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On a kind of uniform distribution of values of multiplicative functions in residue classes

by

W. NARKIEWICZ and J. ŚLIWA (Wrocław)

1. If $f(n)$ is an integer-valued arithmetical function then it is called *weakly uniformly distributed* $(\mathrm{mod}\,N)$, or shortly WUD $(\mathrm{mod}\,N)$ if its values are asymptotically uniformly distributed in residue classes $(\mathrm{mod}\,N)$, prime to N. In [1] a necessary and sufficient condition for a polynomial-like multiplicative function to be WUD $(\mathrm{mod}\,N)$ was established with the use of the tauberian Ikehara–Delange theorem for Dirichlet series. This condition may be reformulated in such a way that it makes sense for arbitrary integer-valued multiplicative functions, not necessarily polynomial-like, and the question arises, whether there exists a connection between this condition and the property WUD $(\mathrm{mod}\,N)$.

In this note we want to point out, that this condition is both necessary and sufficient for the function f to be uniformly distributed in residue classes $(\mathrm{mod}\,N)$ in a certain weaker sense, which for polynomial-like functions coincides with WUD $(\mathrm{mod}\,N)$. Our proof uses only elementary properties of Dirichlet series, and so in particular we obtain a new proof of the necessity part of the result of [1] which avoids the use of deep tauberian theorems.

Let $f(n)$ be a multiplicative, integer-valued function. Denote by $m = m(f, N)$ (where $N \geqslant 3$ is a given integer) the least integer, if it exists, with the property, that the series $\sum p^{-1}$ (where p runs over all primes satisfying $\big(f(p^m), N\big) = 1$) diverges. Let A be the subgroup of the multiplicative group of residue classes $(\mathrm{mod}\,N)$, prime to N, which is generated by residues $r\,(\mathrm{mod}\,N)$ for which the series

$$\sum_{p\in A_r} p^{-1}$$

diverges, where $A_r = \{p\colon f(p^m) \equiv r\,(\mathrm{mod}\,N)\}$.

We shall consider the following condition:

(*) For every non-principal character $\chi(\bmod N)$, trivial on Λ, there exists a prime p such that

(1) $$\sum_{j=0}^{\infty}\chi\big(f(p^j)\big)p^{-j/m} = 0.$$

One sees without trouble, that for polynomial-like multiplicative functions f the condition (*) coincides with that occurring in the main result of [1].

2. Let $f(n)$ be an integer-valued and multiplicative function and let m be defined as in Section 1. We shall say that f is *Dirichlet-WUD* (mod N), provided that for every r prime to N one has

$$\lim_{s\to 1/m+0}\Big(\sum_{\substack{n\\ f(n)\equiv r(\bmod N)}} n^{-s}\Big):\Big(\sum_{\substack{n\\ (f(n),N)=1}} n^{-s}\Big) = \frac{1}{\varphi(N)}$$

when s tends to $1/m$ over real values bigger than $1/m$.

Note that our choice of m implies that the abscissa of absolute convergence of the series

$$\sum_{\substack{n\\ (f(n),N)=1}} n^{-s}$$

equals $1/m$, and that the abscissa of absolute convergence of the other series occurring here does not exceed $1/m$. (It may be smaller, as the trivial example: $f(n)=1$, $N=3$, $r=2$ shows.)

THEOREM. *A multiplicative integer-valued function $f(n)$ is Dirichlet-WUD* (mod N) *if and only if it satisfies the condition* (*).

Proof. Let χ be an arbitrary character (mod N). The function

$$F(s,\chi) = \sum_{n=1}^{\infty}\chi\big(f(n)\big)n^{-s}$$

is defined and regular for $\operatorname{Re} s > 1/m$. One sees easily that in that half-plane we have

(2) $$F(s,\chi) = (s-1/m)^{a(\chi)}g(s,\chi)\exp\Big\{\sum_p \chi\big(f(p^m)\big)p^{-ms}\Big\},$$

where $g(s,\chi)$ is regular for $\operatorname{Re} s \geqslant 1/m$, $g(1/m,\chi)\neq 0$ and $a(\chi)$ is a non-negative integer which is positive if and only if for some prime p the equality (1) holds.

As for $\operatorname{Re} s > 1/m$ and $(j,N)=1$ one has

$$\sum_{\substack{n\\ f(n)\equiv j(\bmod N)}} n^{-s} = \frac{1}{\varphi(N)}\sum_{\chi}\overline{\chi(j)}F(s,\chi)$$

and

$$\sum_{\substack{n\\ (f(n),N)=1}} n^{-s} = F(s,\chi_0)$$

(where χ_0 is the principal character (mod N)) we obtain finally, that f will be Dirichlet-WUD (mod N) if and only if for every j prime to N one has

$$\lim_{s\to 1/m+0}\sum_{\chi\neq\chi_0}\overline{\chi(j)}\,\frac{g(s,\chi)}{g(s,\chi_0)}\Big(s-\frac{1}{m}\Big)^{a(\chi)}\exp\Big\{\sum_{(k,N)=1}\big(\chi(k)-1\big)\sum_{p\in\Lambda_k}p^{-sm}\Big\} = 0$$

and this turns out to be equivalent to

(3) $$\lim_{s\to 1/m+0}\Big\{\sum_{(k,N)=1}\big(\operatorname{Re}\chi(k)-1\big)\sum_{p\in\Lambda_k}p^{-sm} + a(\chi)\log\Big(s-\frac{1}{m}\Big)\Big\} = -\infty$$

for all non-principal characters χ (mod N).

Assume now that f is Dirichlet-WUD (mod N), i.e. (3) holds, but the condition (*) is not satisfied. Then there exists a non-principal character χ (mod N), trivial on Λ for which $a(\chi)=0$. But in this case the bracketed terms of (3) become

$$\sum_{k\in\Lambda}\big(\operatorname{Re}\chi(k)-1\big)\sum_{p\in\Lambda_k}p^{-sm} + \sum_{\substack{k\notin\Lambda\\ (k,N)=1}}\big(\operatorname{Re}\chi(k)-1\big)\sum_{p\in\Lambda_k}p^{-sm} = O(1)$$

as s approaches $1/m$ because for $k\in\Lambda$, $\chi(k)=1$ and for $k\notin\Lambda$ the function

$$\sum_{p\in\Lambda_k}p^{-sm}$$

is regular at $s=1/m$. The obtained evaluation contradicts (3).

Conversely assume that (*) is satisfied. If χ is trivial on Λ then by (*) we have $a(\chi)\geqslant 1$ and (3) follows. If however χ is non-trivial on Λ then we may select $r\in\Lambda$ with $\chi(r)\neq 1$, i.e. $\operatorname{Re}\chi(r)-1<0$, and in view of $\operatorname{Re}\chi(k)-1\leqslant 0$ we obtain for $s>1/m$

$$\sum_{(k,N)=1}\big(\operatorname{Re}\chi(k)-1\big)\sum_{p\in\Lambda_k}p^{-sm} + a(\chi)\log(s-1/m) \leqslant \big(\operatorname{Re}\chi(r)-1\big)\sum_{p\in\Lambda_r}p^{-sm}$$

and the right-hand side of this inequality tends to $-\infty$, hence (3) holds. ∎

3. To show that WUD (mod N) implies Dirichlet-WUD (mod N) one has only to observe that if

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$$

are two Dirichlet series with non-negative and bounded coefficients with their abscissas of convergence equal to a and b respectively ($a \leqslant b$) and moreover for x tending to infinity we have

$$\sum_{n \leqslant x} a_n = \big(1+o(1)\big) \sum_{n \leqslant x} b_n,$$

then $a = b$ and $\lim_{s \to a+0} f(s)/g(s) = 1$. (See e.g. [2], § 8, Satz 8.)

We may finally state a corollary to the theorem proved in Section 2:

COROLLARY. *If f is a multiplicative function which is integer-valued and WUD (mod N), then it satisfies the condition* $(*)$. ■

It would be interesting to determine, whether the Dirichlet-WUD (mod N) is in fact weaker than WUD (mod N) for multiplicative functions.

References

[1] W. Narkiewicz, *On distribution of values of multiplicative function in residue classes*, Acta Arith. 12 (1967), pp. 269–279.

[2] H.-H. Ostmann, *Additive Zahlentheorie*, Berlin 1956.

MATHEMATICAL INSTITUTE, WROCŁAW UNIVERSITY
INSTYTUT MATEMATYCZNY UNIWERSYTETU WROCŁAWSKIEGO IM. B. BIERUTA

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Elementary methods in the theory of L-functions, III
The Deuring-phenomenon

by

J. PINTZ (Budapest)

1. Deuring [2] proved in 1933 that if the class number $h(-D)$ of the imaginary quadratic field belonging to the fundamental discriminant $-D < 0$ is equal to 1 for an infinite sequence of $D_n \to \infty$, then the Riemann hypothesis for $\zeta(s)$ is true. Mordell [5] proved in 1934 that if $h(-D) \nrightarrow \infty$ for $D \to \infty$, then the Riemann hypothesis is true. These striking results showed a curious connection between the possibly existing real zeros of special real L-functions (which exist by the theorem of Hecke (see [4]), if $h(-D) < C_0\sqrt{D}/\log D$) and the non-trivial zeros of the ζ-function.

In [6] we proved that if

$$h(-D) \leqslant \frac{\log D}{2\log\log D} \quad \text{and} \quad \chi(n) = \left(\frac{-D}{n}\right) \tag{1.1}$$

then for the greatest real zero $1-\delta$ of $L(s,\chi) = L(s)$

$$\delta = \frac{L(1)}{\prod_{p|D}\left(1+\frac{1}{p}\right)\frac{\pi^2}{6}}\big(1+o(1)\big) = \frac{6h(-D)}{\prod_{p|D}\left(1+\frac{1}{p}\right)\pi\sqrt{D}}\big(1+o(1)\big).$$

In this paper we shall demonstrate that, assuming a little stronger upper bound for $h(-D)$ than (1.1), we can determine up to a factor $1+o(1)$ the values of the corresponding L-function in a great domain of the critical strip. Our result will also show that except for the real zero $1-\delta$ mentioned above, neither $L(s,\chi)$ nor $\zeta(s)$ has a zero in this domain. As a consequence we have also a weakened form of Mordell's theorem [5], namely that if $h(-D) \nrightarrow \infty$ for $D \to \infty$, then $\zeta(s)$ has no zero in the half-plane $\sigma > \frac{3}{4}$.

Siegel [8] has shown that our assumption (1.1) cannot be valid for infinitely many D's, because by his theorem for an arbitrary $\varepsilon > 0$

$$h(-D) > D^{1/2-\varepsilon} \quad \text{for} \quad D > D_0(\varepsilon). \tag{1.2}$$