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A note on a paper of Franklin

by

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1. Introduction. Let $\wp(z)$ be a Weierstrass elliptic function with complex multiplication and algebraic invariants. Denote by ω, ω' a fundamental pair of periods with $\mathrm{Im}(\omega'/\omega) > 0$, and suppose η, η' are the corresponding quasi-periods of the associated Weierstrass zeta function. Let $\log\sigma$ be an arbitrary determination of a non-zero algebraic number σ. In [2] Franklin attempted to prove that any non-zero linear combination of the numbers $\omega, \omega', \eta, \eta', 2\pi i, \log\sigma$ is transcendental. Unfortunately his proof appears to be invalidated (1) by an error on p. 205. While it is true that the exponential polynomial (23) possesses at least simple zeros at the appropriate points, it seems difficult to obtain information about its derivatives and hence about the multiplicities of these zeros. In this note we complete the proof by using the techniques of [3] (in fact all the ideas required can be found in early papers of Feldman).

The result of Coates in [1] shows that we forfeit no generality by assuming that σ is not a root of unity. Accordingly we shall prove the following theorem.

THEOREM. *If σ is not a root of unity the numbers $1, \omega, \eta, 2\pi i$ and $\log\sigma$ are linearly independent over the field A of algebraic numbers.*

This includes the assertion of Franklin, for it was shown in [3] that the vector space spanned over A by $\omega, \omega', \eta, \eta'$ is actually spanned by ω and η alone. At the same time it extends Theorem III of [3] by adjoining the number $\log\sigma$.

Thus we assume the existence of algebraic numbers $\alpha, \beta, \gamma, \delta \neq 0, \theta$ such that

$$\alpha\omega + \beta\eta + \gamma 2\pi i + \delta\log\sigma = \theta. \tag{1}$$

The extrapolation part of the transcendence proof works for any θ, but the particular determinant argument used depends on whether $\theta \neq 0$ or $\theta = 0$. Contrary to usual expectations in this type of work, the latter case is much easier.

(1) This was first pointed out to me by D. Brownawell and P. Cijsouw.

2. The auxiliary function. For independent variables z_1, z_2, z_3 we put

$$f = f(z_1, z_2, z_3) = a\omega z_1 + \beta\zeta(\omega z_1) + \gamma\cdot 2\pi i z_2 + \delta\log\sigma\cdot z_3,$$

and for an integer k and a small absolute constant ε we write

$$L = L_0 = L_1 = L_2 = [k^{1-2\varepsilon}], \quad L_3 = [k^{8\varepsilon}].$$

Then our auxiliary function $\Phi = \Phi(z_1, z_2, z_3)$ is given by

$$\Phi = \sum_{\lambda_0=0}^{L_0}\sum_{\lambda_1=0}^{L_1}\sum_{\lambda_2=0}^{L_2}\sum_{\lambda_3=0}^{L_3} p(\lambda_0, \ldots, \lambda_3)\big(\wp(\omega z_1)\big)^{\lambda_1} e^{2\pi i\lambda_2 z_2}\sigma^{\lambda_3 z_3} f^{\lambda_0}$$

where naturally σ^z denotes $\exp(z\log\sigma)$. For a differential operator (or just operator, for short)

$$\partial = (\partial/\partial z_1)^{m_1}(\partial/\partial z_2)^{m_2}(\partial/\partial z_3)^{m_3}$$

of order

$$|\partial| = m_1 + m_2 + m_3$$

we put

$$A(z, \partial) = \omega^{-m_1}(2\pi i)^{-m_2}(\log\sigma)^{-m_3}\partial\Phi(z, z, z).$$

We denote by $c, c_1, \ldots$ positive constants depending only on ω', η' and the numbers appearing in (1). Furthermore we assume c is so large that the following lemmas hold for $k > c$. We shall be as brief as possible in the proofs when these are on familiar lines, our principal aim being rather to direct the reader's attention to any unfamiliar features.

LEMMA 1. *There are rational integers $p(\lambda_0, \ldots, \lambda_3)$, not all zero, with absolute values at most $k^{c_1 k}$, such that*

$$A(s+1/2, \partial) = 0$$

for all integers s with $1 \leqslant s \leqslant k^\varepsilon$ and all operators ∂ with $|\partial| < k$.

Proof. Because L_3 is so small the term $\sigma^{\lambda_3 z_3}$ contributes nothing of importance to the estimates. Also

$$\prod_{i=0}^{3}(L_i+1) > k^{3+2\varepsilon},$$

so that the unknowns $p(\lambda_0, \ldots, \lambda_3)$ are sufficiently numerous to be determined from Siegel's version of the Box Principle.

Next we use the Weierstrass sigma function $\sigma(z)$ corresponding to $\wp(z)$; clearly

$$\varphi(z_1, z_2, z_3) = \big(\sigma(\omega z_1)\big)^{3L}\Phi(z_1, z_2, z_3)$$

is an entire function.

LEMMA 2. *For any $R \geqslant 1$ and any operator ∂ with $|\partial| < k$ we have*

$$|\partial\varphi(z, z, z)| < k^{c_2 k} c_2^{LR^2}$$

whenever $|z| \leqslant R$.

Proof. Again the growth of the exponential terms of Φ is negligible beside the growth of the elliptic functions, and the estimate proceeds on well-established lines.

LEMMA 3. *Suppose q, r, s are integers with q even, $(r, q) = 1$ and*

$$1 \leqslant r < q \leqslant k, \quad 1 \leqslant s \leqslant k^2.$$

Then if ∂ is an operator with $|\partial| < k$ such that $A(s+r/q, \partial) \neq 0$, we have

$$|A(s+r/q, \partial)| > k^{-c_3 kq^3} c_3^{-L_3 sq^3}.$$

Proof. As s becomes large the terms $\sigma^{\lambda_3 z_3}$ start to make their presence felt in the size and denominator of $A(s+r/q, \partial)$ to the extent suggested in the lemma. As Franklin points out, the degree of this algebraic number does not exceed $c_4 q^3$, and the lower bound follows immediately.

LEMMA 4. *Let n be any integer with $0 \leqslant n \leqslant 7/\varepsilon$. Then*

$$A(s+r/q, \partial) = 0$$

for all integers q, r, s with q even, $(r, q) = 1$

$$1 \leqslant r < q \leqslant 2k^{n\varepsilon/8}, \quad 1 \leqslant s \leqslant k^{\varepsilon+n\varepsilon/4},$$

and all operators ∂ with $|\partial| < k/2^n$.

Proof. This is true for $n = 0$ by Lemma 1. Let m be an integer with $0 \leqslant m < [7/\varepsilon]$ such that the lemma is true for $n = m$. For a proof by induction it will suffice to obtain a contradiction from the existence of a counterexample

$$A' = A(s'+r'/q', \partial') \neq 0$$

to the lemma for $n = m+1$. As usual, $|\partial'|$ is supposed minimal for this particular choice of q', r', s'. We put

$$Q_n = 2k^{n\varepsilon/8}, \quad S_n = k^{\varepsilon+n\varepsilon/4}, \quad T_n = [k/2^n];$$

then with

$$f(z) = \partial'\varphi(z, z, z)$$

we easily find

$$\Delta f(s+r/q) = 0$$

for all $\Delta = (d/dz)^t$ with $0 \leqslant t < T_{m+1}$ and all triples of integers q, r, s satisfying the conditions of Lemma 4 for $n = m$. If K_m is the total number of such triples, the maximum modulus principle gives in the usual way

$$|f(z)| < k^{c_5 k} c_5^{LS_{m+1}^2} 4^{-T_{m+1}K_m}$$

whenever $|z| \leqslant 2S_{m+1}$. But $K_m > c_6^{-1} S_m Q_m^2$ and $T_{m+1} > c_6^{-1} k$, whence

$$(2) \qquad T_{m+1}K_m > c_7^{-1} k^{1+\varepsilon+m\varepsilon/2}, \qquad LS_{m+1}^2 < c_7 k^{1+\varepsilon/2+m\varepsilon/2},$$

and so

$$|f(s'+r'/q')| < 3^{-T_{m+1}K_m}.$$

From the minimality of $|\partial'|$ we deduce

$$|A'| < 2^{-T_{m+1}K_m}$$

whereas Lemma 3 gives the opposite inequality

$$|A'| > k^{-c_8 k Q_{m+1}^3} c_8^{-L_3 S_{m+1} Q_{m+1}^3}.$$

The inconsistency of these bounds is readily seen from the calculations (2) together with the inequalities

$$kQ_{m+1}^3 < c_9 k^{1+3\varepsilon/8+3m\varepsilon/8}$$

and

$$L_3 S_{m+1} Q_{m+1}^3 < c_{10} k^{10\varepsilon+5m\varepsilon/8} < c_{10} k^{7/8+10\varepsilon+m\varepsilon/2}$$

after recalling that $m < 7/\varepsilon$. This completes the proof of Lemma 4.

3. Completion of the proof for $\theta \neq 0$. Let $\mathscr{E}$ be the set of points close to 1/4 modulo the periods of $\wp(\omega z_1)$ in the sense of Lemma 1.4 of [3]. For brevity we write $\mathscr{E}(r)$ for the points of $\mathscr{E}$ with absolute values at most r.

LEMMA 5. *For all z in $\mathscr{E}(k^{3/2})$ and all operators ∂ with $|\partial| \leqslant L_3$ we have*

$$|\partial\Phi(z, z, z)| < e^{-k^{9/2}}.$$

Proof. Let $n = [7/\varepsilon]$ in Lemma 4, and let ∂ be any operator of order at most L_3. Then with

$$f(z) = \partial\varphi(z, z, z)$$

we find that

$$\Delta f(s+r/q) = 0$$

for $\Delta = (d/dz)^t$ with $0 \leqslant t < T_{n+1}$ and all triples q, r, s of integers satisfying the conditions of Lemma 4 with this value of n. As before, the maximum modulus principle gives

$$|f(z)| < 3^{-T_{n+1}K_n}$$

whenever $|z| \leqslant 2S_{n+1}$. The lemma now follows from the inequalities

$$k^{3/2} < 2S_{n+1}, \qquad T_{n+1}K_n > c_{11}^{-1} k^{9/2+\varepsilon/2},$$

and the elementary lower bounds for the sigma function established in Lemma 7.1 of [3].

For $0 \leqslant \lambda_3 \leqslant L_3$ we set

$$F(\lambda_3) = F(\lambda_3, z_1, z_2, z_3) = \sum_{\lambda_0=0}^{L_0} \sum_{\lambda_1=0}^{L_1} \sum_{\lambda_2=0}^{L_2} p(\lambda_0, \ldots, \lambda_3)\big(\wp(\omega z_1)\big)^{\lambda_1} e^{2\pi i \lambda_2 z_2} f^{\lambda_0}.$$

The functions appearing in $F(\lambda_3)$ are algebraically independent because $\delta \log\sigma \neq 0$, and so by Lemma 1 at least one of the functions $F(0), \ldots, F(L_3)$ is not identically zero. Let $M \leqslant L_3$ be the non-negative integer such that $r_0, \ldots, r_M$ are all the distinct values of λ_3 for which $F(\lambda_3)$ is not identically zero. Thus

$$\Phi = \sum_{\mu=0}^{M} F(r_\mu)\sigma^{r_\mu z_3}.$$

For $0 \leqslant \nu \leqslant M$ we put

$$\Phi(\nu) = \Phi(\nu, z_1, z_2, z_3) = \sum_{\mu=0}^{M} (r_\mu \log\sigma)^\nu F(r_\mu)\sigma^{r_\mu z_3},$$

and we define the functions

$$G(\mu, \nu) = G(\mu, \nu, z_1, z_2, z_3)$$

by the relations

$$G(\mu, 0) = F(r_\mu) \qquad (0 \leqslant \mu \leqslant M)$$

$$G(\mu, \nu+1) = (\partial/\partial z_3)G(\mu, \nu) + r_\mu \log\sigma \cdot G(\mu, \nu) \qquad (0 \leqslant \nu < M).$$

Then Lemma 2.1 of [3] shows that the Wronskian $W = W(z_1, z_2, z_3)$ of the functions $\Phi(0), \ldots, \Phi(M)$ with respect to z_3 is given by

$$W = \Xi e^{\xi z_3} \det_{0 \leqslant \mu, \nu \leqslant M} G(\mu, \nu)$$

where

$$\Xi = (\log\sigma)^{M(M+1)/2} \prod_{0 \leqslant \nu < \mu \leqslant M} (r_\mu - r_\nu)$$

and

$$\xi = (\log\sigma) \sum_{\mu=0}^{M} r_\mu.$$

It follows without much difficulty that

$$\Psi = \Psi(z_1, z_2, z_3) = (\log\sigma)^{-M(M+1)/2} \Xi^{-1} e^{-\xi z_3} W(z_1, z_2, z_3)$$

is a polynomial in $\wp(\omega z_1)$, $e^{2\pi i z_2}$ and f of degree at most

$$N \leqslant c_{12} L L_3.$$

Furthermore its coefficients are algebraic numbers of a fixed field and have sizes and denominators not exceeding $k^{c_{13}kL_3}$.

LEMMA 6. *For all z in $\mathcal{E}(k^{3/2})$ we have*

$$|\Psi(z, z, z)| < e^{-k^4}.$$

Proof. Let z be in $\mathcal{E}(k^{3/2})$; it is clearly enough to prove that $|W(z, z, z)| < e^{-k^{17/4}}$. But the first row of this Wronskian is formed from the numbers $(\partial/\partial z_3)^{\mu}\Phi(z, z, z)$ whose absolute values do not exceed $e^{-k^{9/2}}$ as a consequence of the preceding lemma. The remaining entries can be estimated by noting that since z stays away from the poles of $\wp(\omega z_1)$ we have

$$|\Phi(\nu, z, z, z_3)| < c_{14}^{Lk^{3/2}}$$

for any complex number z_3 such that $|z_3 - z| = 1$. We then employ Cauchy's integral formula to differentiate with respect to z_3 as in the proof of Lemma 2.10 of [3]. Thus

$$|W(z, z, z)| < c_{15}^{LL_3k^{3/2}} e^{-k^{9/2}} < e^{-k^{17/4}}.$$

Our eventual aim is to show that Ψ vanishes identically using only the inequalities of Lemma 6 and the algebraic nature of its coefficients. We can write

$$\Psi = \sum_{\nu_0=0}^{N}\sum_{\nu_1=0}^{N}\sum_{\nu_2=0}^{N} q(\nu_0, \nu_1, \nu_2)\big(\wp(\omega z_1)\big)^{\nu_1} e^{2\pi i\nu_2 z_2} f^{\nu_0}$$

from which we extract the simpler functions

$$\Omega(\nu_0) = \Omega(\nu_0, z_1, z_2) = \sum_{\nu_1=0}^{N}\sum_{\nu_2=0}^{N} q(\nu_0, \nu_1, \nu_2)\big(\wp(\omega z_1)\big)^{\nu_1} e^{2\pi i\nu_2 z_2} \qquad (0 \leqslant \nu_0 \leqslant N)$$

so that

$$\Psi = \sum_{\nu_0=0}^{N} \Omega(\nu_0) f^{\nu_0}.$$

LEMMA 7. *For all z in $\mathcal{E}(k^{5/4})$ and all ν_0 we have*

$$|\Omega(\nu_0, z, z)| < e^{-k^3}.$$

Proof. Let z be a fixed point of $\mathcal{E}(k^{5/4})$; then $z(m) = z + m$ lies in $\mathcal{E}(k^{3/2})$ for all non-negative integers $m \leqslant N$. Also if $x = f(z, z, z)$ the relation (1) implies that

$$f\big(z(m), z(m), z(m)\big) = x + m\theta$$

whence the polynomial

$$Q(X) = \sum_{\nu_0=0}^{N} \Omega(\nu_0, z, z) X^{\nu_0}$$

satisfies

$$|Q(x+m\theta)| = \big|\Psi\big(z(m), z(m), z(m)\big)\big| < e^{-k^4} \qquad (0 \leqslant m \leqslant N).$$

Now if $\theta \neq 0$ the numbers $x + m\theta$ are distinct; furthermore

$$|x + m\theta| \leqslant |x| + N|\theta| < c_{16}k^{5/4}$$

and thus the interpolation formula of Lagrange (see Lemma 1.3 of [3] for a convenient version) immediately gives

$$|\Omega(\nu_0, z, z)| < (c_{17}k^{5/4})^N e^{-k^4} < e^{-k^3}$$

for all ν_0.

The treatment of $\Omega(\nu_0)$ is roughly similar. We observe that for any non-negative integers n, m' not exceeding N the point

$$z(n, m') = 1/4 + n/N^2 + m'\omega'/\omega$$

lies in $\mathcal{E}(k^{5/4})$. For fixed n we put

$$r(\nu_0, \nu_2) = \sum_{\nu_1=0}^{N} q(\nu_0, \nu_1, \nu_2)\big(\wp(\omega/4 + n\omega/N^2)\big)^{\nu_1} \qquad (0 \leqslant \nu_0, \nu_2 \leqslant N).$$

It follows that the polynomial

$$R(X) = \sum_{\nu_2=0}^{N} r(\nu_0, \nu_2) X^{\nu_2}$$

satifies

$$\big|R\big(x(m')\big)\big| = \big|\Omega\big(\nu_0, z(n, m'), z(n, m')\big)\big| < e^{-k^3} \qquad (0 \leqslant m' \leqslant N)$$

with

$$x(m') = e^{2\pi i z(n, m')}.$$

Now

$$|x(m')| = t^{m'}$$

where

$$t = \exp\big(-2\pi \operatorname{Im}(\omega'/\omega)\big) < 1;$$

in particular the $x(m')$ are distinct and have absolute values at most unity, while for $m_1' < m_2'$ clearly

$$|x(m_1') - x(m_2')| \geqslant |x(m_1')| - |x(m_2')| = t^{m_1'} - t^{m_2'} \geqslant c_{18}^{-N}.$$

Thus again from the interpolation formula we conclude that

$$|r(\nu_0, \nu_2)| < c_{19}^{N^2} e^{-k^3} < e^{-k^2}$$

for all ν_0, ν_2. Finally on considering the polynomial

$$\sum_{\nu_1=0}^{N} q(\nu_0, \nu_1, \nu_2) X^{\nu_1}$$

at the points

$$X = \wp(\omega/4 + n\omega/N^2) \qquad (0 \leqslant n \leqslant N)$$

it is evident that

$$|q(\nu_0, \nu_1, \nu_2)| < e^{-k^{3/2}}$$

for all ν_0, ν_1, ν_2. From the algebraic properties of these coefficients recorded earlier this immediately forces their vanishing and therefore the identical vanishing of Ψ.

Hence W also vanishes identically, which means that there exist functions $H(\nu) = H(\nu, z_1, z_2)$ independent of z_3 and not all zero, such that

$$\sum_{\nu=0}^{M} H(\nu)\Phi(\nu) = 0.$$

Thus we have

$$\sum_{\mu=0}^{M} K(\mu) F(r_\mu) \sigma^{r_\mu z_3} = 0$$

where

(3) $$K(\mu) = K(\mu, z_1, z_2) = \sum_{\nu=0}^{M} (r_\mu \log\sigma)^\nu H(\nu).$$

But since σ^{z_3} is not an algebraic function of z_3, this compels the vanishing of the coefficients $K(\mu)F(r_\mu)$ and hence of all the functions $K(\mu)$. Now to (3) is associated a non-zero Vandermonde determinant. Therefore all the functions $H(\nu)$ are identically zero, and this contradiction concludes the proof of the impossibility of (1) when $\theta \neq 0$.

4. Completion of the proof for $\theta = 0$. All we require from Lemma 4 are the equations

(4) $$\partial\Phi(s+1/4, s+1/4, s+1/4) = 0$$

for all integers s with $1 \leqslant s \leqslant L_3+1$ and all operators ∂ of order at most $3L$. Recall that

$$\Phi = \sum_{\lambda_3=0}^{L_3} F(\lambda_3)\sigma^{\lambda_3 z_3}.$$

Now

$$f(z, z, z) = \beta\big(\zeta(\omega z) - \eta z\big)$$

has period 1; thus $F(\lambda_3, z, z, z)$ also has period 1 and it follows that for any integer s

$$\Phi(s+1/4, s+1/4, s+1/4) = \sum_{\lambda_3=0}^{L_3} F(\lambda_3, 1/4, 1/4, 1/4)\sigma^{\lambda_3(s+1/4)}.$$

Therefore from (4) with the identity operator ∂ the polynomial

$$S(X) = \sum_{\lambda_3=0}^{L_3} F(\lambda_3, 1/4, 1/4, 1/4) X^{\lambda_3}$$

of degree at most L_3 satisfies

$$S(\sigma^{s+1/4}) = 0 \qquad (1 \leqslant s \leqslant L_3+1).$$

Since σ is not a root of unity the numbers $\sigma^{s+1/4}$ are distinct and so

$$F(\lambda_3, 1/4, 1/4, 1/4) = 0$$

for all λ_3. Now assume that $m < 3L$ is a non-negative integer such that

(5) $$\partial F(\lambda_3, 1/4, 1/4, 1/4) = 0$$

for all λ_3 and all operators ∂ of order at most m. Let $|\partial'| = m+1$; then for all integers s as above

$$\partial'\Phi(s+1/4, s+1/4, s+1/4) = \sum_{\lambda_3=0}^{L_3} \partial' F(\lambda_3, 1/4, 1/4, 1/4)\sigma^{\lambda_3(s+1/4)}$$

and a similar argument with (4) for $\partial = \partial'$ shows that

$$\partial' F(\lambda_3, 1/4, 1/4, 1/4) = 0$$

for all λ_3. Hence by induction (5) holds for all λ_3 and all operators ∂ of order at most $3L$.

The final contradiction now follows from a reformulation of the original argument of Baker. We can write $F(\lambda_3)$ as a polynomial $P(\lambda_3)$ in the functions

$$f_1 = \wp(\omega z_1), \qquad f_2 = e^{2\pi i z_2}, \qquad f_0 = f(z_1, z_2, z_3).$$

The Jacobian matrix of f_1, f_2, f_0 with respect to z_1, z_2, z_3 at $z_1 = z_2 = z_3 = 1/4$ is readily verified to be triangular with diagonal elements

$$\omega\wp'(\omega/4), \quad -2\pi, \quad \delta\log\sigma.$$

It is therefore non-singular; and so z_1, z_2, z_3 are locally holomorphic functions of f_1, f_2, f_0 at the point

$$f_1 = \wp(\omega/4), \quad f_2 = i, \quad f_0 = f(1/4, 1/4, 1/4).$$

It follows from (5) that the polynomial $P(\lambda_3)$ has a zero of order at least $3L+1$ at this point. However, its total degree does not exceed $3L$, whence $P(\lambda_3)$ vanishes identically for all λ_3. This contradicts the choice of coefficients made at the outset, and thereby completes the proof.

The same techniques will establish the linear independence over A of $1, \omega, \eta$ and $\log\sigma$ without any hypotheses of complex multiplication.

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Further applications of Turán's methods to the distribution of prime ideals in ideal classes mod $\mathfrak{f}$

by

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1. Let K be an algebraic number field, ν and Δ degree and discriminant of the field K respectively, $\mathfrak{f}$ a given ideal of K, $\mathfrak{N}\mathfrak{a}$ the norm of an ideal $\mathfrak{a}$ of K and $\mathfrak{p}$ a prime ideal of K (see [2]).

Denote further by $\mathscr{H}$ (mod $\mathfrak{f}$) an ideal-class mod $\mathfrak{f}$ ([3], Def. VIII), by $\mathscr{H}_0$ (mod $\mathfrak{f}$) the principal class mod $\mathfrak{f}$ and by $h(\mathfrak{f})$ the class-number. Let $\chi(\mathscr{H})$ be a character of the abelian group of ideal-classes $\mathscr{H}$ (mod $\mathfrak{f}$), $\chi(\mathfrak{a})$ the extension of $\chi(\mathscr{H})$ ([3], Def. X) and χ_0 — the principal character mod $\mathfrak{f}$.

Denote by $\zeta_K(s)$ the Dedekind Zeta-function and by $\zeta(s, \chi)$ the Hecke–Landau Zeta-functions ([3], Def. XVII).

Denote further

$$\gamma(n, \mathscr{H}) = \sum_{(\mathfrak{N}\mathfrak{p})^m = n, \mathfrak{p}^m \in \mathscr{H} (\mathrm{mod}\ \mathfrak{f})} \log \mathfrak{N}\mathfrak{p},$$

(1.1) $$\psi(x, \mathscr{H}) = \sum_{n \leqslant x} \gamma(n, \mathscr{H}),$$

(1.2) $$\Delta(x, \mathscr{H}_1, \mathscr{H}_2) = \psi(x, \mathscr{H}_1) - \psi(x, \mathscr{H}_2).$$

2. In this paper we shall establish an exact correspondence between the order of magnitude of the expressions (1.2) and the regions in which some $\zeta(s, \chi)$-functions do not vanish (compare [8] and [9], Th. XXXVI). In the following C_i, $i = 1, 2, \ldots$ denote positive constants independent of K.

Theorem 1. *Suppose* $\mathscr{H}_1, \mathscr{H}_2$ *denote any fixed ideal-classes* (mod $\mathfrak{f}$), $\mathscr{H}_1 \neq \mathscr{H}_2$. *If* $h(\mathfrak{f}) \geqslant 2$,

$$\prod_{\chi, \chi(\mathscr{H}_1) \neq \chi(\mathscr{H}_2)} \zeta(s, \chi) \neq 0$$

in the region

(2.1) $$\sigma > 1 - C_0 \eta(|t|), \quad 0 < C_0 \leqslant \tfrac{1}{2},$$