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The sequences of prime divisors of integers

by

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1. Introduction. Let $2 = p_1 < p_2 < \ldots$ denote the sequence of prime numbers and let $q_1(n) < q_2(n) < \ldots < q_\omega(n)$ be those primes which divide n; that is, with some positive integers a_j,

$$n = q_1^{a_1}(n) q_2^{a_2}(n) \ldots q_\omega^{a_\omega}(n). \tag{1}$$

The present investigation concerns the order of magnitude of $q_j(n)$ where j may also depend on n. The results will be average type of statements, that is, we consider $q_j(n)$ for $n \leqslant x$ and we let $x \to +\infty$. We then decide whether a set $n \leqslant x$, specified by a condition on $q_j(n)$, is "large" or not and, in a particular, if its density exists. Here we apply the natural density.

We say that a set A of positive integers has density $d(A)$ if, as $x \to +\infty$

$$\lim \nu_x(n\colon\ n \in A) = d(A)$$

exists, where $\nu_x(n\colon \ldots)$ denotes the number of integers $0 < n \leqslant x$ which satisfy the condition stated in the dotted space.

In the representation (1), $\omega = \omega(n)$ obviously depends on n and thus, if $j = j(n)$ is chosen in advance, $q_j(n)$ may have no meaning. In order to avoid the need for distinguishing several cases in our statements, we extend the definition of $q_j(n)$ as follows.

DEFINITION. $q_j(n)$ is the *j-th term* in (1) if $1 \leqslant j \leqslant \omega(n)$. If $j \leqslant 0$, $q_j(n) = 1$ and $q_j(n) = +\infty$ for $j > \omega(n)$.

This extended meaning of $q_j(n)$ does not affect the fact that our statements concern actual prime divisors, that is, elements of (1). This is made possible by $\omega(n)$ being close to the monotonic function $\log\log n$ for "almost all n". More precisely, it is well known ([3], p. 41) that, if

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$A(y)$ denotes the set of integers n for which $\omega(n) \geqslant y \log\log n$, then

$$d(A(1-\varepsilon)) = 1 \quad \text{and} \quad d(A(1+\varepsilon)) = 0. \tag{2}$$

(2) will always be one of the guides in our choice of $j = j(n)$.

The present investigation was induced by a letter from P. Erdös to the author, for which I would like to express my appreciation.

2. The results and discussion. Let us, first of all, quote results on $q_{\omega-k}(n)$, $k = 0, 1, 2, \ldots$ The case $k = 0$ is due to de Bruijn [1] and arbitrary k to Levin and Fainleib [4].

THEOREM A. *For any fixed integer $k \geqslant 0$, there is a differentiable distribution function $F_k(z)$ such that, as $x \to +\infty$,*

$$\lim \nu_x\big(n\colon \log q_{\omega-k}(n) < z \log x\big) = F_k(z). \tag{3}$$

It is evident that the above statement cannot hold for $q_{\omega-k}(n)$ with $k = k(n) \to +\infty$ with n. As a matter of fact, reversing the inequality in (3), the limit will be $1 - F_k(z) > 0$ for $0 < z < 1$. But for any $0 < z < 1$, there are only a finite number of prime divisors of $n \leqslant x$ which are larger than x^z. For this case the following result holds.

THEOREM 1. *Let $\varepsilon > 0$. Then, for $k = k(x) \to +\infty$ with x and for which $k(x) = o(\log\log x)$, as $x \to +\infty$,*

$$\lim \nu_x\big(n\colon e^{-(1+\varepsilon)k} \log x < \log q_{\omega-k}(n) < e^{-(1-\varepsilon)k} \log x\big) = 1.$$

In addition to Theorem 1, we shall prove the following results.

THEOREM 2. *Let $j = j(x) \to +\infty$ with x in such a way that $j(x) < \log\log x$, and*

$$|j(x) - \log\log x| \geqslant (\log\log x)^h \quad \textit{with some } h > \tfrac{1}{2}.$$

Then, as $x \to +\infty$,

$$\lim \nu_x\big(n\colon \log\log q_j(n) - j < z j^{1/2}\big) = (2\pi)^{-1/2} \int_{-\infty}^{z} e^{-t^2/2}\, dt. \tag{4}$$

THEOREM 3. *For $j(x) \leqslant (1-\varepsilon)\log\log x$, where $\varepsilon > 0$ is an arbitrary number, as $x \to +\infty$ and as $j(x) = j \to +\infty$,*

$$\lim \nu_x\big(n\colon \log q_{j+1}(n)/\log q_j(n) < z\big) = 1 - 1/z, \quad z > 1.$$

Some comments are in order. First of all, we wish to emphasize that the assumptions on $k(x)$ and $j(x)$ are to serve only to guarantee that the results equally apply to elements of (1) as well as to the extended values of $q_j(n)$. As a guide, (2) can be consulted and, in the course of the proof, further references will be provided.

It is interesting to look at the statements with some detail. If we take $\log\log q_{\omega-k}(n)$ in Theorem 1 and apply (2) we get, roughly speaking,

that $\log\log q_{\omega-k}(n)$ is "about $\omega(n) - k(n)$". Notice that a similar asymptotic property of $\log\log q_j(x)$ is implied by (4) and therefore Theorems 1 and 2 show a common property of "large" and "small" prime divisors. Besides Theorem A, this weaker form of Theorem 2 seems to be the only explicit statement in the literature on $q_j(n)$, which appears in Erdös [2]. Similarly, a common property of small and large prime divisors is expressed in Theorem A and Theorem 3. If for bounded k we write

$$\frac{\log q_{\omega-k}(n)}{\log q_{\omega-k-1}(n)} = \frac{\log q_{\omega-k}(n)}{\log x} : \frac{\log q_{\omega-k-1}(n)}{\log x},$$

then (3) suggests, and indeed, from the arguments of Levin and Fainleib [4] it follows, that the left-hand side has a limiting distribution. This is the exact statement of Theorem 3 for $k \to +\infty$ with x. Of course, the forms of the limits differ.

We conclude this section with a simple remark. It can easily be seen that, for fixed j and t,

$$d(\{n\colon q_j(n) = p_t\}) = a_{jt}$$

exists and is positive. This remark is added only to cover the whole range of possible values of j in the statements of this section.

3. Proofs. The proof of Theorems 1 and 2 is based on the following relation. For $t = 1, 2, \ldots,$ let

$$\varepsilon_t(n) = \begin{cases} 1 & \text{if} \quad p_t | n, \\ 0 & \text{otherwise}. \end{cases}$$

Then, putting

$$\eta_T(n) = \varepsilon_1(n) + \varepsilon_2(n) + \ldots + \varepsilon_T(n), \tag{5}$$

we evidently have

$$\{n\colon \eta_T(n) < j\} = \{n\colon q_j(n) > p_T\}. \tag{6}$$

We can now turn to the details of proof.

Proof of Theorem 1. We apply the Turán–Kubilius inequality ([3], p. 35) in the following form. Let $T = T(x) = x^{\alpha(x)}$, where $0 < \alpha(x) \to 0$ as $x \to +\infty$. Then, as $x \to +\infty$,

$$\lim \nu_x\Big(n\colon \Big|\sum_{t=T+1}^{x} \varepsilon_t(n) + \log \alpha(x)\Big| < \log^{3/4} 1/\alpha(x)\Big) = 1. \tag{7}$$

Now, since for $n \leqslant x$, $\omega(n) = \eta_x(n)$, we get by (6),

$$\Big\{n\colon \sum_{t=T+1}^{x} \varepsilon_t(n) > k\Big\} = \{n\colon \eta_T < \omega(n) - k\} = \{n\colon q_{\omega-k}(n) > p_T\}. \tag{8}$$

Guided by (2), we take $k = k(x) \to +\infty$ with x in such a way that $k(x) = o(\log\log x)$. Then, since for any $\delta > 0$ (see [5], p. 106)

(9) $$\log T < \log p_T < (1+\delta)\log T,$$

(8) implies

(10) $$\nu_x\Big(n\colon \sum_{t=T+1}^{x} \varepsilon_t(n) > k\Big) = \nu_x\big(n\colon \log q_{\omega-k}(n) > \log p_T\big) \leqslant \nu_x\big(n\colon \log q_{\omega-k}(n) > \log T\big).$$

Let us now set $a(x) = \exp\big(-(1+\varepsilon)k(x)\big)$ and $T(x) = x^{a(x)}$. Then $a(x) \to 0$ and $T(x) \to +\infty$ as $x \to +\infty$ and thus (7) and (10) imply that, as $x \to +\infty$,

(11) $$\lim \nu_x\{n\colon \log q_{\omega-k}(n) > (\log x)\exp\big(-(1+\varepsilon)k\big)\} = 1.$$

On the other hand, using the upper inequality in (9), we get from (8)

(12) $$\nu_x\Big(n\colon \sum_{t=T+1}^{x} \varepsilon_t(n) > k\Big) \geqslant \nu_x\big(n\colon \log q_{\omega-k}(n) > (1+\delta)\log T\big).$$

Since with the choice of $a(x) = \exp\big(-(1-\varepsilon)k(x)\big)$ and $T(x) = x^{a(x)}$, (7) is again applicable, we get that, as $x \to +\infty$,

(13) $$\lim \nu_x\{n\colon \log q_{\omega-k}(n) > (1+\delta)(\log x)\exp\big(-(1-\varepsilon)k\big)\} = 0.$$

Because both $\delta > 0$ and $\varepsilon > 0$ are arbitrary, δ plays no role in (13) and thus Theorem 1 is established.

Proof of Theorem 2. Here we need the asymptotic normality of $\eta_T(n)$ as $T = T(x) \to +\infty$ with x. More precisely, by a theorem of probabilistic number theory (see [3], p. 61), if $T = T(x) \to +\infty$ with x, and $T(x) \leqslant x$, then for each fixed real z,

(14) $$\lim \nu_x\big(n\colon \eta_T(n) - \log\log T < z(\log\log T)^{1/2}\big) = \Phi(z),$$

where $\Phi(z)$ signifies the standard normal distribution function occurring on the right-hand side of (4). (6) and (14) yield that if, for a given function $j = j(x)$, we determine $T = T(x)$ from the relation

(15) $$j(x) = \log\log T(x) + z\big(\log\log T(x)\big)^{1/2},$$

then, as $x \to +\infty$,

(16) $$\lim \nu_x\big(n\colon \log\log q_j(n) > \log\log p_T\big) = \Phi(z).$$

In view of (9), $\log\log p_T = \log\log T + O(1)$ and thus by (15)

(17) $$\log\log p_T - j(x) = -zj^{1/2} + O(1).$$

(16) and (17) imply that, as $x \to +\infty$,

$$\lim \nu_x\big(n\colon \log\log q_j(x) - j > -zj^{1/2}\big) = \Phi(z),$$

what is equivalent to (4) by $\Phi(z) = 1 - \Phi(-z)$. This completes the proof of Theorem 2.

Let us remark that we did not make use of our assumptions on $j(x)$ in the course of the proof, except that $T(x) \leqslant x$ and thus by (15), $j(x) \leqslant \log\log x$. The additional assumptions were made only to guarantee that the conclusion remain the same if $q_j(n)$ is restricted to its values occurring in (1).

Proof of Theorem 3. First of all note that

$$\{n\colon q_{j+1}(n) > s;\ q_j(n) = p_k\} = \{n\colon \varepsilon_t(n) = 0,\ p_k < p_t \leqslant s;\ q_j(n) = p_k\}.$$

Since $\{n\colon q_j(n) = p_k\}$ can be expressed in terms of $\varepsilon_t(n)$, $1 \leqslant t \leqslant k$, by Lemma 1.4 of [3], p. 5, for $s \leqslant x^{a(x)}$, where $a(x) > 0$ and $a(x) \to 0$ as $x \to +\infty$,

(18) $$\nu_x\big(n\colon q_{j+1}(n) > s,\ q_j(n) = p_k\big) = \big(1+o(1)\big)\Big\{\prod_{p_k < p \leqslant s}(1 - 1/p)\Big\}\,\nu_x\big(n\colon q_j(n) = p_k\big),$$

where $o(1)$ is uniform in s and k for $p_k < s \leqslant x^{a(x)}$. For a fixed real number $z > 1$, let us choose s so that $\log s \sim z\log p_k$. We then have, as a consequence of the prime number theorem (see [5], p. 106)

(19) $$\prod_{p_k < p \leqslant s}\Big(1 - \frac{1}{p}\Big) = \frac{\log s}{\log p_k} + O\big(\exp\big(-\beta(\log s)^{1/10}\big)\big) = \big(1+o(1)\big)z^{-1},$$

where $\beta > 0$ is a uniform constant and thus $o(1)$ can be made tending to 0 uniformly for $p_k \geqslant U(x) \to +\infty$ with x.

Writing now

(20) $$\nu_x\Big(n\colon \frac{\log q_{j+1}(n)}{\log q_j(n)} > z\Big) = \sum_{k=1}^{+\infty} \nu_x\Big(n\colon \frac{\log q_{j+1}(n)}{\log q_j(n)} > z,\ q_j(n) = p_k\Big)$$

we apply to the tails of the above sum that

(21) $$\sum_{p_k \leqslant U} \nu_x\Big(n\colon \frac{\log q_{j+1}(n)}{\log q_j(n)} > z,\ q_j(n) = p_k\Big) \leqslant \nu_x\big(n\colon q_j(n) \leqslant U\big)$$

and

(22) $$\sum_{p_k \geqslant W} \nu_x\Big(n\colon \frac{\log q_{j+1}(n)}{\log q_j(n)} > z,\ q_j(n) = p_k\Big) \leqslant \nu_x\big(n\colon q_j(n) \geqslant W\big).$$

For a given $\varepsilon > 0$, we choose

$$U = \exp\{\exp[(1-\varepsilon)j]\} \quad \text{and} \quad W = \exp\{\exp[(1+\varepsilon)j]\}.$$

By Theorem 2 we get that the left-hand sides of (21) and (22) tend to 0 as $x \to +\infty$. Since the above $U = U(x) \to +\infty$ with x, for $U < p_k < W$, we can apply (18) and (19) and we get

$$\sum_{U<p_k<W}{}' \nu_x\left(n: \frac{\log q_{j+1}(n)}{\log q_j(n)} > z,\ q_j(n) = p_k\right) = \big(1+o(1)\big)z^{-1}\nu_x\big(n:\ U < q_j(n) < W\big). \tag{23}$$

Another appeal to Theorem 2 yields that, for our j, as $x \to +\infty$,

$$\nu_x\big(n:\ U < q_j(n) < W\big) \to 1,$$

and thus (20)–(23) lead to the conclusion of Theorem 3. This completes the proof.

References

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Über die maximale Norm der Idealteiler des Polynoms $\alpha x^m + \beta y^n$ mit den algebraischen Koeffizienten

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1. Einleitung. K. Mahler bewies [16], daß der allergrößte Primteiler P des Polynoms

$$G(x, y) = ax^m + by^n, \tag{1}$$

wo $m \geqslant 2$, $n \geqslant 3$, $a \neq 0$, $b \neq 0$ ganz und rational sind, unbegrenzt bei $X = \max(|x|, |y|) \to \infty$, $(x, y) = 1$, wächst. Um das Verhalten von P zu erforschen, betrachtete K. Mahler Ergebnisse von C. Parry [17] über die p-adische Verallgemeinerung vom Thue–Siegelschen Satz. Da sie nichteffektiv sind, war es unmöglich auch im Prinzip die Geschwindigkeit des Wachstums von P bei $X \to \infty$ festzusetzen.

Zum ersten Male brachten A. Vinogradov und V. Sprindžuk [2] ([1]) das prinzipielle Schema der Effektivisierung des Ergebnisses von K. Mahler für $m = n \geqslant 3$, wenn $G(x, y)$ die irreduzibele binäre Form ist. Später, wenn J. Coates [13] und V. Sprindžuk [7] die Gleichung von Thue–Mahler analysierten, bekamen sie explizite Abschätzungen für P im Falle, daß $G(x, y)$ eine binäre Form ist (sieh ausführlicher [4] und [1] auf den Seiten 209–211).

Auf die Möglichkeit der Effektivisierung (1) deutete V. Sprindžuk in [8] hin. J. Coates [14] realisierte diese Möglichkeit, wenn $m = 2$, $n = 3$. Er setzte fest, daß der allergrößte Primteiler die Zahl

$$10^{-3}(\ln\ln X)^{1/4}$$

übertrifft, wenn $(x, y) = 1$. In diesem Artikel bringen wir einen effektiven Beweis des Ergebnisses von K. Mahler [16] und lösen die ähnliche Aufgabe für relative Körper. Sei $\boldsymbol{K}$ ein gewisser Körper von algebraischen Zahlen des Grades $[\boldsymbol{K}:\boldsymbol{Q}] = d$ über den Körper der Rationalzahlen $\boldsymbol{Q}$, wo $G(x, y) = \alpha x^m + \beta y^n$ $(m \geqslant 2,\ n \geqslant 3,\ m \neq n)$ ein Polynom mit Koeffizienten aus $\boldsymbol{Z}_{\boldsymbol{K}}$-Ring der ganzen Zahlen des Körpers $\boldsymbol{K}$, $x, y \in \boldsymbol{Z}_{\boldsymbol{K}}$ und $(x, y) = 1$ ist.

([1]) Konkret, wurde die irreduzibele binäre Form $G(x, y)$ in allgemeiner Form betrachtet.