It is therefore non-singular; and so $z_1, z_2, z_3$ are locally holomorphic functions of $f_1, f_2, f_0$ at the point

$$f_1 = \wp(\omega/4), \quad f_2 = i, \quad f_0 = f(1/4, 1/4, 1/4).$$

It follows from (5) that the polynomial $P(\lambda_3)$ has a zero of order at least $3L+1$ at this point. However, its total degree does not exceed $3L$, whence $P(\lambda_3)$ vanishes identically for all $\lambda_3$. This contradicts the choice of coefficients made at the outset, and thereby completes the proof.

The same techniques will establish the linear independence over $A$ of $1, \omega, \eta$ and $\log\sigma$ without any hypotheses of complex multiplication.

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# Further applications of Turán's methods to the distribution of prime ideals in ideal classes mod $\mathfrak{f}$

by

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**1.** Let $K$ be an algebraic number field, $\nu$ and $\Delta$ degree and discriminant of the field $K$ respectively, $\mathfrak{f}$ a given ideal of $K$, $\mathfrak{N}\mathfrak{a}$ the norm of an ideal $\mathfrak{a}$ of $K$ and $\mathfrak{p}$ a prime ideal of $K$ (see [2]).

Denote further by $\mathscr{H}$ (mod $\mathfrak{f}$) an ideal-class mod $\mathfrak{f}$ ([3], Def. VIII), by $\mathscr{H}_0$ (mod $\mathfrak{f}$) the principal class mod $\mathfrak{f}$ and by $h(\mathfrak{f})$ the class-number. Let $\chi(\mathscr{H})$ be a character of the abelian group of ideal-classes $\mathscr{H}$ (mod $\mathfrak{f}$), $\chi(\mathfrak{a})$ the extension of $\chi(\mathscr{H})$ ([3], Def. X) and $\chi_0$ — the principal character mod $\mathfrak{f}$.

Denote by $\zeta_K(s)$ the Dedekind Zeta-function and by $\zeta(s, \chi)$ the Hecke–Landau Zeta-functions ([3], Def. XVII).

Denote further

$$\gamma(n, \mathscr{H}) = \sum_{(\mathfrak{N}\mathfrak{p})^m = n, \mathfrak{p}^m \in \mathscr{H} (\mathrm{mod}\ \mathfrak{f})} \log \mathfrak{N}\mathfrak{p},$$

$$\psi(x, \mathscr{H}) = \sum_{n \leqslant x} \gamma(n, \mathscr{H}), \tag{1.1}$$

$$\Delta(x, \mathscr{H}_1, \mathscr{H}_2) = \psi(x, \mathscr{H}_1) - \psi(x, \mathscr{H}_2). \tag{1.2}$$

**2.** In this paper we shall establish an exact correspondence between the order of magnitude of the expressions (1.2) and the regions in which some $\zeta(s, \chi)$-functions do not vanish (compare [8] and [9], Th. XXXVI). In the following $C_i$, $i = 1, 2, \ldots$ denote positive constants independent of $K$.

Theorem 1. *Suppose* $\mathscr{H}_1, \mathscr{H}_2$ *denote any fixed ideal-classes* (mod $\mathfrak{f}$), $\mathscr{H}_1 \neq \mathscr{H}_2$. *If* $h(\mathfrak{f}) \geqslant 2$,

$$\prod_{\chi, \chi(\mathscr{H}_1) \neq \chi(\mathscr{H}_2)} \zeta(s, \chi) \neq 0$$

*in the region*

$$\sigma > 1 - C_0 \eta(|t|), \quad 0 < C_0 \leqslant \tfrac{1}{2}, \tag{2.1}$$

*where $C_0$ is a constant depending on the ideal $\mathfrak{f}$ and on the field $K$, $\eta(t)$ is for $t \geqslant 0$ a decreasing function, having a continuous derivative $\eta'(t)$ and satisfying the conditions:*

(a) $$0 < \eta(t) \leqslant \tfrac{1}{2},$$

(b) $$\eta'(t) \to 0 \quad as \quad t \to \infty,$$

(c) $$\frac{1}{\eta(t)} = O(\log t) \quad as \quad t \to \infty,$$

*and $a$ denotes a fixed number, $0 < a < 1$, then*

(2.2) $$|\Delta(x, \mathcal{H}_1, \mathcal{H}_2)| < C_1 C_0^{-1} \nu \log(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}+2) x \exp\left(-\frac{aC_0}{2}\omega(x)\right)$$

*for*

$$x \geqslant \omega^{-1}\left(\frac{24}{aC_0}\log\frac{24}{aC_0}\right),$$

*where*

$$\omega(x) = \min_{t\geqslant 1}\{\eta(t)\log x + \log t\},$$

*and $\omega^{-1}(x)$ denotes the function inverse to $\omega(x)$, $C_1$ is a constant depending on $a$ and $\eta(t)$ only.*

THEOREM 2. *Suppose $h(\mathfrak{f}) \geqslant 2$, $\mathcal{H}$ is any fixed ideal class* (mod $\mathfrak{f}$), *$\mathcal{H} \neq \mathcal{H}_0$, $0 < a < 1$, $\eta_1(t)$ is a function satisfying except* (a), (b), (c) *also the additional condition*

(d) $$\eta_1(t) \leqslant C_2 \quad for \quad t > C_3,$$

*where $C_2$ is a sufficiently small positive number and*

$$\omega_1(x) = \min_{t\geqslant 1}\{\eta_1(t)\log x + \log t\}.$$

*If*

(2.3) $$|\Delta(x, \mathcal{H}, \mathcal{H}_0)| < C_4 C_0^{-1} \nu \log(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}+2) x \exp\left(-\frac{aC_0}{2}\omega_1(x)\right)$$

*for $x \geqslant \varphi(a, C_0)$, where $C_0$ is a constant depending on $a$ and $\eta_1(t)$, $\varphi(a, C_0) > 1$, then*

$$\prod_{\chi,\chi(\mathcal{H})\neq 1} \zeta(s,\chi) \neq 0$$



*in the region*

(2.4) $$\sigma > 1 - \frac{\log t}{400\log\omega_1^{-1}(\log t^{4/aC_0})},$$

$$t > \max\Big\{C_5, A(C_0, K), \eta_1^{-1}\big(\exp(-\nu^2 h^2(\mathfrak{f}))\big), (|\Delta|+2)\mathfrak{N}\mathfrak{f}, \exp\left(\frac{aC_0}{4}\omega_1(\varphi(a, C_0))\right)\Big\} = \mathcal{B}(a, C_0, K, \eta_1, \varphi),$$

*where $\omega_1^{-1}(x)$, $\eta_1^{-1}(x)$ denote functions inverse to $\omega_1(x)$, $\eta_1(x)$ respectively,*

$$A(C_0, K) = \big(C_0^{-1}\nu^2 h(\mathfrak{f})\log^2(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}+2)\big)^5$$

*and $C_5$ depends on $C_4$ only.*

THEOREM 3. *Under the conditions of Theorem 2, we have*

$$\prod_{\chi,\chi(\mathcal{H})\neq 1} \zeta(s,\chi) \neq 0$$

*in the region*

(2.5) $$\sigma > 1 - \frac{aC_0}{(40)^2}\eta_1(t^{4/aC_0}), \quad t > \mathcal{B}(a, C_0, K, \eta_1, \varphi).$$

Choosing $\eta_1(t) = \eta(t) = 1/\log^\gamma(t+2)$, $0 < \gamma \leqslant 1$, we obtain from Theorems 1 and 2 the following

THEOREM 4. *If $\gamma_1$ denotes the supremum of the numbers $\gamma$ for which*

(2.6) $$\Delta(x, \mathcal{H}, \mathcal{H}_0) = O\big(x\exp(-C_6\log^\gamma x)\big)$$

*and $\gamma_2$ is the infimum of the numbers $\gamma'$ for which*

$$\prod_{\chi,\chi(\mathcal{H})\neq 1} \zeta(s,\chi) \neq 0$$

*in the region*

(2.7) $$\sigma > 1 - \frac{C_7}{\log^{\gamma'}|t|}, \quad |t| \geqslant C_8,$$

*then*

$$\gamma_1 = \frac{1}{1+\gamma_2}.$$

*The constants depend on $\gamma$, $\mathfrak{f}$ and the field $K$.*

**3.** There are well known the following properties of $\zeta(s,\chi)$ functions (see [3]).

I. For $\chi \neq \chi_0$, $\zeta(s,\chi)$ is a regular function in the whole complex plane.

II. $\zeta(s,\chi)$ has an infinity of roots in the strip $0<\sigma<1$ of the complex plane.

III. For $\sigma>1$

$$\zeta(s,\chi)=\prod_{\mathfrak{p}}\left(1-\frac{\chi(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^s}\right)^{-1}. \tag{3.1}$$

Hence for $\sigma>1$,

$$\zeta(s,\chi)\neq 0. \tag{3.2}$$

IV. For $\sigma>1$

$$-\frac{\zeta'}{\zeta}(s,\chi)=\sum_{n}\frac{G(n,\chi)}{n^s}, \tag{3.3}$$

where

$$G(n,\chi)=\sum_{(\mathfrak{N}\mathfrak{p})^m=n}\chi(\mathfrak{p}^m)\log\mathfrak{N}\mathfrak{p}. \tag{3.4}$$

V. We have the estimate

$$|G(n,\chi)|\leqslant\frac{\nu}{\log 2}\log^2 n \tag{3.5}$$

(see [6], Lemma 2).

The proofs of Theorems 1 and 2 will rest on the following lemmas.

LEMMA 1 (Turán's second main theorem). *Let $z_1, z_2, \ldots, z_h$ be complex numbers such that*

$$|z_1|\geqslant|z_2|\geqslant\ldots\geqslant|z_h|,\qquad |z_1|\geqslant 1$$

*and let $b_1, b_2, \ldots, b_h$ be any complex numbers.*

*Then, if $m$ is positive and $N\geqslant h$, there exists an integer $l$ such that $m\leqslant l\leqslant m+N$,*

$$|b_1z_1^l+b_2z_2^l+\ldots+b_hz_h^l|\geqslant\left(\frac{N}{48e^2(2N+m)}\right)^N\min_{1\leqslant j\leqslant h}|b_1+b_2+\ldots+b_j| \tag{3.6}$$

(see [9], p. 52).

LEMMA 2. *In the strip $0<\sigma\leqslant 4$ of the complex plane we have the estimate*

$$|\zeta(s,\chi)|\leqslant\left(|\Delta|\mathfrak{N}\mathfrak{f}(|t|+2)^\nu\right)^{C_9},\qquad \chi\neq\chi_0, \tag{3.7}$$

*where $C_9$ is a numerical constant* (see [6]).

LEMMA 3. *For $\sigma>1$, $\chi\neq\chi_0$,*

$$\frac{1}{|\zeta(s,\chi)|}\leqslant(|\Delta|2^\nu)^{C_{10}}\frac{1}{\sigma-1}. \tag{3.8}$$



This inequality follows from (3.1) and an upper estimate of $|\zeta_K(s)|$ (see [5], Lemma 7).

LEMMA 4. *If $s_0=1+\mu+it'$, $0<\mu\leqslant 1/40$, $t'\geqslant 10$ and $N_1$ stands for the number of roots of $\zeta(s,\chi)$, $\chi\neq\chi_0$ in the circle $|s-s_0|\leqslant 8\mu$, then*

$$N_1<1+\frac{C_{11}\nu\log(|\Delta|\mathfrak{N}\mathfrak{f}t')}{\log(8\mu)^{-1}}. \tag{3.9}$$

This lemma follows from the estimates (3.7), (3.8) by the use of Jensen inequality. The constant $C_{11}$ is purely numerical.

LEMMA 5. *Denote by $V(T)$ the number of roots of $\zeta(s,\chi)$, $\chi\neq\chi_0$ in the rectangle $1/16\leqslant\sigma\leqslant 1$, $T\leqslant t\leqslant T+1$. Then for $-\infty<T<+\infty$*

$$V(T)<C_{12}\log\left(|\Delta|\mathfrak{N}\mathfrak{f}(|T|+2)^\nu\right), \tag{3.10}$$

*where $C_{12}$ is a numerical constant.*

LEMMA 6. *There exists a broken line $L$ in the vertical strip $1/4\leqslant\sigma\leqslant 1/2$, symmetrical to the real axis, and consisting of horizontal and vertical segments alternately, having the following property: if we denote by $T_m$ the ordinates of horizontal segments, so for each integer $m$ there exists only one such $T_m=T_m(\chi)$ that $m<T_m<m+1$ and*

$$\left|\frac{\zeta'}{\zeta}(s,\chi)\right|<C_{13}\log^2\left(|\Delta|\mathfrak{N}\mathfrak{f}(|t|+2)^\nu\right) \tag{3.11}$$

*holds for $s\in L$.*

*If $1/4\leqslant\sigma\leqslant 3$, $t=T_m$, $|m|\geqslant 2$, then*

$$\left|\frac{\zeta'}{\zeta}(s,\chi)\right|<C_{14}\log^2\left(|\Delta|\mathfrak{N}\mathfrak{f}(|t|+2)\right). \tag{3.12}$$

For the proofs of Lemmas 5 and 6 see [5] and [6].

LEMMA 7. *If $1<\sigma\leqslant 3/2$, $\xi>1$ and $l\geqslant 2$ is a positive integer, then*

$$\left|(-1)^l\sum_{n\geqslant\xi}\frac{\gamma(n,\mathscr{H}_0)-\gamma(n,\mathscr{H})}{n^s}\cdot\frac{\log^{l+1}(n/\xi)}{(l+1)!}+\right. \left.+\frac{1}{h(\mathfrak{f})}\sum_{\chi}\left(1-\overline{\chi}(\mathscr{H})\right)\sum_{\varrho=\varrho(\chi)}\frac{\xi^{\varrho-s}}{(\varrho-s)^{l+2}}\right| <C_{15}\frac{\xi^{1/2-\sigma}\log^2\left(|\Delta|\mathfrak{N}\mathfrak{f}(|t|+2)^\nu\right)}{(\sigma-\frac{1}{2})^{l+2}}, \tag{3.13}$$

*where the sum is taken over all roots of the function $\prod_{\chi,\chi(\mathscr{H})\neq 1}\zeta(s,\chi)$ lying to the right of the line $L$ defined by Lemma 6.*

For the method of the proof see [9], Appendix V.

Lemma 8. *Let $R > 0$ and the function*

$$f(s) = \sum_{n=0}^{\infty} a_n (s-s_0)^n$$

*be regular in the circle $|s-s_0| \leqslant R$ and satisfy the inequality $\mathrm{Re} f(s) \leqslant M$ for $|s-s_0| = R$. Then in the circle $|s-s_0| \leqslant r < R$,*

$$(3.14) \qquad |f'(s)| \leqslant \frac{2R}{(R-r)^2}\big(M - \mathrm{Re} f(s_0)\big)$$

*holds* (see [4], p. 383).

Lemma 9. *Let the series*

$$(3.15) \qquad f(s) = \sum_n a_n n^{-s}$$

*be absolutely convergent for $\sigma > 1$, and*

$$(3.16) \qquad |a_n| < CH\Phi(n),$$

*where $C > 0$, $H$ is a positive parameter and $\Phi(x)$ is a positive non decreasing function (for large values of $x$). Let*

$$(3.17) \qquad \sum_n |a_n| n^{-\sigma} = O\big(H(\sigma-1)^{-a}\big), \qquad a > 0$$

*for $\sigma \to 1_{+0}$. Suppose further $b > 1$, $T > 0$, $x > 1$. Then*

$$(3.18) \qquad \sum_{n \leqslant x} a_n = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s)\frac{x^s}{s}\,ds + O\left(\frac{Hx^b}{T(b-1)^a}\right) + O\big(H\Phi(2x)T^{-1}x\log 2x\big) + O\big(H\Phi(2x)\big),$$

*where the constant implied by the $O$ notation is independent of $x$, $T$ and $H$ but depends on $b$ only* (see [4], p. 376).

Lemma 10. *If $\zeta(s,\chi) \neq 0$, $\chi \neq \chi_0$ in the region*

$$(3.19) \qquad \sigma > 1 - C_0\eta(|t|),$$

*where $\eta(t)$ and $C_0$ are defined by (2.1), then there exists $T_0 \geqslant 1$, such that*

$$(3.20) \qquad \frac{\zeta'}{\zeta}(s,\chi) = O\Big(C_0^{-1}\nu\log\big(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}(|t|+2)\big)\log(|t|+2)\Big)$$

*in the region*

$$(3.21) \qquad \begin{aligned} &1 - aC_0\eta(t) \leqslant \sigma \leqslant 2, \quad t \geqslant T_0,\\ &1 - aC_0\eta(T_0) \leqslant \sigma \leqslant 2, \quad 0 \leqslant t \leqslant T_0, \end{aligned}$$

*$0 < a < 1$, where $T_0$ and the constant implied by the $O$ notation depend on $\eta(t)$ and $a$ only.*



Proof. Using Lemma 8 we shall prove that

$$(3.22) \qquad \left|\frac{\zeta'}{\zeta}(s,\chi)\right| = O\Big(C_0^{-1}\nu\log\big(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}(|t|+2)\big)\log(|t|+2)\Big)$$

for $\sigma \geqslant 1 + aC_0\eta(t)$. Let $t_0 \geqslant 0$, $s_0 = \sigma_0 + it_0$, $1 + aC_0\eta(t_0) \leqslant \sigma_0 \leqslant 2$, $R = aC_0\eta(t_0)$, $f(s) = \log\zeta(s,\chi)$.

From Lemma 2 it follows that in the circle $|s-s_0| \leqslant R$ and for $\chi \neq \chi_0$;

$$(3.23) \qquad \mathrm{Re} f(s) = \log|\zeta(s,\chi)| \leqslant C_9\log\big(|\Delta|\mathfrak{N}\mathfrak{f}(|t|+3)^{\nu}\big).$$

Similarly, from Lemma 3 we have

$$(3.24) \qquad -\mathrm{Re} f(s_0) = \log\frac{1}{|\zeta(s_0,\chi)|} \leqslant C_{16}\log\big((aC_0)^{-1}|\Delta|2^{\nu}\log(t_0+2)\big).$$

From Lemma 8 and owing to (3.23), (3.24) we get (3.22). Thus all that remains to complete the proof is to show that (3.20) holds in the remaining part of the region (3.21).

Since $\zeta(s,\chi) \neq 0$ in the region $\sigma > 1 - C_0\eta(|t|)$, there exists a single-valued branch of the logarithm

$$f(s) = \log\zeta(s,\chi)$$

in this region.

Let $t_0 \geqslant 0$, $s_0 = 1 + aC_0\eta(t_0) + it_0$, $r = 2aC_0\eta(t_0)$, $R = aC_0\eta(t_0) + \frac{1}{2}(1+a)C_0\eta(t_0)$.

Owing to (b) def. $\eta(t)$, the circle with radius $R$ lies in the region $\sigma > 1 - C_0\eta(|t|)$, for $t_0 \geqslant T_0$ where $T_0$ depends on $a$ and $\eta(t)$ only. Hence the circle with radius

$$R' = aC_0\eta(t_0) + \tfrac{1}{2}C_0(1+a)\min\big(\eta(t_0),\eta(T_0)\big)$$

and the centre at $s_0$ is contained in the region (3.19).

Applying Lemma 2 and 3 to the function $f(s)$ and to the circle $|s-s_0| \leqslant R'$, we get in virtue of Lemma 8 the estimate (3.20) in the strip

$$1 - \min\big(\eta(|t|),\eta(T_0)\big) \leqslant \sigma \leqslant 1 + aC_0\eta(|t|).$$

**4.** Proof of Theorem 1. We apply Lemma 9 to the function

$$f(s) = -\frac{\zeta'}{\zeta}(s,\chi) = \sum_n \frac{G(n,\chi)}{n^s}$$

with $b = 1+1/\log x$, $3 \leqslant T \leqslant x$, $H = \nu$, $\Phi(n) = \log^2 n$. From (3.5) it follows

$$\sum_n \frac{|G(n,\chi)|}{n^\sigma} \leqslant \frac{\nu}{\log 2} \sum_n \frac{\log^2 n}{n^\sigma} = O\big(\nu(\sigma-1)^{-3}\big).$$

Hence, from Lemma 9, we get

$$(4.1)\qquad \sum_{n\leqslant x} G(n,\chi) = -\frac{1}{2\pi i}\int_{b-iT}^{b+iT} \frac{\zeta'}{\zeta}(s,\chi)\frac{x^s}{s}\,ds = O(\nu T^{-1} x \log^3 x)$$

and

$$\sum_{n\leqslant x} \gamma(n,\mathscr{H}) = \sum_{n\leqslant x}\frac{1}{h(\mathfrak{f})}\sum_{\chi}\frac{1}{\chi(\mathscr{H})}G(n,\chi)$$
$$= -\frac{1}{2\pi i h(\mathfrak{f})}\sum_{\chi}\frac{1}{\chi(\mathscr{H})}\int_{b-iT}^{b+iT}\frac{\zeta'}{\zeta}(s,\chi)\frac{x^s}{s}\,ds + O(\nu T^{-1}x\log^3 x).$$

Therefore owing to (1.2) we have

$$(4.2)\qquad \Delta(x,\mathscr{H}_1,\mathscr{H}_2) = \frac{1}{2\pi i h(\mathfrak{f})}\sum_{\chi(\mathscr{H}_1)\neq\chi(\mathscr{H}_2)}\left(\frac{1}{\chi(\mathscr{H}_2)}-\frac{1}{\chi(\mathscr{H}_1)}\right)\int_{b-iT}^{b+iT}\frac{\zeta'}{\zeta}(s,\chi)\frac{x^s}{s}\,ds + O(\nu T^{-1}x\log^3 x),$$

where $b = 1+1/\log x$ and the constant implied by the $O$ notation is numerical.

We estimate the integral in (4.2) by the use of Lemma 10. We define the contour of integration $L$ consisting of the following parts

$L_0$: $s = b+it$, $-T\leqslant t\leqslant T$,
$L_1$: $s = \sigma+iT$, $1-\alpha C_0\eta(T)\leqslant\sigma\leqslant b$,
$L_2$: $s = 1-\alpha C_0\eta(t)+it$, $T_0\leqslant t\leqslant T$,
$L_3$: $s = 1-\alpha C_0\eta(T_0)+it$, $-T_0\leqslant t\leqslant T_0$,
$L_4$: $s = 1-\alpha C_0\eta(t)-it$, $T_0\leqslant t\leqslant T$,
$L_5$: $s = \sigma-iT$, $1-\alpha C_0\eta(T)\leqslant\sigma\leqslant b$.

In virtue of I and (2.1) it follows that

$$(4.3)\qquad \int_L \frac{\zeta'}{\zeta}(s,\chi)\frac{x^s}{s}\,ds = 0,\qquad \chi\neq\chi_0.$$

Denote

$$F(s) \stackrel{\text{Def}}{=} \frac{\zeta'}{\zeta}(s,\chi)\frac{x^s}{s}.$$



From Lemma 10 it follows that

$$(4.4)\qquad \int_{L_1} F(s)\,ds = O\Big(C_0^{-1}\nu T^{-1}x\log\big(C_0^{-1}|\Delta|\,\mathfrak{N}\mathfrak{f}(T+2)\big)\log(T+2)\Big).$$

$$(4.5)\qquad \int_{L_2} F(s)\,ds$$
$$= O\Big(C_0^{-1}\nu\int_{T_0}^{T} t^{-1}x^{1-\alpha C_0\eta(t)}\log\big(C_0^{-1}|\Delta|\,\mathfrak{N}\mathfrak{f}(t+2)\big)\log(t+2)\,dt\Big)$$
$$= O\Big(C_0^{-1}\nu x\int_{T_0}^{T} t^{1-\alpha C_0}\exp\big(-\alpha C_0(\eta(t)\log x+\log t)\big)\log\big(C_0^{-1}|\Delta|\,\mathfrak{N}\mathfrak{f}(t+2)\big)\times \log(t+2)\,dt\Big)$$
$$= O\Big(C_0^{-1}\nu x\exp\big(-\alpha C_0\,\omega(x)\big)\cdot\log\big(C_0^{-1}|\Delta|\,\mathfrak{N}\mathfrak{f}(t+2)\big)\log(T+2)\,\frac{T^{\alpha C_0}-1}{\alpha C_0}\Big)$$
$$= O\Big(C_0^{-1}\nu x\exp\big(-\alpha C_0\,\omega(x)\big)\cdot\log\big(C_0^{-1}|\Delta|\,\mathfrak{N}\mathfrak{f}(t+2)\big)\log^2(T+2)\,T^{\alpha C_0}\Big).$$

$$(4.6)\qquad \int_{L_3} F(s)\,ds$$
$$= O\Big(\int_{-T_0}^{T}(|t|+1)^{-1}x^{1-\alpha C_0\eta(T_0)}\log\big(C_0^{-1}|\Delta|\,\mathfrak{N}\mathfrak{f}(|t|+2)\big)\log(|t|+2)\Big)$$
$$= O\Big(C_0^{-1}\nu x\exp\big(-\alpha C_0\,\omega(x)\big)\log\big(C_0^{-1}|\Delta|\,\mathfrak{N}\mathfrak{f}(T+2)\big)\log(T+2)\Big).$$

Similarly we estimate the integrals over $L_4$ and $L_5$. The constants implied by the $O$-notation depend only on $\alpha$ and $\eta(t)$. Owing to (4.4)–(4.6) and (4.3), (4.2) we get

$$(4.7)\qquad \Delta(x,\mathscr{H}_1,\mathscr{H}_2) = O\Big(C_0^{-1}\nu T^{-1}x\log\big(C_0^{-1}|\Delta|\,\mathfrak{N}\mathfrak{f}(T+2)\big)\log(T+2)\Big) + O\Big(C_0^{-1}\nu T^{\alpha C_0}x\exp\big(-\alpha C_0\,\omega(x)\big)\log\big(C_0^{-1}|\Delta|\,\mathfrak{N}\mathfrak{f}(T+2)\big)\log^2(T+2)\Big) + O(\nu T^{-1}x\log^3 x).$$

Putting

$$T \stackrel{\text{Def}}{=} \exp\big(\tfrac{1}{4}\,\omega(x)\big)$$

we have

$$(4.8)\qquad \Delta(x,\mathscr{H}_1,\mathscr{H}_2) = O\Big(C_0^{-1}\nu x\exp\big(-\tfrac{1}{4}\,\omega(x)\big)\log\big(C_0^{-1}|\Delta|\,\mathfrak{N}\mathfrak{f}+2\big)\,\omega^2(x)\Big) + O\Big(C_0^{-1}\nu x\exp\big(-\tfrac{3}{4}\alpha C_0\,\omega(x)\big)\log\big(C_0^{-1}|\Delta|\,\mathfrak{N}\mathfrak{f}+2\big)\,\omega^3(x)\Big) + O\Big(\nu x\exp\big(-\tfrac{1}{4}\,\omega(x)\big)\log^3 x\Big).$$

Under $C_0\leqslant 1/2$, $x\geqslant\omega^{-1}\big(24(\alpha C_0)^{-1}\log\big(24(\alpha C_0)^{-1}\big)\big)$ and the definition of $\eta(t)$ — condition (c) — we get (2.2) and Theorem 1 is thus proved.

**5. Proof of Theorem 2** (compare [9], pp. 149–157). Let $t \geq 2$. From (1.1), (1.2) it follows

$$\gamma(n, \mathscr{H}) - \gamma(n, \mathscr{H}_0) = \Delta(n, \mathscr{H}, \mathscr{H}_0) - \Delta(n-1, \mathscr{H}, \mathscr{H}_0).$$

Hence

$$(5.1) \quad \Big|\sum_{N_1 \leq n \leq N_2} \big(\gamma(n, \mathscr{H}) - \gamma(n, \mathscr{H}_0)\big)\exp(-it\log n)\Big| \leq |\Delta(N_2, \mathscr{H}, \mathscr{H}_0)| + |\Delta(N_1-1, \mathscr{H}, \mathscr{H}_0)| + \sum_{N_1 \leq n \leq N_2-1} |\Delta(n, \mathscr{H}, \mathscr{H}_0)| \Big|1 - \exp\Big(-it\log\Big(1+\frac{1}{n}\Big)\Big)\Big|.$$

Let $N_1, N_2$ be as large that

$$(5.2) \quad \max\{\omega_1^{-1}(\log t^{4(aC_0)^{-1}}), \varphi(C_0)\} \leq \frac{N}{2} < N_1 < N_2 \leq N.$$

But

$$(5.3) \quad \omega_1(1+t^2) < \log(1+t^2) < \log(t^{4(aC_0)^{-1}})$$

and owing to (2.3) we get

$$(5.4) \quad \Big|\sum_{N_1 \leq n \leq N_2} \big(\gamma(n, \mathscr{H}) - \gamma(n, \mathscr{H}_0)\big)\exp(-it\log n)\Big| \leq C_{17} MNt^{-1},$$

where

$$M = C_0^{-1}\nu\log(C_0^{-1}|\Delta|\mathfrak{N}\mathfrak{f}+2).$$

The constant $C_{17}$ and further constants depend on $a$ and $\eta_1(t)$ at most.

Suppose

$$(5.5) \quad 1 < \sigma \leq 3/2.$$

By partial summation and (5.4) we have

$$(5.6) \quad \Big|\sum_{N_1 \leq n \leq N_2} \big(\gamma(n, \mathscr{H}) - \gamma(n, \mathscr{H}_0)\big)n^{-s}\Big| \leq C_{18} MNt^{-1}.$$

We choose

$$(5.7) \quad \eta \geq \max\{\omega_1^{-1}(\log t^{4(aC_0)^{-1}}), \varphi(C_0)\}$$

and apply the inequality (5.6) for

$$N_1^j = \eta \cdot 2^j, \quad N_2^j = \eta \cdot 2^{j+1}, \quad j = 0, 1, 2, \ldots$$

Therefore

$$(5.8) \quad \Big|\sum_{n \geq \eta} \big(\gamma(n, \mathscr{H}) - \gamma(n, \mathscr{H}_0)\big)n^{-s}\Big| \leq C_{19} Mt^{-1}(\sigma-1)^{-1}\eta^{1-\sigma}.$$



We choose further

$$(5.9) \quad \xi \geq \max\{\omega_1^{-1}(\log t^{4(aC_0)^{-1}}), \varphi(C_0)\}.$$

Denoting by $l$ a positive integer, $l \geq 2$ and following [9], p. 154, we get

$$(5.10) \quad \Big|\sum_{n \geq \xi} \big(\gamma(n, \mathscr{H}) - \gamma(n, \mathscr{H}_0)\big)n^{-s}\log^{l+1}\Big(\frac{n}{\xi}\Big)\Big| < C_{20} Mt^{-1}(\sigma-1)^{-(l+2)}(l+1)!\,\xi^{1-\sigma}.$$

Hence by Lemma 7 and (5.3), (5.9) we get the estimate

$$(5.11) \quad \Big|\sum_{\chi(\mathscr{H}) \neq 1} \big(1-\overline{\chi}(\mathscr{H})\big) \sum_{\varrho = \varrho(\chi)} \frac{\xi^{\varrho - s}}{(\varrho - s)^{l+2}}\Big| < C_{21} M_1 \frac{\xi^{1-\sigma}}{t(\sigma-1)^{l+2}} \log^2 t,$$

where $M_1 = C_0 h(\mathfrak{f}) \cdot M^2$ and $\varrho = \varrho(\chi)$ runs through all roots of $\zeta(s, \chi)$ lying to the right of the broken line $L$ from Lemma 6.

Let us suppose now that our theorem is not true. Hence there exist such roots

$$\varrho^* = \sigma^* + it^*, \quad t^* \to \infty$$

of the function $\prod_{\chi(\mathscr{H}) \neq 1} \zeta(s, \chi)$ that then

$$(5.12) \quad \sigma^* > 1 - \frac{\log t^*}{400\log\omega_1^{-1}(\log t^{*4(aC_0)^{-1}})},$$

$$(5.13) \quad t^* > \max\Big\{e^{23}, A(C_0, K), \eta_1^{-1}\big(\exp(-\nu^2 h^2(\mathfrak{f}))\big), (|\Delta|+2)\mathfrak{N}\mathfrak{f}, \exp\Big(\frac{aC_0}{4}\omega_1(\varphi(C_0))\Big)\Big\} = T_0.$$

Putting in (5.11)

$$(5.14) \quad s = s_1 = 1 + \frac{\log t^*}{10\log\omega_1^{-1}(\log t^{*4(aC_0)^{-1}})} + it^* = \sigma_1 + it^*,$$

$$(5.15) \quad \xi = \exp\big((l+2)\lambda\big),$$

where

$$(5.16) \quad \log t^* \leq l+2 \leq \tfrac{5}{4}\log t^*,$$

$$(5.17) \quad \lambda = \frac{\log\omega_1^{-1}(\log t^{*4(aC_0)^{-1}})}{\log t^*},$$

it follows without difficulty that then (5.5) and (5.9) are satisfied. Multiplying both sides of (5.11) by

$$|\xi^{\varrho_1 - \varrho^*}(s_1 - \varrho^*)^{l+2}| = \xi^{\sigma_1 - \sigma^*}(\sigma_1 - \sigma^*)^{l+2},$$

we have

$$(5.18)\qquad \Big|\sum_{\chi}\big(1-\bar\chi(\mathscr{H})\big)\sum_{\varrho=\varrho(\chi)}\xi^{\varrho-\varrho^*}\Big(\frac{s_1-\varrho^*}{s_1-\varrho}\Big)^{l+2}\Big| < C_{22}t^{*-2/3}\xi^{1-\sigma^*},$$

with (5.13) and $t^*>C_{23}(C_4)$.

Similarly to [9], p. 156, by the use of Lemma 5, we get from (5.18) the estimate

$$(5.19)\quad V=\Big|\sum_{\chi}\big(1-\bar\chi(\mathscr{H})\big)\sum_{\substack{\varrho=\varrho(\chi)\\ |t_\varrho(\chi)-t^*|<6(\sigma_1-\sigma^*)\\ \sigma_\varrho(\chi)>1-3(\sigma_1-\sigma^*)}}\xi^{\varrho-\varrho^*}\Big(\frac{s_1-\varrho^*}{s_1-\varrho}\Big)^{l+2}\Big| < t^{*-2/3}\xi^{1-\sigma^*}$$

with $t^*>C_{24}(C_4)$.

We estimate the sum $V$ from below by the use of Lemma 1. We choose

$$z_j=\frac{s_1-\varrho^*}{s_1-\varrho}\exp\big(\lambda(\varrho-\varrho^*)\big),$$

$$(5.20)\qquad m=\log t^*,\qquad \mu=\frac{\log t^*}{10\log\omega_1^{-1}(\log t^{*4(aC_0)^{-1}})}.$$

We have now to determine the number $N$. It is easily to notice that the region

$$1-3(\sigma_1-\sigma^*)\leqslant\sigma<1,\qquad |t-t^*|\leqslant 6(\sigma_1-\sigma^*)$$

is contained in the circle $|s-s_1|\leqslant 8(\sigma_1-1)$.

If we denote by $N_1$ the number of roots of the function $\prod_{\chi(\mathscr{H})\neq 1}\zeta(s,\chi)$ in this circle and if $C_2\leqslant\exp\big(-(28C_{11})^2\big)$ then from Lemma 4 and owing to (5.13), we get

$$N_1<\frac{1}{14}\log t^*$$

with $t^*>\max\big(T_0, C_{25}(C_4)\big)$.

Therefore we can determine $N$ from Lemma 1 putting

$$(5.21)\qquad N=\frac{1}{14}\log t^*.$$

The numbers $b_j$ from Lemma 1, are in our case of the form

$$b_j=1-\chi(\mathscr{H}).$$

It can easily be proved that

$$\min_{1\leqslant j\leqslant h}|b_1+b_2+\ldots+b_j|\geqslant\min_{1\leqslant j\leqslant h}\mathrm{Re}(b_1+b_2+\ldots+b_j)\geqslant\frac{8}{h^2(\mathfrak{f})}.$$



In virtue of Lemma 1, there exists an exponent $l+2$, that then

$$(5.22)\qquad V>t^{*-0.66}.$$

By comparision of (5.19) and (5.22) we get

$$1-\sigma^*>\frac{\log t^*}{400\log\omega_1^{-1}(\log t^{*4(aC_0)^{-1}})}$$

contrary to the assumption (5.12). This completes the proof of Theorem 2.

The proof of Theorem 3 easily follows from the inequality (8.5) stated in [7].

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