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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

New proofs of a theorem of Edmund Landau

by

BŘETISLAV NOVÁK (Praha)

1. Introduction. Let $Q(u) = \sum_{j,l=1}^{r} a_{jl} u_j u_l$ be a positive definite quadratic form with an integral symmetric coefficient matrix of determinant D. Further, let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be real numbers and let $M_1 > 0$, $M_2 > 0, \ldots, M_r > 0$, $b_1, b_2, \ldots, b_r$ be integers. For $x \geqslant 0$ denote by $A(x)$ the sum

$$\sum e^{2\pi i \sum_{j=1}^{r} \alpha_j u_j} = \sum_{n \leqslant x} a_n,$$

where the summation runs over all r-tuples $u = (u_1, u_2, \ldots, u_r)$ of real numbers such that $Q(u) \leqslant x$ and $u_j \equiv b_j \pmod{M_j}$, $j = 1, 2, \ldots, r$. Let

$$V(x) = \frac{\pi^{r/2} x^{r/2}}{\sqrt{D}\,\Gamma(\frac{1}{2}r+1)}$$

be the volume of the ellipsoid $Q(u) \leqslant x$ and let

$$P(x) = A(x) - \frac{e^{2\pi i \sum_{j=1}^{r} \alpha_j b_j}\,\delta}{M_1 M_2 \ldots M_r} V(x) = A(x) - K x^{r/2}$$

be the corresponding lattice remainder term. Here and in the sequel we put $\delta = 1$ if all numbers $\alpha_j M_j$ are integers, and $\delta = 0$ otherwise. The first definitive result in the theory of lattice points in high-dimensional ellipsoids follows from the works of Jarník (see [1], p. 162), Landau ([1], p. 148) and Walfisz ([5]):

If $r > 4$, $\alpha_j = b_j = 0$, $M_j = 1$, $j = 1, 2, \ldots, r$ then

$$P(x) = O(x^{r/2-1}) \tag{1}$$

and

$$P(x) = \Omega(x^{r/2-1}). \tag{2}$$

Moreover, the estimate (1) holds even for rational values of a_j and arbitrary integers b_j, M_j, $M_j > 0$ (Landau and Walfisz loc. cit.) and this result was extended in [2] for arbitrary real numbers a_j. The result (2) was proved for arbitrary rational numbers a_j in the nonsingular case in [6] (for the definition of the nonsingular case see § 2) and by a rather different method in [3].

In the sequel let n be a nonnegative integer. It is very easy to see that (1) and the relation

$$P(n) = O(n^{r/2-1})$$

are equivalent. In 1925, E. Landau (see [1], p. 163) raised the question whether instead of (2) the even stronger result

$$P(n) = \Omega(n^{r/2-1}) \tag{3}$$

holds. Landau ([1], pp. 163–165) solved this problem (using a relatively artificial method) provided $a_j = 0$. Walfisz ([6], pp. 44–46) showed the relation $(r > 4)$

$$\sum_{n\leqslant x} |a_n|^2 = K_1 x^{r-1} + O(x^{r-2}), \tag{4}$$

where K_1 is a positive constant depending only on Q, a_j, b_j and M_j provided that all the numbers a_j are rational and that the nonsingular case takes place. From (4), $a_n = \Omega(n^{r/2-1})$ immediately follows and thus relation (3) also holds.

Some times ago, B. Diviš raised the question whether the estimate (3) can be proved by using the method of [3]. The aim of our remark is to present this proof. It is easy to see that our proof is much simpler than the earlier ones.

2. Theorem and remarks. Let $s = \sigma + it$ still denote a complex number with a positive real part σ. Clearly,

$$\Theta(s) = \sum_{n=0}^{\infty} a_n e^{-ns} = s\int_0^{\infty} e^{-xs} A(x)\,dx \tag{5}$$

is the theta function corresponding to our problem. Let us recall the following basic lemma ([3], p. 266):

Lemma 1. *Let $a_1, a_2, \ldots, a_r$ be rational numbers. Further, let h and k be integers, $k > 0$, $(h, k) = 1$. Then*

$$\sigma^{r/2}\Theta\left(\sigma + \frac{2\pi i h}{k}\right) = \frac{\pi^{r/2} S_{h,k}}{\sqrt{D} M_1 M_2 \ldots M_r k^r} + O(e^{-c/k^2\sigma}) \text{ (1)} \tag{6}$$

for $\sigma \to 0_+$ (the constants involved in the symbol O are of the "type" c), where

$$S_{h,k} = \sum_{a_1, a_2, \ldots, a_r = 1}^{k} e^{-\frac{2\pi i h}{k} Q(a_j M_j + b_j) + 2\pi i \sum_{j=1}^{r} a_j (a_j M_j + b_j)}$$

if all the numbers $a_j M_j k$ are integers, and $S_{h,k} = 0$ otherwise.

We now define the singular case for rational values of a_j by the following conditions:

$$A(x) \not\equiv 0 \text{ and } S_{h,k} = 0 \quad \text{for all integers } h, k \text{ from the lemma.}$$

Let us recall that if $\delta = 1$ we obviously have the nonsingular case ($S_{0,1} \neq 0$).

Remark 1. It is well known that the function $\Theta(-2\pi i\tau)$ represents, for rational a_j, a modular form of dimension $-r/2$. Our lemma also arises from the well-known expansion of $\Theta(-2\pi i\tau)$ in a neighbourhood of the cusp $-h/k$ and our singular case corresponds to non-zero cusp form. For examples see [6].

The following easy lemma of a technical nature will be useful in what follows.

Lemma 2. *Let β be a real number, $\beta \geqslant 1$. Then*

$$\sum_{n=0}^{\infty} n^{\beta} e^{-n\sigma} = \frac{\Gamma(\beta+1)}{\sigma^{\beta+1}} + O\left(\frac{1}{\sigma^{\beta-1}}\right) \tag{7}$$

for $\sigma \to 0_+$. If, in addition, a_n $(n = 0, 1, 2, \ldots)$ is a sequence of complex numbers, $a_n = o(n^{\beta})$, $s = \sigma + it$ then

$$\sum_{n=0}^{\infty} a_n e^{-ns} = o\left(\frac{1}{\sigma^{\beta+1}}\right) \tag{8}$$

for $\sigma \to 0_+$ uniformly in t.

Proof. We put $\varrho(x) = \frac{1}{2} - x + [x]$, $[x]$ means the integral part of x, $\omega(x) = \int_0^x \varrho(t)\,dt$. Then, using the Euler–MacLaurin (or Sonin) sum formula, we obtain

$$\sum_{n=0}^{\infty} n^{\beta} e^{-n\sigma} = \int_0^{\infty} e^{-\sigma x} x^{\beta}\,dx + \int_0^{\infty} \omega(x) \frac{d^2}{dx^2}(e^{-\sigma x} x^{\beta})\,dx$$

and thus (7). In order to prove (8), let $\varepsilon > 0$ be given. Then there exists an $n_0 = n_0(\varepsilon)$, such that $|a_n| < \varepsilon n^{\beta}$ for $n \geqslant n_0$. Hence according to (7)

$$\left|\sum_{n=0}^{\infty} a_n e^{-ns}\right| \leqslant \sum_{n=0}^{n_0} |a_n| + \varepsilon \sum_{n=0}^{\infty} n^{\beta} e^{-n\sigma} = \sum_{n=0}^{n_0} |a_n| + \frac{\varepsilon\Gamma(\beta+1)}{\sigma^{\beta+1}} + O\left(\frac{1}{\sigma^{\beta-1}}\right)$$

(1) The letter c still means (various) positive constants depending only on Q, a_j, b_j and M_j.

and thus

$$\limsup_{\sigma\to 0_+} \sigma^{\beta+1}\Big|\sum_{n=0}^{\infty} a_n e^{-ns}\Big| \leqslant \varepsilon\Gamma(\beta+1)$$

which proves (8).

THEOREM. *Let $a_1, a_2, \ldots, a_r$ be rational numbers. Let the nonsingular case take place. Then for $r \geqslant 4$*

$$P(n) = \Omega(n^{r/2-1}).$$

Proof. Using (5), we obtain

$$\Theta(s) = \sum_{n=0}^{\infty} s\int_{n}^{n+1} e^{-xs}A(n)\,dx = (1-e^{-s})\Big(K\sum_{n=0}^{\infty} n^{r/2}e^{-ns} + \sum_{n=0}^{\infty} P(n)e^{-ns}\Big).$$

Let us suppose $P(n) = o(n^{r/2-1})$. Using (8), we easily obtain

$$\frac{\Theta(s)}{1-e^{-s}} = K\sum_{n=0}^{\infty} n^{r/2}e^{-ns} + o(\sigma^{-r/2}) \tag{9}$$

for $s = \sigma+it$, $\sigma\to 0_+$ uniformly in t.

First, let $\delta = K = 0$. Then there is a couple h, k of integers, $h > 0$, $k > 1$, $(h, k) = 1$ with $S_{h,k} \neq 0$ (recall that $S_{h,k} = S_{h',k}$ for $h \equiv h' \pmod k$). Putting $t = 2\pi ih/k$, we obtain from (9), according to Lemma 1, for $\sigma\to 0_+$

$$\frac{1}{1-e^{-\sigma-2\pi ih/k}}\Big(\frac{\pi^{r/2}S_{h,k}}{\sqrt{D}M_1M_2\ldots M_rk^r} + O(e^{-c/k^2\sigma})\Big) = o(1)$$

and then

$$S_{h,k} = 0,$$

which is a contradiction. If $\delta = 1$, we put $t = 0$ and we obtain from (9), using (7) and (6) for $h = 0$, $k = 1$ and having in mind that $S_{0,1} = e^{2\pi i\sum_{j=1}^{r} a_jb_j} \neq 0$,

$$\frac{1}{1-e^{-\sigma}}\Big(\frac{\pi^{r/2}S_{0,1}}{\sqrt{D}M_1M_2\ldots M_r} + O(e^{-c/\sigma})\Big) = \frac{\pi^{r/2}S_{0,1}}{\sqrt{D}M_1M_2\ldots M_r\sigma} + o(1)$$

and then

$$\frac{1}{1-e^{-\sigma}} - \frac{1}{\sigma} = o(1)$$

for $\sigma\to 0_+$, which is again a contradiction.

Remark 2. The stronger result of Walfisz mentioned in the introduction, i.e., $a_n = \Omega(n^{r/2-1})$ for $\delta = 0$, $r > 4$ and in the nonsingular case,

can be proved as follows: Let $s = \sigma+it$, $a_n = o(n^{r/2-1})$. Then, using (8), we obtain

$$\Theta(\sigma+it) = \sum_{n=0}^{\infty} a_n e^{-ns} = o(\sigma^{-r/2})$$

for $\sigma\to 0_+$ uniformly in t, which is a contradiction of (6).

Remark 3. In the singular case we have only the following results so far:

$$P(x) = \Omega(x^{r/4-1/4})$$

([1], pp. 71–84).

$$P(x) = O(x^{r/4-1/10})$$

for $r > 4$ ([6], p. 62).

Remark 4. Our results can be extended to any rational form by an elementary transformation. If at least one of the numbers a_j or b_j is irrational or if the irrational form Q has an "almost diagonal" shape, then

$$P(x) = o(x^{r/2-1})$$

(for $r > 4$) and we know (in many important cases) the exact order of the function $P(x)$ up to an arbitrary $\varepsilon > 0$ only (see [4] or [3]). In those cases, generally, E. Landau's problem under considerations loses its meaning. The method presented above gives many results for "discrete" Ω-estimates under more general assumptions.

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