2} \end{cases}$$

$$\geqslant c \quad \text{if} \quad \begin{cases} a \geqslant 2c \text{ and } 0 < a < \frac{1}{2}x_{k+2} \text{ or} \\ a \leqslant x_{k+2} - 2c \text{ and } \frac{1}{2}x_{k+2} \leqslant a < x_{k+2}. \end{cases}$$

Hence $\beta < c$ at most for $a = 1, \ldots, [2c]$ and for $a = [x_{k+2} - 2c] + 1, \ldots \ldots, [x_{k+2}] - 1$, and these comprise at most $4c$ values of a.

Proof of II. For simplicity, rename the two successive left quasi-convergents as P_1/Q_1 and P_2/Q_2, so that $P_2Q_1 - P_1Q_2 = 1$, and suppose that

$$\frac{P_1}{Q_1} < \frac{p}{q} < \frac{P_2}{Q_2} < x, \quad x - \frac{p}{q} < \frac{c}{q^2}.$$

Then

$$\frac{P_2}{Q_2} - \frac{p}{q} < x - \frac{p}{q} < \frac{c}{q^2},$$

so

$$q < \frac{cQ_2}{\begin{vmatrix} P_2 & p \\ Q_2 & q \end{vmatrix}},$$

while

$$\frac{p}{q} - \frac{P_1}{Q_1} < \frac{P_2}{Q_2} - \frac{P_1}{Q_1}$$

so

$$q > \frac{Q_2 \begin{vmatrix} p & P_1 \\ q & Q_1 \end{vmatrix}}{\begin{vmatrix} P_2 & P_1 \\ Q_2 & Q_1 \end{vmatrix}}.$$

Hence

$$(9) \qquad \begin{vmatrix} p & P_1 \\ q & Q_1 \end{vmatrix} \cdot \begin{vmatrix} P_2 & p \\ Q_2 & q \end{vmatrix} < c \begin{vmatrix} P_2 & P_1 \\ Q_2 & Q_1 \end{vmatrix} = c.$$

By another standard property of Farey sequences, the reduced fractions p/q between neighboring Farey points P_1/Q_1 and P_2/Q_2 are given by equations

$$p = uP_1 + vP_2,$$
$$q = uQ_1 + vQ_2,$$

where u and v are coprime positive integers. Introducing these expressions into (9) gives

$$v \cdot u < c,$$

and there are fewer than

$$c \int_1^{c+1} x^{-1} dx = c \log(c+1)$$

such pairs u, v.

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Received on 17. 1. 1975
and in revised form on 2. 7. 1975 (660)