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The distribution of $f(d)$ (mod 1)

by

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Introduction. Let f be a real-valued function defined on the positive integers. I shall be concerned with the distribution (mod 1) of $f(d)$ as d runs through the divisors of an integer n. As usual, $\tau(n)$ denotes the number of these divisors. We write

$$D_f(n, z) = \operatorname{card}\{d:\ d\,|\,n \text{ and } f(d) \leqslant z \ (\mathrm{mod}\ 1)\}$$

and say that f is *uniformly distributed* if there exists a sequence of natural numbers n with asymptotic density 1 on which $D_f(n, z) \sim \tau(n)z$ uniformly for $z \in [0, 1]$. The discrepancy of the distribution is defined as

$$\Delta(n) = \Delta_f(n) = \sup_{0 \leqslant z_1 < z_2 \leqslant 1} |D_f(n, z_2) - D_f(n, z_1) - \tau(n)(z_2 - z_1)|$$

and so f is uniformly distributed if $\Delta_f(n) = o\big(\tau(n)\big)$ for almost all n.

In order to see whether f is uniformly distributed a natural approach is to look at the Weyl sums

$$\sigma_\nu(n, f) = \sum_{d|n} e^{2i\pi\nu f(d)}, \qquad \nu \in Z.$$

For uniform distribution we require that this is small compared to $\tau(n)$ for small, non-zero integers ν; more precisely, a well known theorem of Erdös and Turán [1] states that (in the present context)

$$\Delta_f(n) \ll \frac{\tau(n)}{T} + \sum_{\nu=1}^{T} \frac{1}{\nu}\, |\sigma_\nu(n, f)|$$

where T is any positive integer, and the constant implied by Vinogradov's notation $\ll$ is absolute, here and throughout the paper.

The function $f(d) = \log d$ was studied in Hall [3]–[5] and Erdös and Hall [2]. It is uniformly distributed. In this case the Weyl sums are multiplicative since $\log d$ is additive: this helps considerably, and I understand that Professor Kátai has found a compact necessary and sufficient condition that an additive f should be uniformly distributed.

We therefore look at the simplest non-additive functions defined analytically rather than by number theoretic properties, such as $(\log\log d)^a$, $(\log d)^a$, d^a. At present I can only deal with functions similar to $(\log d)^a$, and I give the following result.

THEOREM. *Let f satisfy the following conditions*

(i) *$uf'(u)$ is monotonic,*

(ii) $\log^{-\beta} u \ll |uf'(u)| \ll \log^{\gamma} u$, $\max(\beta,\gamma) < \log\frac{4}{3}$.

Then f is uniformly distributed. (The conditions need only hold for sufficiently large u.)

I think that $(\log d)^a$ is uniformly distributed for all $a > 0$, thus the correct condition in (ii) should probably be simply $\beta < 1$. There are a number of losses in the present method, which involves several partial summations. One complication is that it is not the average order, but the normal order, of the Weyl sums which concerns us: substantially this means that we have to neglect numbers with abnormally many prime factors.

It seems likely that in fact $(\log\log d)^a$ is uniformly distributed provided $a > 1$, and this would be best possible. To see that $\log\log d$ is not uniformly distributed, notice that if n has a prime factor $p \geqslant n^{1-\delta}$, which happens for a positive density of the integers, then for the divisors d of n which are multiples of p, $\log\log p \leqslant \log\log d \leqslant \log\log p + O(\delta)$. Thus half the values of $\log\log d \pmod 1$ are in a sub-interval of $[0,1)$ of length $\ll \delta$.

Before embarking on the proof of the theorem, we establish two lemmas. Note: At various points I have written down the functions $\log x$, $\log\log x$, and implicitly assumed that their values are $\geqslant 1$ when x is too small for this to be the case. This could be avoided by writing $\log(x+3)$, $\log\log(x+30)$ and so on at each occasion.

LEMMA 1. *Let $\omega(n)$ denote the number of prime factors of n counted according to multiplicity. Then there exists a function $C(y)$ such that for each fixed y, $0 < y < 2$ and uniformly for $x > 0$,*

$$\sum_{n\leqslant x} y^{\omega(n)} = C(y)x\log^{y-1}x + O\left(\frac{x}{\log^{2-y}x}\right).$$

This is straightforward and I omit the details.

LEMMA 2. *For each fixed y, $0 < y \leqslant 1$, and uniformly for $x > 0$ and $k = 2, 3, 4, \ldots$ we have*

$$\sum_{\substack{n\leqslant x\\(n,k)=1}} y^{\omega(n)} = C(y,k)x\log^{y-1}x + O\left(\frac{x\log\log x}{\log^{2-y}x}(\log\log k)^A\right)$$

where A is an absolute constant and

$$C(y,k) = C(y)\prod_{p|k}\left(1-\frac{y}{p}\right) < C(y).$$

Proof. The result is trivial if $(\log\log k)^A > \log x$ since the error term is then of greater order than the sum on the left, by Lemma 1. We therefore assume that $(\log\log k)^A \leqslant \log x$ and choose r and H so that

$$r = \frac{1}{\log\log k} \leqslant 1, \quad r\log H = 2\log\log x.$$

If $A > 1$ and x is sufficiently large, as we may assume, then $\sqrt{H} < x$. Next,

$$\sum_{\substack{n\leqslant x\\(n,k)=1}} y^{\omega(n)} = \sum_{d|k}\mu(d)y^{\omega(d)}\sum_{m\leqslant x/d} y^{\omega(m)} = S_1 + S_2$$

say, where

$$S_1 = \sum_{\substack{d|k\\d\leqslant H}}\mu(d)y^{\omega(d)}\left\{C(y)\frac{x}{d}\log^{y-1}\frac{x}{d} + O\left(\frac{x}{d}\log^{y-2}\frac{x}{d}\right)\right\}$$

$$= \sum_{\substack{d|k\\d\leqslant H}}\mu(d)y^{\omega(d)}\left\{C(y)\frac{x}{d}\log^{y-1}x + O\left(\frac{x\log H}{d\log^{2-y}x}\right)\right\}$$

$$= \left\{C(y,k) + O\left(\sum_{\substack{d|k\\d>H}}\frac{|\mu(d)|}{d}\right)\right\}x\log^{y-1}x + O\left(\frac{xk\log H}{\varphi(k)\log^{2-y}x}\right).$$

Also

$$S_2 = \sum_{\substack{d|k\\d>H}}\mu(d)y^{\omega(d)}\sum_{m\leqslant x/d} y^{\omega(m)} \ll x\sum_{\substack{d|k\\d>H}}\frac{|\mu(d)|}{d}.$$

But

$$\sum_{\substack{d|k\\d>H}}\frac{|\mu(d)|}{d} \leqslant \frac{1}{H^r}\prod_{p|k}\left(1+\frac{1}{p^{1-r}}\right) \leqslant \frac{1}{\log^2 x}\prod_{p|k}\left(1+\frac{1}{p^{1-r}}\right),$$

and the product on the right would be greatest if k were the product of small primes, say if

$$k = \prod_{p\leqslant z} p \geqslant \exp B_1 z$$

for some absolute positive constant B_1. But then

$$\prod_{p|k}\left(1+\frac{1}{p^{1-r}}\right) = \prod_{p\leqslant z}\left(1+\frac{1}{p^{1-r}}\right) \leqslant \prod_{p}\left(1+\frac{z^{2r}}{p^{1+r}}\right)$$
$$\leqslant \exp\{z^{2r}\log\zeta(1+r)\} \ll (\log\log k)^{\Delta}.$$

Therefore

$$S_1+S_2 = C(y,k)x\log^{y-1}x+O\left(\frac{xk\log H}{\varphi(k)\log^{2-y}x}+\frac{x(\log\log k)^{\Delta}}{\log^2 x}\right).$$

Substituting the value of $\log H$, and noting that $k/\varphi(k) \ll \log\log k$, we obtain our result.

Proof of the theorem. By the result of Erdös and Turán [1], we have

$$\Delta(n) \ll \Delta_1(n)+\frac{\tau(n)}{T}, \quad \text{where} \quad \Delta_1(n) = \sum_{\nu=1}^{T}\frac{1}{\nu}|\sigma_\nu(n,f)|.$$

We choose $T = T(x)$ where $T(x)\to\infty$ as $x\to\infty$: therefore to show that f is uniformly distributed it will be sufficient to show that $\Delta_1(n) \leqslant \varepsilon(x)\tau(n)$ for all but $o(x)$ integers $n\leqslant x$, where $\varepsilon(x)\to 0$. By the Cauchy–Schwarz inequality,

$$\Delta_1^2(n) \ll (\log T)\sum_{\nu=1}^{T}\frac{1}{\nu}\sum_{d_1|n}\sum_{d_2|n}\lambda(d_1,\nu)\bar{\lambda}(d_2,\nu),$$

where $\lambda(d,\nu) = e^{2i\pi\nu f(d)}$, and so for $0<y\leqslant 1$, we have

$$\sum_{n\leqslant x}\Delta_1^2(n)y^{\omega(n)} \ll (\log T)\sum_{\nu=1}^{T}\frac{1}{\nu}S(x,\nu)$$

where $S(x,\nu)$ is defined below. We require a good estimate for

$$\begin{aligned} S(x,\nu) &= \sum_{n\leqslant x}y^{\omega(n)}\sum_{d_1|n}\sum_{d_2|n}\lambda(d_1,\nu)\bar{\lambda}(d_2,\nu) \\ &= \sum_{d_1\leqslant x}\sum_{d_2\leqslant x}y^{\omega([d_1,d_2])}\lambda(d_1,\nu)\bar{\lambda}(d_2,\nu)\sum_{m\leqslant x/[d_1,d_2]}y^{\omega(m)} \\ &= \sum_{r\leqslant x}y^{\omega(r)}\sum_{[d_1,d_2]=r}\lambda(d_1,\nu)\bar{\lambda}(d_2,\nu)\sum_{m\leqslant x/r}y^{\omega(m)}. \end{aligned}$$

But

$$\sum_{[d_1,d_2]=r}1 = \prod_{p^\alpha\|r}(2\alpha+1)\leqslant 3^{\omega(r)}$$

so that provided $y\leqslant 1/3$ we have

$$S(x,\nu) = C(y)\sum_{r\leqslant x}y^{\omega(r)}\frac{x}{r}\log^{y-1}\frac{x}{r}\sum_{[d_1,d_2]=r}\lambda(d_1,\nu)\bar{\lambda}(d_2,\nu)+O\left(\sum_{r\leqslant x}\frac{x}{r}\log^{y-2}\frac{x}{r}\right).$$

We write $d_2 = d_1m/k$, $r = d_1m$ where $k|d_1$ and $(m,k)=1$ so that

$$S(x,\nu) = C(y)\sum_{d_1\leqslant x}\lambda(d_1,\nu)\sum_{k|d_1}\sum_{\substack{m\leqslant x/d_1\\(m,k)=1}}\frac{xy^{\omega(md_1)}}{md_1}\log^{y-1}\frac{x}{md_1}\bar{\lambda}\left(\frac{md_1}{k}\right)+$$
$$+O(x\log\log x),$$

the constant implied by the O-notation being independent of ν. Thus

$$|S(x,\nu)| \ll x\sum_{d\leqslant x}\frac{y^{\omega(d)}}{d}\sum_{k|d}\left|\sum_{\substack{m\leqslant x/d\\(m,k)=1}}\frac{y^{\omega(m)}}{m}\left(\log\frac{x}{md}\right)^{y-1}\bar{\lambda}\left(\frac{md}{k}\right)\right|+x\log\log x,$$

where we have replaced d_1 by d. We estimate the inner sum by partial summation employing Lemma 2, noting that if

$$g(m) = \frac{1}{m}\left(\log\frac{x}{md}\right)^{y-1}\bar{\lambda}\left(\frac{md}{k}\right)$$

then since $|f'(u)| \ll u^{-1}(\log u)^{\gamma}$ $(0<\gamma<\gamma_0)$ we have

$$|g(m)-g(m+1)| \ll \frac{\nu}{m^2}(\log x)^{\gamma}\left(\log\frac{x}{md}\right)^{y-1}.$$

The partial summation gives

$$\sum_{\substack{m\leqslant x/d\\(m,k)=1}}y^{\omega(m)}g(m)$$
$$= C(y,k)\sum_{m\leqslant x/d}\log^{y-1}mg(m)+O\left(\nu(\log x)^{\gamma}\left(\log\frac{x}{d}\right)^{y-1}(\log\log x)^{\Delta+1}\right).$$

Now

$$\begin{aligned} x\sum_{d\leqslant x}\frac{y^{\omega(d)}}{d}\tau(d)\log^{y-1}\frac{x}{d} &\ll \sum_{d\leqslant x}(2y)^{\omega(d)}\sum_{m\leqslant x/d}\log^{y-1}m \\ &\ll \sum_{m\leqslant x}\log^{y-1}m\sum_{d\leqslant x/m}(2y)^{\omega(d)} \\ &\ll \sum_{m\leqslant x}\frac{x}{m}\log^{2y-1}\frac{x}{m}\log^{y-1}m \ll x\log^{3y-1}x. \end{aligned}$$

Therefore

$$|S(x,\nu)| \ll x \sum_{d\leqslant x} \frac{y^{\omega(d)}}{d} \sum_{k|d} \Big| \sum_{m\leqslant x/d} \frac{\log^{y-1} m}{m} \log^{y-1}\frac{x}{md}\,\bar{\lambda}\Big(\frac{md}{k}\Big)\Big| + \nu x \log^{3y-1+\gamma} x (\log\log x)^{\Delta+1} + x\log\log$$

Again, the inner sum of the right is to be estimated by partial su tion. We have

$$\sum_{m\leqslant z} \frac{1}{m}\bar{\lambda}\Big(\frac{md}{k}\Big) = \int_0^{\log z} \exp\{-2i\pi\nu f(e^w d/k)\}\,dw + O(\nu\log^\gamma x)$$

and we notice that if $uf'(u)$ is monotonic and $|uf'(u)| \ll \log^{-\beta} u$ (… then by Lemma 4.2 of Titchmarsh [6], the integral on the right i

$$\ll \frac{1}{\nu}\log^\beta \frac{zd}{k}$$

and so

$$\sum_{m\leqslant x/d} \frac{\log^{y-1} m}{m}\log^{y-1}\frac{x}{md}\,\bar{\lambda}\Big(\frac{md}{k}\Big) \ll \Big(\frac{\log^\beta x}{\nu} + \nu\log^\gamma x\Big)\log^{y-1}\frac{x}{d}.$$

Therefore

$$|S(x,\nu)| \ll \nu x\log^{3y-1+\gamma} x(\log\log x)^{\Delta+1} + \frac{1}{\nu} x\log^{3y-1+\beta} x + x\log\log x$$

and

$$\sum_{\nu=1}^{T} \frac{1}{\nu} S(x,\nu) \ll Tx\log^{3y-1+\gamma} x(\log\log x)^{\Delta+1} + x\log^{3y-1+\beta} x + x(\log T)\log\log$$

We select $y = 1/3$ and $T = T(x) = [\log\log x]$. This gives

$$\sum_{n\leqslant x} \frac{\Delta_1^2(n)}{3^{\omega(n)}} \ll x(\log x)^\eta (\log\log x)^{\Delta+2},$$

where $\eta = \max(\beta,\gamma)$. We may neglect those integers $n \leqslant x$ whi not satisfy

$$\log\log x - (\log\log x)^{2/3} < \nu(n) \leqslant \omega(n) < \log\log x + (\log\log x)^{2/3}$$

as they have zero density. For the remaining integers, either $\Delta_1(n) \leqslant \varepsilon(x)\tau(n)$, with $\varepsilon(x) = (\log x)^{-\varepsilon}$, or

$$\frac{\Delta_1^2(n)}{3^{\omega(n)}} > (\log x)^{\log(4/3)-3\varepsilon}.$$

From the above, the number of such integers is

$$\ll x(\log x)^{\eta+3\varepsilon-\log(4/3)}(\log\log x)^{\Delta+2},$$

and this is $o(x)$ provided $\eta < \log\frac{4}{3}$. This completes the proof.

References

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