The sequences of prime divisors of integers

by

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1. Introduction. Let \(2 = p_1 < p_2 < \ldots\) denote the sequence of prime numbers and let \(q_j(n) < q_{j+1}(n) < \ldots < q_{j+k}(n)\) be those primes which divide \(n\); that is, with some positive integers \(a_j,\)

\[
(1) \quad n = q_1^{a_1}(n) q_2^{a_2}(n) \ldots q_k^{a_k}(n).
\]

The present investigation concerns the order of magnitude of \(q_j(n)\) where \(j\) may also depend on \(n\). The results will be average type of statements, that is, we consider \(q_j(n)\) for \(n \leq x\) and we let \(x \to +\infty\). We then decide whether a set \(n \leq x\), specified by a condition on \(q_j(n)\), is "large" or not and, in a particular, if its density exists. Here we apply the natural density.

We say that a set \(A\) of positive integers has density \(d(A)\) if, as \(x \to +\infty\)

\[
\lim \tau_x(n; n \in A) = d(A)
\]

exists, where \(\tau_x(n; \ldots)\) denotes the number of integers \(0 < n \leq x\) which satisfy the condition stated in the dotted space.

In the representation (1), \(x = \omega(n)\) obviously depends on \(n\) and thus, if \(j = j(n)\) is chosen in advance, \(q_j(n)\) may have no meaning. In order to avoid the need for distinguishing several cases in our statements, we extend the definition of \(q_j(n)\) as follows.

**Definition.** \(q_j(n)\) is the \(j\)-th term in (1) if \(1 \leq j \leq \omega(n)\). If \(j < 0, q_j(n) = 1\) and \(q_j(n) = +\infty\) for \(j > \omega(n)\).

This extended meaning of \(q_j(n)\) does not affect the fact that our statements concern actual prime divisors, that is, elements of (1). This is made possible by \(\omega(n)\) being close to the monotonic function \(\log \log n\) for "almost all \(n\)". More precisely, it is well known ([3], p. 41) that, if

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A (y) denotes the set of integers n for which $\omega(n) \geq y \log \log n$, then

$$d(A(1-e)) = 1 \quad \text{and} \quad d(A(1+e)) = 0.$$  

(2) will always be one of the guides in our choice of $j = f(n)$.

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2. The results and discussion. Let us, first of all, quote results on $q_{\omega-k}(n)$, $k = 0, 1, 2, \ldots$. The case $k = 0$ is due to de Bruijn [1] and arbitrary k to Levin and Fehmleb [4].

**Theorem A.** For any fixed integer $k \geq 0$, there is a differentiable distribution function $F_k(x)$ such that, as $x \to +\infty$,

$$\lim_{x \to +\infty} P_k(n; \log q_{\omega-k}(n) < x \log x) = F_k(x).$$

It is evident that the above statement cannot hold for $q_{\omega-k}(n)$ with $k = k(n) + \infty$ with x and for which $k(n) = o(\log \log n)$, as $x \to +\infty$,

$$\lim_{x \to +\infty} P_k(n; e^{-\alpha k \log k} \log q_{\omega-k}(n) < e^{-\alpha k \log k} \log x) = 1.$$  

In addition to Theorem 1, we shall prove the following results.

**Theorem 2.** Let $j = f(x) \to +\infty$ with $x$ and for which $k = o(\log \log x)$, as $x \to +\infty$,

$$\lim_{x \to +\infty} P_k(n; \log |j(x) - \log \log x| \geq (\log \log x)^h \quad \text{with some} \quad h > \frac{1}{2}.$$

Then, as $x \to +\infty$,

$$\lim_{x \to +\infty} P_k(n; (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-\alpha x} d\alpha) = 1.$$  

**Theorem 3.** For $j(x) \leq (1-x) \log \log x$, where $\epsilon > 0$ is an arbitrary number, as $x \to +\infty$ and as $j(x) = j + \infty$,

$$\lim_{x \to +\infty} P_k(n; \log q_{\omega-k}(n) \log q_j(n) < x) = 1 - 1/x, \quad x > 1.$$

Some comments are in order. First of all, we wish to emphasize that the assumptions on $k(x)$ and $j(x)$ are to serve only to guarantee that the results equally apply to elements of (1) as well as to the extended values of $q_j(n)$. As a guide, (2) can be consulted and, in the course of the proof, further references will be provided.

It is interesting to look at the statements with some detail. If we take $\log \log q_{\omega-k}(n)$ in Theorem 1 and apply (2) we get, roughly speaking,

that $\log \log q_{\omega-k}(n)$ is "about $\omega(n) - k(n)$". Notice that a similar asymptotic property of $\log \log q_j(n)$ is implied by (4) and therefore Theorems 1 and 2 show a common property of "large" and "small" prime divisors. Besides Theorem A, this weaker form of Theorem 2 seems to be the only explicit statement in the literature on $q_j(n)$, which appears in Erdős [2]. Similarly, a common property of small and large prime divisors is expressed in Theorem A and Theorem 3. If for bounded k we write

$$\log q_{\omega-k}(n) \log q_{\omega-k-1}(n) \log q_{\omega-k-2}(n) \log q_{\omega-k-3}(n) \ldots$$

then (3) suggests, and indeed, from the arguments of Levin and Fehmleb [4] it follows, that the left-hand side has a limiting distribution. This is the exact statement of Theorem 3 for $k \to +\infty$ with x. Of course, the forms of the limits differ.

We conclude this section with a simple remark. It can easily be seen that, for fixed j and $x$, $d([n; q_j(n) = p_1]) = \alpha_j$

exists and is positive. This remark is added only to cover the whole range of possible values of j in the statements of this section.

3. Proofs. The proof of Theorems 1 and 2 is based on the following relation. For $t = 1, 2, \ldots$, let

$$e_t(n) = \begin{cases} 1 & \text{if } p_t \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

Then, putting

$$\eta_t(n) = e_1(n) + e_2(n) + \ldots + e_t(n),$$

we evidently have

$$\{n: \eta_t(n) < j\} = \{n: q_j(n) > \prod_{t} \}.$$  

We can now turn to the details of proof.

**Proof of Theorem 1.** We apply the Turán–Kubilius inequality ([3], p. 35) in the following form. Let $T = T(x) = x^{1/2}$, where $0 < \alpha(x) \to 0$ as $x \to +\infty$. Then, as $x \to +\infty$,

$$\lim_{x \to +\infty} P_k(n; \sum_{t=0}^{x} e_t(n) + \log \alpha(x) < \log^{1/2} \alpha(x) = 1.$$  

Now, since for $n \leq x$, $\omega(n) = \eta_x(n)$, we get by (6),

$$\{n: \sum_{t=0}^{x} e_t(n) > k\} = \{n: \omega(n) - k > \prod_{t} \} = \{n: q_{\omega-k}(n) > \prod_{t} \}.$$
Guided by (2), we take \( k = k(x) \to +\infty \) with \( x \) in such a way that \( k(x) = o(\log \log x) \). Then, since for any \( \delta > 0 \) (see [5], p. 106)
\[
\log T < \log p_T < (1 + \delta) \log T,
\]
(8) implies

\[
\nu_x \left( n: \sum_{i=1}^{x} q_i(n) > k \right) = \nu_x \left( n: \log q_{x,k}(n) > \log T \right) \leq \nu_x \left( n: \log q_{x,k}(n) > \log T \right).
\]

Let us now set \( a(x) = \exp \left( -(1 + \varepsilon) k(x) \right) \) and \( T(x) = x^{\omega(x)} \). Then \( a(x) \to 0 \) and \( T(x) \to +\infty \) as \( x \to +\infty \) and thus (7) and (10) imply that, as \( x \to +\infty \),
\[
\lim \nu_x \left( n: \log q_{x,k}(n) > \log x \exp \left( -(1 + \varepsilon) k \right) \right) = 1.
\]

On the other hand, using the upper inequality in (9), we get from (8)
\[
\nu_x \left( n: \sum_{i=1}^{x} q_i(n) > k \right) \geq \nu_x \left( n: \log q_{x,k}(n) > (1 + \delta) \log T \right)
\]

Since with the choice of \( a(x) = \exp \left( -(1 + \varepsilon) k(x) \right) \) and \( T(x) = x^{\omega(x)} \), (7) is again applicable, we get that, as \( x \to +\infty \),
\[
\lim \nu_x \left( n: \log q_{x,k}(n) > (1 + \delta) \log x \exp \left( -(1 + \varepsilon) k \right) \right) = 0.
\]

Because both \( \delta > 0 \) and \( \varepsilon > 0 \) are arbitrary, \( \delta \) plays no role in (13) and thus Theorem 1 is established.

Proof of Theorem 2. Here we need the asymptotic normality of \( \nu_x(n) \) as \( T = T(x) \to +\infty \) with \( x \). More precisely, by a theorem of probabilistic number theory (see [3], p. 61), if \( T = T(x) \to +\infty \) with \( x \), and
\[
T(x) = x^w,
\]
then, for each fixed real \( \sigma \),
\[
\lim \nu_x \left( n: \nu_x(n) - \log \log T < \sigma (\log \log T)^{1/2} \right) = \Phi(x),
\]
where \( \Phi(x) \) signifies the standard normal distribution function occurring on the right-hand side of (4). (6) and (14) yield that if, for a given function \( j = j(x) \), we determine \( T = T(x) \) from the relation
\[
j(x) = \log \log T(x) + \sigma (\log \log T(x))^{1/2},
\]
(15) then, as \( x \to +\infty \),
\[
\lim \nu_x \left( n: \log q_j(n) > \log p_T \right) = \Phi(x).
\]

In view of (9), \( \log p_T = \log T + O(1) \) and thus by (15)
\[
\log p_T - j(x) = -\frac{j^{1/2}}{\sqrt{\pi}} + O(1).
\]

(16) and (17) imply that, as \( x \to +\infty \),
\[
\lim \nu_x \left( n: \log \log q_j(n) - j > -\frac{j^{1/2}}{\sqrt{\pi}} \right) = \Phi(x),
\]
what is equivalent to (4) by \( \Phi(x) = 1 - \Phi(-x) \). This completes the proof of Theorem 2.

Let us remark that we did not make use of our assumption on \( j(x) \) in the course of the proof, except that \( T(x) = x^w \) and thus by (15), \( j(x) < \log \log x \). The additional assumptions were made only to guarantee that the conclusion remain the same if \( q_j(n) \) is restricted to its values occurring in (1).

Proof of Theorem 3. First of all note that
\[
\nu_x \left( n: q_{s+1}(n) > s, q_s(n) = p_k \right) = \nu_x \left( n: q_s(n) = 0, p_k < q_s < s, q_s(n) = p_k \right).
\]

Since \( [n: q_s(n) = p_k] \) can be expressed in terms of \( q_s(n) \), \( 1 \leq s \leq k \), by Lemma 1.4 of [3], p. 61, for \( s \leq x^{\omega(x)} \), where \( a(x) > 0 \) and \( a(x) \to 0 \) as \( x \to +\infty \),
\[
\nu_x \left( n: q_{s+1}(n) > s, q_s(n) = p_k \right)
\]

\[
= (1 + o(1)) \left( \prod_{p_k < q_p \leq n} \frac{1}{1 - 1/p} \right) \nu_x \left( n: q_s(n) = p_k \right),
\]
where \( o(1) \) is uniform in \( s \) and \( k \) for \( p_k < s \leq n^{\omega(x)} \). For a fixed real number \( \sigma > 1 \), let us choose \( s \) so that \( \log s \sim \log p_k \). We then have, as a consequence of the prime number theorem (see [5], p. 106)
\[
\frac{1}{\log p_k} \log s\left( 1 - \frac{1}{p} \right) = \log s + O \left( \log \log p_k \right) - \log s \left( 0, s \right),
\]
where \( \beta > 0 \) is a uniform constant and thus \( o(1) \) can be made tending to 0 uniformly for \( p_k \to U(x) \to +\infty \) with \( x \).

Writing now
\[
\nu_x \left( n: \frac{\log q_{s+1}(n)}{\log q_s(n)} > x \right) = \sum_{s=1}^{\infty} \nu_x \left( n: \frac{\log q_{s+1}(n)}{\log q_s(n)} > x, q_s(n) = p_k \right)
\]
we apply to the tails of the above sum that
\[
\sum_{p_k < q_s \leq n} \nu_x \left( n: \frac{\log q_{s+1}(n)}{\log q_s(n)} > x, q_s(n) = p_k \right) \leq \nu_x \left( n: q_s(n) \leq U \right)
\]
and
\[
\sum_{p_k > U} \nu_x \left( n: \frac{\log q_{s+1}(n)}{\log q_s(n)} > x, q_s(n) = p_k \right) \leq \nu_x \left( n: q_s(n) \geq W \right).
\]
Für einen gegebenen \( \alpha > 0 \), wählen wir
\[
U = \exp\{\exp[(1 - \alpha)J]\} \quad \text{und} \quad W = \exp\{\exp[(1 + \alpha)J]\}.
\]

By Theorem 2 we get that the left-hand sides of (21) and (22) tend to \( 0 \) as \( \sigma \to +\infty \). Since the above \( U = U(\sigma) \to +\infty \) with \( \sigma \), for \( U < p_k < W \), we can apply (18) and (19) and we get
\[
\sum_{U < p_k < W} \nu_\sigma(n) \frac{\log g_{i+1}(n)}{\log g_i(n)} > \alpha, \quad g_i(n) = p_k
\]
\[
= (1 + o(1)) \sigma^{-1} \nu_\sigma(n: U < g_i(n) < W).
\]

Another appeal to Theorem 2 yields that, for our \( j \), as \( \sigma \to +\infty \),
\[
\nu_\sigma(n: U < g_i(n) < W) \to 1,
\]
and thus (20)–(23) lead to the conclusion of Theorem 3. This completes the proof.

References


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Über die maximale Norm der Idealteiler des Polynoms \( ax^n + by^n \) mit den algebraischen Koeffizienten

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1. Einleitung. K. Mahler bewies [16], daß der allergrößte Primteiler \( P \) des Polynoms
\[
G(x, y) = ax^n + by^n,
\]
wo \( m \geq 2, n \geq 3, a \neq 0, b \neq 0 \) ganz und rational sind, unbegrenzt bei \( X = \max(|x|, |y|) \to \infty \), \( (x, y) = 1 \), wächst. Um das Verhalten von \( P \) zu erforschen, betrachtete K. Mahler Ergebnisse von C. Parry [17] über die \( p \)-adische Verallgemeinerung vom Thue–Siegel–Eisenzatz. Da sie nichteffektiv sind, war es unmöglich auch im Prinzip die Geschwindigkeit des Wachstums von \( P \) bei \( X \to \infty \) festzusetzen.


\[
10^{-3}(\ln \ln X)^4
\]
übertrifft, wenn \( (x, y) = 1 \). In diesem Artikel bringen wir einen effektiven Beweis des Ergebnisses von K. Mahler [16] und lösen die ähnliche Aufgabe für relative Körper. Sei \( K \) ein gewisser Körper von algebraischen Zahlen des Grades \( [K: Q] = d \) über dem Körper der Rationalzahlen \( Q \), wo \( G(x, y) = ax^n + by^n \) \( (m \geq 2, n \geq 3, \ell \neq n) \) ein Polynom mit Koeffizienten aus \( \mathbb{Z}_K \)-Ring der ganzen Zahlen des Körpers \( K \), \( x, y \in \mathbb{Z}_K \) und \( (x, y) = 1 \) ist.

(1) Konkret, wurde die irreduzible binäre Form \( G(x, y) \) in allgemeiner Form betrachtet.