3) *Il existe une constante réelle C' telle que, pour tout P n'appartenant pas aux fibres dégénérées*:

$$h_\iota(P) \leqslant \Phi(P) + 2h_\pi(P) + C'.$$

4) *L'encadrement obtenu en 2) et 3) est le plus fin possible, en ce sens que, dans le cas $q = 3$, s'il existe r_1, r_2, C dans $\mathbf{R}$ tels que, sur le complémentaire Ω' des fibres dégénérées, on ait*:

$$\forall P \in \Omega'(\mathbf{Q}) \qquad \Phi(P) + r_1 h_\pi(P) - C \leqslant h_\iota(P) \leqslant \Phi(P) + r_2 h_\pi(P) + C$$

alors $r_1 \leqslant 1$ et $r_2 \geqslant 2$.

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On basis problem for Siegel modular forms of degree 2

by

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1. Introduction. In the theory of modular forms of a complex variable there is a famous problem so called as "basis problem" (for the details see [3]). In [1] van der Blij treated the special case of the above problem. His main result can be stated as "the space of modular forms of level one and of weight k is spanned by the theta-series attached to positive definite even integral quadratic forms of determinant unity if and only if the weight k is a multiple of 4". In this paper we shall treat the corresponding problem in the case of Siegel modular forms of degree two. Our main result is the following:

THEOREM. *Let $M(2, k)$ be the linear space of Siegel modular forms of degree 2 and of weight k (k is an even non-negative integer), then $M(2, k)$ is spanned by theta-series attached to positive definite even integral quadratic forms of determinant unity if and only if k is a multiple of 4.*

The proof of this theorem rests partly on equipment and precise observation of certain positive definite even integral quadratic lattices of determinant unity and partly on the work of Igusa [5] which determines the graded structure of Siegel modular forms of degree 2.

2. Some preliminaries. Let $\mathfrak{H}_2$ be Siegel upper-half space of degree 2 and $\varphi(\tau)$ be a Siegel modular form of degree 2 and of weight k, then $\varphi(\tau)$ can be expanded in a Fourier series

$$\varphi(\tau) = \sum_T a(T) e^{2\pi i \sigma(T\tau)}, \tag{1}$$

where T runs over the set $\mathfrak{T}$ of all positive semi-definite semi-integral matrices of size 2 and $\sigma(T\tau)$ means the trace of the matrix $T\tau$ ([11], [5]).

PROPOSITION 2.1. *Let*

$$\varphi_1(\tau) = \sum_T a(T) e^{2\pi i \sigma(T\tau)} \quad \textit{and} \quad \varphi_2(\tau) = \sum_T b(T) e^{2\pi i \sigma(T\tau)}$$

be Fourier expansions of Siegel modular forms of degree 2 with weights k_1 and k_2 respectively, then the product $\varphi_1(\tau)\varphi_2(\tau)$ is of weight k_1+k_2 and its Fourier expansion is given by:

$$\varphi_1(\tau)\varphi_2(\tau) = \sum_T c(T)e^{2\pi i\sigma(T\tau)}, \tag{2}$$

where $c(T) = \sum_{T_1+T_2=T} a(T_1)b(T_2)$ and T_1 and T_2 run over all possible pairs of solutions $\{T_1, T_2\}$ of $T_1+T_2 = T$ with T_1, T_2 and T in $\mathfrak{T}$.

The proof of this proposition is clear and we omit it.

After Witt [12] and Eichler [4] we shall consider positive definite integral quadratic forms in the language of lattices in the linear space over the field of rational numbers Q with positive metric. We shall assume this settings throughout this paper. We shall say a lattice L is integral if we have $(x, y)\in \boldsymbol{Z}$ for any pair x and y in L, where $(\ , \ )$ means the positive metric of L and $\boldsymbol{Z}$ is the ring of rational integers. If L is an integral lattice, then (x, x) is a positive integer for any $x\in L$ other than zero vector and we shall call x in L as m-vector when x satisfies $(x, x) = m$ with some positive integer m. We shall denote by $\mathrm{Aut}(L)$ the group of all automorphisms of the lattice L. An integral lattice L is called even integral if we have $(x, x) \equiv 0 \pmod 2$ for any $x\in L$. The determinant of a lattice L is defined by the determinant of the quadratic form corresponding to L. It is known that if L is an even integral quadratic lattice with determinant unity then the rank of L is neccessarily divisible by 8. Now we define theta-series $\vartheta(2, L)$ of degree 2 associated with even integral lattice L which is essentially the same thing as theta-series of degree 2 for positive definite even integral quadratic form. Let x and y be in L, then we denote by $[x, y]$ the matrix

$$\begin{pmatrix} (x, x) & (x, y) \\ (x, y) & (y, y) \end{pmatrix}$$

and theta-series $\vartheta(2, L)$ is defined by:

$$\vartheta(2, L) = \sum_{x,y} e^{\pi i\sigma([x,y]\tau)}, \tag{3}$$

where x and y run over on L independently and τ is a variable on $\mathfrak{H}_2$. By Satz D of Witt [12] $\vartheta(2, L)$ becomes a Siegel modular form of degree 2 with weight equal to the half of the rank of L when L is an even integral lattice of determinant unity. Since L is even integral lattice with positive metric, $[x, y]$ is always positive semi-definite even integral matrix of degree 2 for any x and y in L. Let T be in $\mathfrak{T}$ and $a(T, L)$ be the number of solutions of pairs x and y in L such that $[x, y] = 2T$, then $\vartheta(2, L)$ can be expanded as:

$$\vartheta(2, L) = \sum_{T\in\mathfrak{T}} a(T, L)e^{2\pi i\sigma(T\tau)}. \tag{4}$$

It is clear that $a(T, L)$ is a finite non-negative integer for each T and L and (4) is Fourier expansion of $\vartheta(2, L)$. From now on we shall restrict ourselves to the case where L is even integral.

We shall equip some suitable lattices. Let A_n, D_n $(n \geqslant 4)$ and E_8 be even integral lattices given in [6] or [8]. We shall conventionally use the following notations:

$$A_n = [e_1-e_2, e_2-e_3, \ldots, e_n-e_{n+1}]_{\boldsymbol{Z}},$$
$$D_n = [e_1-e_2, e_2-e_3, \ldots, e_{n-1}-e_n, e_n+e_n]_{\boldsymbol{Z}},$$
$$E_8 = \Big[e_1-e_2, \ldots, e_6-e_7, e_6+e_7, \tfrac{1}{2}\sum_{i=1}^{8} e_i\Big]_{\boldsymbol{Z}}$$

where $e_1, \ldots, e_n$ are orthonormal vectors and $[e_1-e_2, e_2-e_3, \ldots, e_n-e_{n+1}]_{\boldsymbol{Z}}$ means the lattice spanned by $e_1-e_2, \ldots, e_n-e_{n+1}$ over $\boldsymbol{Z}$ and so on.

PROPOSITION 2.2. *When and only when $n+1$ is a square integer congruent to one modulo 8 there exists an even integral lattice $\tilde{A}_n$ of rank n and of determinant unity containing A_n.*

Proof. After Niemeier [8] we use the dual lattice $A_n^\#$ of A_n. Since $A_n^\#/A_n$ is a cyclic group of order $n+1$ with its generator $u = \frac{1}{n+1}\sum_{i=1}^{n} e_i - \frac{n}{n+1}e_{n+1}$ modulo A_n ([8]), it is easy to see that $n+1$ be a square (say r^2) if $\tilde{A}_n$ exists at all because such $\tilde{A}_n$ must satisfy the isomorphicity condition $A_n^\#/\tilde{A}_n \cong \tilde{A}_n/A_n$. In this case $\tilde{A}_n/A_n$ is a cyclic group of order r and the representative of $\tilde{A}_n/A_n$ must be of the form:

$$jru \quad \text{with} \quad 1 \leqslant j < r \text{ and } (j, r) = 1.$$

But there holds $A_n+\boldsymbol{Z}j_1ru = A_n+\boldsymbol{Z}j_2ru$ for each pair of integers j_1 and j_2 such that $(j_1, r) = (j_2, r) = 1$. So we can take $\tilde{A}_n$ as $A_n+\boldsymbol{Z}ru$. Since A_n is even integral, the necessary and sufficient condition that $\tilde{A}_n$ is even integral is (ru, ru) is a positive even integer and we see that

$$(ru, ru) = \frac{nr^2}{(n+1)^2} + \frac{n^2r^2}{(n+1)^2} = n = r^2-1.$$

Thus r^2-1 is an even integer and we can say that $n+1$ is a square integer congruent to one modulo 8. Conversely, if $n+1$ is such a number, then $\tilde{A}_{n+1} = A_n + Zru$ with $r^2 = n+1$ and

$$u = \frac{1}{n+1}\sum_{i=1}^{n} e_i - \frac{n}{n+1} e_{n+1}$$

is an even integral lattice of rank n and of determinant unity. ∎

PROPOSITION 2.3. *When and only when n is a multiple of 8 $(n \geqslant 8)$ there exists an even integral lattice $\tilde{D}_n$ of determinant unity containing D_n.*

The content of this proposition is already discussed at pp. 330–331 in [12] and we omit its proof. We only give the basis of $\tilde{D}_n$ $(n \equiv 0 \bmod 8)$ by:

$$\tilde{D}_n = \Big[e_1 - e_2, \ldots, e_{n-2} - e_{n-1}, e_{n-1} + e_{n-1}, \tfrac{1}{2}\sum_{i=1}^{n} e_i\Big]_Z$$

$\tilde{D}_8$ is nothing else but E_8.

3. Some auxiliary lemmas and propositions. Let L be an even integral lattice, then we denote by $V(m, L)$ the set of m-vectors in L for each even integer m. We shall use the symbol $\|S\|$ for the cardinality of a finite set S.

LEMMA 3.1. (i) $V(2, \tilde{A}_{24}) = V(2, A_{24})$ *and* $\|V(2, \tilde{A}_{24})\| = 600$, (ii) $V(4, \tilde{A}_{24})$ *consists of two transitive classes C_1 and C_2 under the action of* $\mathrm{Aut}(\tilde{A}_{24})$, *where C_1 and C_2 are given by:*

$$C_1 = \{e_{i_1} + e_{i_2} - e_{i_3} - e_{i_4} \mid 1 \leqslant i_1, i_2, i_3, i_4 \leqslant 25,$$
$$i_1, i_2, i_3 \text{ and } i_4 \text{ are different from each other}\},$$

$$C_2 = \Big\{\pm\Big[\tfrac{1}{5}\sum_{i=1}^{25} e_i - (e_{i_1} + e_{i_2} + e_{i_3} + e_{i_4} + e_{i_5})\Big] \Big| 1 \leqslant i_1 < i_2 < i_3 < i_4 < i_5 \leqslant 25\Big\}.$$

Proof of (i). Using the fact that

$$5u = \frac{1}{5}\sum_{i=1}^{24} e_i - \frac{24}{5} e_{25} \equiv \frac{1}{5}\sum_{i=1}^{25} e_i - \sum_{i=21}^{25} e_i \mod A_{24},$$

we see that $\tilde{A}_{24}$ is also expressed as $A_{24} + Zw$ with $w = \frac{1}{5}\sum_{i=1}^{25} e_i - \sum_{i=21}^{25} e_i$. As Niemeier remarked p. 150 in [8] w has the following property:

$$(5) \qquad (w, w) \leqslant (w+v, w+v) \quad \text{for all } v \in A_{24}.$$

w has the order 5 modulo A_{24}, that is, it holds that we have $\lambda_1 w \equiv \lambda_2 w \bmod A$ with integers λ_1 and λ_2 if and only if $\lambda_1 \equiv \lambda_2 \pmod 5$.

It can be observed that:

$$2w \equiv \tfrac{2}{5}\sum_{i=1}^{25} e_i - \sum_{i=16}^{25} e_i \mod A_{24},$$
$$3w \equiv \tfrac{3}{5}\sum_{i=1}^{25} e_i - \sum_{i=11}^{25} e_i \mod A_{24},$$
$$4w \equiv \tfrac{4}{5}\sum_{i=1}^{25} e_i - \sum_{i=6}^{25} e_i \mod A_{24}$$

and

$$5w \equiv 0 \mod A_{24}.$$

$w_2 = \frac{2}{5}\sum_{i=1}^{25} e_i - \sum_{i=16}^{25} e_i$, $w_3 = \frac{3}{5}\sum_{i=1}^{25} e_i - \sum_{i=11}^{25} e_i$ and $w_4 = \frac{4}{5}\sum_{i=1}^{25} e_i - \sum_{i=6}^{25} e_i$ have the property (5) (see also pp. 148–150 in [8]) and we see that $(w, w) = 4$, $(w_2, w_2) = 6$, $(w_3, w_3) = 6$ and $(w_4, w_4) = 4$. By the above discussion we can say that $V(2, \tilde{A}_{24}) = V(2, A_{24})$. By calculating combinatorially, we get $\|V(2, \tilde{A}_{24})\| = 600$.

Proof of (ii). Since the series $\sum_{u \in \tilde{A}_{24}} e^{\pi i (u,u) z}$, where z is a complex variable with positive imaginary part, is a modular form of weight 12 of level 1, $\|V(4, \tilde{A}_{24})\|$, the number of 4-vectors in $\tilde{A}_{24}$, is given by the formula (11) of [9] with $n = 2$ and $1104 - 384s = 600$. Hence $\|V(4, \tilde{A}_{24})\|$ is 182160. Clearly $\tilde{A}_{24}$ contains the following two types of 4-vectors:

$$C_1 = \{e_{i_1} + e_{i_2} - e_{i_3} - e_{i_4} \mid 1 \leqslant i_1, i_2, i_3, i_4 \leqslant 25\},$$

$$C_2 = \Big\{\pm\Big[\tfrac{1}{5}\sum_{i=1}^{25} e_i - (e_{i_1} + e_{i_2} + e_{i_3} + e_{i_4} + e_{i_5})\Big] \Big| 1 \leqslant i_1 < i_2 < i_3 < i_4 < i_5 \leqslant 25\Big\}.$$

It is clear that C_1 and C_2 are disjoint sets. By calculating combinatorially we get $\|C_1\| = 75900$ and $\|C_2\| = 106260$ and $\|C_1\| + \|C_2\| = \|V(4, \tilde{A}_{24})\|$. This means that $V(4, \tilde{A}_{24}) = C_1 \cup C_2$ (disjoint union). Since $\mathrm{Aut}(\tilde{A}_{24})$ contains $\mathrm{Aut}(A_{24})$ as a subgroup it can be seen that C_1 (resp. C_2) is transitive under the action of $\mathrm{Aut}(\tilde{A}_{24})$. If there exists an element ϱ of $\mathrm{Aut}(\tilde{A}_{24})$ such that ϱ carries an element u of C_1 into an element v of C_2, then without loss of generality we can assume that $u = e_1 + e_2 - e_3 - e_4$, $v = \frac{1}{5}\sum_{i=1}^{25} e_i - \sum_{i=1}^{5} e_i$ and $\varrho u = v$. Consider the 2-vector $e_2 - e_3$, and we have $(u, e_2 - e_3) = 2$. But $\big(\varrho u, \varrho(e_2 - e_3)\big) = \big(v, \varrho(e_2 - e_3)\big)$ is not equal to 2 because of the shape of v and because $\varrho(e_2 - e_3)$ is another 2-vector in A_{24} and has the form $e_{j_1} - e_{j_2}$ with $1 \leqslant j_1, j_2 \leqslant 25$ and $j_1 \neq j_2$. This means that such $\varrho \in \mathrm{Aut}(\tilde{A}_{24})$ does not exist. ∎

LEMMA 3.2. (i) *$V(2, E_8)$ is transitive under the action of* $\mathrm{Aut}(E_8)$ *and* $\|V(2, E_8)\| = 240$, (ii) *$V(4, E_8)$ is transitive under the action of* $\mathrm{Aut}(E_8)$ *and* $\|V(4, E_8)\| = 2160$.

The transitivity statement of this lemma is asserted by Hilfssatz (5.4) of [8] and $\|V(2, E_8)\| = 240$ and $\|V(4, E_8)\| = 2160$ are merely calculations and we omit those.

LEMMA 3.3. (i) *$V(2, \tilde{D}_{8n}) = V(2, D_{8n})$ $(n \geqslant 2)$ is transitive under the action of* $\mathrm{Aut}(\tilde{D}_{8n})$ *and* $\|V(2, \tilde{D}_{8n})\| = 16n(8n-1)$, (ii) *$V(4, \tilde{D}_{16})$ consists of three transitive classes C_3, C_4 and C_5 under the action of* $\mathrm{Aut}(\tilde{D}_{16})$, *where C_3, C_4 and C_5 are given by*:

$$C_3 = \{\pm 2e_i \mid 1 \leqslant i \leqslant 16\},$$
$$C_4 = \{\pm e_{i_1} \pm e_{i_2} \pm e_{i_3} \pm e_{i_4} \mid 1 \leqslant i_1 < i_2 < i_3 < i_4 \leqslant 16\},$$
$$C_5 = \Big\{\tfrac{1}{2}\sum_{i=1}^{16} s_i e_i \mid s_i = \pm 1, \prod_{i=1}^{16} s_i = 1\Big\},$$

(iii) *$V(4, \tilde{D}_{8n})$ $(n \geqslant 3)$ consists of two transitive classes $C_6(8n)$ and $C_7(8n)$ under the action of* $\mathrm{Aut}(\tilde{D}_{8n})$, *where $C_6(8n)$ and $C_7(8n)$ are given by*:

$$C_6(8n) = \{\pm 2e_i \mid 1 \leqslant i \leqslant 8n\},$$
$$C_7(8n) = \{\pm e_{i_1} \pm e_{i_2} \pm e_{i_3} \pm e_{i_4} \mid 1 \leqslant i_1 < i_2 < i_3 < i_4 \leqslant 8n\}.$$

Proof of (i). Since $V(2, \tilde{D}_{8n})$ $(n \geqslant 2)$ consists of $\pm e_i \pm e_j$ with $1 \leqslant i < j \leqslant 8n$, the transitivity is clear. By calculating we get $\|V(2, \tilde{D}_{8n})\| = 16n(8n-1)$.

Proof of (ii). Since the series $\sum_{u \in \tilde{D}_{16}} e^{\pi i(u,u)z}$ is a modular form of weight 8 and of level 1 and the dimension of the space of modular forms of weight 8 and of level 1 is one, $\sum_{u \in \tilde{D}_{16}} e^{\pi i(u,u)z}$ must be equal to primitive Eisenstein series of weight 8, namely, to

$$1 + 480\sum_{n=1}^{\infty} \sigma_7(n) e^{2\pi i n z}, \quad \text{where} \quad \sigma_7(n) = \sum_{d|n} d^7.$$

So we can say that $\|V(4, \tilde{D}_{16})\| = 480\sigma_7(2) = 61920$. It can be seen that C_3, C_4 and C_5 are mutually disjoint subsets of $V(4, \tilde{D}_{16})$. An easy computation shows that $\|C_3\| = 32$, $\|C_4\| = 29120$ and $\|C_5\| = 32768$ and that $\|C_3\| + \|C_4\| + \|C_5\| = \|V(4, \tilde{D}_{16})\|$. This means that $V(4, \tilde{D}_{16}) = C_3 \cup C_4 \cup C_5$ (disjoint union). Since $\mathrm{Aut}(\tilde{D}_{16})$ is generated by reflections with respect to 2-vectors in $\tilde{D}_{16}$, it can be observed that C_3, C_4 and C_5 are transitive classes under the action of $\mathrm{Aut}(\tilde{D}_{16})$.

Proof of (iii). Since we know $\tilde{D}_{8n} = D_{8n} + \mathbf{Z}u$ with $u = \frac{1}{2}\sum_{i=1}^{8n} e_i$ and u has the property (see p. 150 of [8]):

$$(u, u) \leqslant (u+v, u+v) \qquad \forall v \in D_{8n},$$

we can say that $V(4, \tilde{D}_{8n}) = V(4, D_{8n})$ for $n \geqslant 3$. It is easy to verify that $V(4, \tilde{D}_{8n})$ is a disjoint union of $C_6(8n)$ and $C_7(8n)$ and that they are transitive classes of $\mathrm{Aut}(\tilde{D}_{8n})$. ∎

It should be remarked that

$$\|C_6(8n)\| = 16n \quad \text{and} \quad \|C_7(8n)\| = \tfrac{1}{3}16n(8n-1)(8n-2)(8n-3).$$

Now we shall describe the process of calculating Fourier coefficients $a(T, L)$ of $\vartheta(2, L)$ for some $T \in \mathfrak{T}$ and for some even integral lattice L. For the later convenience we set:

$$T_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$
$$T_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$
$$T_6 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix}, \quad T_7 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad T_8 = \begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix},$$
$$T_9 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

It is obvious that $a(T_1, L) = \|V(2, L)\|$ and $a(T_2, L) = \|V(4, L)\|$ for each even integral L. To calculate $a(T_3, L)$ we need the number of pairs of 2-vectors x and y in L such that $(x, y) = 0$. $a(T_4, L)$ is the number of pairs of 2-vectors x and y such that $(x, y) = 1$. $a(T_5, L)$ is the number of pairs of 2-vector x and 4-vector y such that $(x, y) = 0$ and so on. We shall number theta-series attached to special even integral lattices of determinant unity as follows:

$$\vartheta_j = \vartheta(2, L_j) = \sum_{T \in \mathfrak{T}} a(T, L_j) e^{2\pi i \sigma(T\tau)} = \sum_{T \in \mathfrak{T}} a_j(T) e^{2\pi i \sigma(T\tau)},$$

where $L_1 = E_8$, $L_2 = \tilde{D}_{16}$, $L_3 = E_8 \oplus E_8 \oplus E_8$ (orthogonal sum), $L_4 = \tilde{D}_{24}$, $L_5 = \tilde{A}_{24}$, $L_6 = E_8 \oplus E_8 \oplus E_8 \oplus E_8$, $L_7 = \tilde{D}_{24} \oplus E_8$, $L_8 = \tilde{A}_{24} \oplus E_8$, $L_9 = \tilde{D}_{32}$, $L_{10} = E_8 \oplus E_8 \oplus E_8 \oplus E_8 \oplus E_8$, $L_{11} = \tilde{D}_{24} \oplus E_8 \oplus E_8$, $L_{12} = \tilde{A}_{24} \oplus E_8 \oplus E_8$, $L_{13} = \tilde{D}_{32} \oplus E_8$, $L_{14} = \tilde{D}_{40}$. We should keep in mind that ϑ_1 is Siegel modular form of weight 4, ϑ_2 is Siegel modular form of weight 8 (i.e. $\vartheta_2 \in M(2, 8)$), ϑ_3, ϑ_4 and ϑ_5 are in $M(2, 12)$, ϑ_6, ϑ_7, ϑ_8 and ϑ_9 are in $M(2, 16)$, ϑ_{10}, ϑ_{11}, ϑ_{12}, ϑ_{13} and ϑ_{14} are in $M(2, 20)$. As typical calculations of $a(T, L)$ we show how $a_1(T_3)$, $a_1(T_6)$, $a_2(T_7)$ and $a_5(T_9)$ are calculated.

Taking 2-vector $e_1-e_2 \in E_8$, then we must look for all 2-vectors $y \in E_8$ such that $(e_1-e_2, y) = 0$. Solutions of such y's are given by:

$$\pm(e_1+e_2), \quad \pm e_i \pm e_j, \quad 3 \leqslant i < j \leqslant 8$$

and

$$\pm \tfrac{1}{2}\Big(e_1+e_2+\sum_{i=3}^{8} s_i e_i\Big), \quad s_i = \pm 1, \quad \prod_{i=3}^{8} s_i = 1.$$

(It is understood henceforth that the indices $i, j, \ldots$ of $e_i, e_j, \ldots$ are mutually different in their range of running.) The number of such y's is 126. By Lemma 3.2(i) we can say that to any 2-vector $x \in E_8$ the number of 2-vectors y such that $(x, y) = 0$ is equal to 126, so we have $a_1(T_3) = 240 \times \times 126$. Taking 2-vector $e_1-e_2 \in E_8$ and looking for all 4-vectors $y \in E_8$ such that $(e_1-e_2, y) = 1$, we get solutions y as follows:

$$e_1 \pm e_{i_1} \pm e_{i_2} \pm e_{i_3}, \quad 3 \leqslant i_1 < i_2 < i_3 \leqslant 8,$$
$$-e_2 \pm e_{i_1} \pm e_{i_2} \pm e_{i_3}, \quad 3 \leqslant i_1 < i_2 < i_3 \leqslant 8,$$
$$\tfrac{1}{2}\Big(3e_1+e_2+\sum_{i=3}^{8} s_i e_i\Big), \quad s_i = \pm 1, \quad \prod_{i=3}^{8} s_i = -1,$$
$$\tfrac{1}{2}\Big(-e_1-3e_2+\sum_{i=3}^{8} s_i e_i\Big), \quad s_i = \pm 1, \quad \prod_{i=3}^{8} s_i = -1$$

and

$$\tfrac{1}{2}\Big(e_1-e_2+3s_{i_3}e_{i_3}+\sum_{t=4}^{8} s_{i_t} e_{i_t}\Big), \quad s_{i_t} = \pm 1, \quad \prod_{t=3}^{8} s_{i_t} = -1.$$

The number of such y's is 576. By Lemma 3.2(i), (ii) we can say that to any 2-vector $x \in E_8$ the number of 4-vectors y such that $(x, y) = 1$ is equal to 576, so we have $a_1(T_6) = 240 \times 576$. Calculation of $a_2(T_7)$ is a little complicated. We pick up $2e_1 \in C_3$, $e_1+e_2+e_3+e_4 \in C_4$ and $\frac{1}{2}\sum_{i=1}^{16} e_i \in C_5$ as the representatives of transitive classes C_3, C_4 and C_5 in Lemma 3.3(ii). 4-vectors $y_3 \in \tilde{D}_{16}$ such that $(2e_1, y_3) = 0$ are given by:

$$\pm 2e_i, \quad 2 \leqslant i \leqslant 16$$

and

$$\pm e_{i_1} \pm e_{i_2} \pm e_{i_3} \pm e_{i_4}, \quad 2 \leqslant i_1 < i_2 < i_3 < i_4 \leqslant 16.$$

The number of such y_3's is 21870. 4-vectors $y_4 \in \tilde{D}_{16}$ such that $(e_1+e_2+ +e_3+e_4, y_4) = 0$ are given by:

$$\pm 2e_i, \quad 5 \leqslant i \leqslant 16,$$
$$\pm(e_{i_1}-e_{i_2}) \pm e_{i_3} \pm e_{i_4}, \quad 1 \leqslant i_1 < i_2 \leqslant 4, \quad 5 \leqslant i_3 < i_4 \leqslant 16,$$
$$\pm e_{i_1} \pm e_{i_2} \pm e_{i_3} \pm e_{i_4}, \quad 5 \leqslant i_1 < i_2 < i_3 < i_4 \leqslant 16,$$
$$(e_{i_1}+e_{i_2}-e_{i_3}-e_{i_4}), \quad \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$$

and

$$\tfrac{1}{2}\Big(\sum_{i=1}^{4} e_i + \sum_{i=5}^{16} s_i e_i\Big) - (e_{i_1}+e_{i_2}),$$
$$1 \leqslant i_1 < i_2 \leqslant 4, \quad s_i = \pm 1, \quad \prod_{i=5}^{16} s_i = 1.$$

The number of such y_4's is 23406. 4-vectors $y_5 \in \tilde{D}_{16}$ such that $(\frac{1}{2}\sum_{i=1}^{16} e_i, y_5) = 0$ are given by:

$$e_{i_1}+e_{i_2}-e_{i_3}-e_{i_4}, \quad 1 \leqslant i_1, i_2, i_3, i_4 \leqslant 16$$

and

$$\tfrac{1}{2}\sum_{i=1}^{16} s_i e_i, \quad s_i = \pm 1, \quad \sum_{i=1}^{16} s_i = 0.$$

The number of such y_5's is 23790. It can be seen that to any 4-vector $x \in C_i$ $(i = 3, 4, 5)$ the number of 4-vectors y_i such that $(x, y_i) = 0$ is same and we have:

$$a_2(T_7) = 21870 \times \|C_3\| + 23406 \times \|C_4\| + 23790 \times \|C_5\| = 1461833280.$$

We pick up

$$e_1+e_2-e_3-e_4 \in C_1 \quad \text{and} \quad \tfrac{1}{5}\sum_{i=1}^{25} e_i - \sum_{i=1}^{5} e_i \in C_2$$

as the representatives of transitive classes C_1 and C_2 in Lemma 3.1(ii). 4-vectors $y_1 \in \tilde{A}_{24}$ such that $(e_1+e_2-e_3-e_4, y_1) = 2$ are given by:

$$e_1+e_2-(e_{i_3}+e_{i_4}), \quad 5 \leqslant i_3, i_4 \leqslant 25,$$
$$-(e_3+e_4)+(e_{i_5}+e_{i_6}), \quad 5 \leqslant i_5, i_6 \leqslant 25,$$
$$e_{i_1}-e_{i_2} \pm (e_{i_5}-e_{i_6}), \quad 1 \leqslant i_1 \leqslant 2, \ 3 \leqslant i_2 \leqslant 4, \ 5 \leqslant i_5, i_6 \leqslant 25,$$
$$\tfrac{1}{5}\sum_{i=1}^{25} e_i - (e_3+e_4+e_{i_5}+e_{i_6}+e_{i_7}), \quad 5 \leqslant i_5, i_6, i_7 \leqslant 25,$$
$$-\tfrac{1}{5}\sum_{i=1}^{25} e_i + (e_1+e_2+e_{i_5}+e_{i_6}+e_{i_7}), \quad 5 \leqslant i_5, i_6, i_7 \leqslant 25.$$

The number of such y_1's is 4760. 4-vectors $y_2 \in \tilde{A}_{24}$ such that

$$\Big(\tfrac{1}{5}\sum_{i=1}^{25} e_i - \sum_{i=1}^{5} e_i, y_2\Big) = 2$$

are given by:

$$-(e_{i_1}+e_{i_2})+e_{i_3}+e_{i_4}, \qquad 1 \leqslant i_1, i_2 \leqslant 5, \; 6 \leqslant i_3, i_4 \leqslant 25$$

and

$$\tfrac{1}{5}\sum_{i=1}^{25} e_i-(e_{i_1}+e_{i_2}+e_{i_3}+e_{i_4}+e_{i_5}), \qquad 1 \leqslant i_1, i_2, i_3 \leqslant 5, \; 6 \leqslant i_4, i_5 \leqslant 25.$$

The number of such y_2's is 3800. We have

$$a_5(T_9) = 4760 \times \|C_1\| + 3800 \times \|C_2\| = 765072000.$$

Lemmas 3.1–3.3 will be sufficient for calculating $a_i(T_j)$ for $0 \leqslant j \leqslant 9$ and $i = 1, 2, 4, 5, 9, 14$. To calculate $a_i(T_j)$ for $0 \leqslant j \leqslant 9$ and $i = 3, 6, 7, 8, 10, 11, 12, 13$ we have only to utilize Proposition 2.1 and the following facts (I), (II) and (III).

(I) $\vartheta_3 = \vartheta_1^3,\ \vartheta_6 = \vartheta_1^4,\ \vartheta_7 = \vartheta_4\vartheta_1,\ \vartheta_8 = \vartheta_5\vartheta_1,\ \vartheta_{10} = \vartheta_1^5,$
$\vartheta_{11} = \vartheta_4\vartheta_1^2,\ \vartheta_{12} = \vartheta_5\vartheta_1^2,\ \vartheta_{13} = \vartheta_9\vartheta_1.$

(II) The decompositions:

$$T_3 = \begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix}, \qquad T_4 = \begin{pmatrix}1&\frac12\\\frac12&1\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix},$$

$$T_5 = \begin{pmatrix}1&0\\0&2\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&2\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix},$$

$$T_6 = \begin{pmatrix}1&\frac12\\\frac12&2\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}1&\frac12\\\frac12&1\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix},$$

$$\begin{aligned}T_7 &= \begin{pmatrix}2&0\\0&2\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}\\ &=\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}1&0\\0&2\end{pmatrix}=\begin{pmatrix}2&0\\0&1\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}2&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&2\end{pmatrix}\\ &=\begin{pmatrix}1&-1\\-1&1\end{pmatrix}+\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}1&\frac12\\\frac12&1\end{pmatrix}+\begin{pmatrix}1&-\frac12\\-\frac12&1\end{pmatrix},\end{aligned}$$

$$\begin{aligned}T_8 &= \begin{pmatrix}2&\frac12\\\frac12&2\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}1&\frac12\\\frac12&1\end{pmatrix}=\begin{pmatrix}1&-\frac12\\-\frac12&1\end{pmatrix}+\begin{pmatrix}1&1\\1&1\end{pmatrix}\\ &=\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}1&\frac12\\\frac12&2\end{pmatrix}=\begin{pmatrix}2&\frac12\\\frac12&1\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix},\end{aligned}$$

$$\begin{aligned}T_9 &= \begin{pmatrix}2&1\\1&2\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}1&1\\1&1\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&\frac12\\\frac12&1\end{pmatrix}+\begin{pmatrix}1&\frac12\\\frac12&1\end{pmatrix}\\ &=\begin{pmatrix}2&1\\1&1\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}1&1\\1&2\end{pmatrix}\end{aligned}$$

are all possible decompositions of T_j $(3 \leqslant j \leqslant 9)$ by elements of $\mathfrak{T}$.

(III) Let $\varphi(\tau) = \sum_T a(T)e^{2\pi i\sigma(T\tau)}$ be Fourier expansion of Siegel modular form $\varphi(\tau)$ of degree 2, then it holds that $a(T) = a({}^tUTU)$ for any

unimodular matrix U of degree 2, where ${}^tU$ denotes the transpose of U (formula (48) in [11]). As special case we have

$$a\begin{pmatrix}1&1\\1&1\end{pmatrix}=a\begin{pmatrix}1&0\\0&0\end{pmatrix}=a\begin{pmatrix}0&0\\0&1\end{pmatrix}=a\begin{pmatrix}1&-1\\-1&1\end{pmatrix},$$

$$a\begin{pmatrix}1&-\frac12\\-\frac12&1\end{pmatrix}=a\begin{pmatrix}1&\frac12\\\frac12&1\end{pmatrix}, \quad a\begin{pmatrix}2&\frac12\\\frac12&1\end{pmatrix}=a\begin{pmatrix}2&-\frac12\\-\frac12&1\end{pmatrix}=a\begin{pmatrix}1&\frac12\\\frac12&2\end{pmatrix}=a\begin{pmatrix}1&-\frac12\\-\frac12&2\end{pmatrix},$$

$$a\begin{pmatrix}0&0\\0&2\end{pmatrix}=a\begin{pmatrix}2&0\\0&0\end{pmatrix} \quad\text{and}\quad a\begin{pmatrix}1&1\\1&2\end{pmatrix}=a\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Along the above way we get the following table of $a_i(T_j)$.

$i \backslash j$	0	1	2	3	4	5
1	1	240	2160	30240	13440	181440
2	1	480	61920	175680	26880	15914880
3	1	720	179280	436320	40320	88672320
4	1	1104	170064	1022304	97152	131300928
5	1	600	182160	303600	27600	74685600
6	1	960	354240	812160	53760	259925760
7	1	1344	437184	1582464	110592	453420288
8	1	840	328320	621840	41040	210889440
9	1	1984	575424	3456128	238080	876967680
10	1	1200	586800	1303200	67200	571147200
11	1	1584	761904	2257824	124032	981862848
12	1	1080	532080	1055280	54480	466325280
13	1	2224	1053744	4438688	251520	1909003200
14	1	3120	1462320	8779680	474240	3702628800

$i \backslash j$	6	7	8	9
1	138240	1239840	967680	604800
2	6727680	1461833280	953948160	225388800
3	19768320	15579220320	7503805440	757296000
4	27202560	14744809824	6283791360	777313152
5	17001600	15928677600	7844538240	765072000
6	39260160			
7	65495040			
8	31827840			
9	128880640	198995164288	60702781440	5226570240
10	65203200	226106935200	55973606400	3074256000
11	110238720	383278632864	92830417920	5668117632
12	53105280	185566037280	46232150400	2485756800
13	212823040	734060529568	176848696320	11095370880
14	421125120	1422569435040	335355970560	22126615680

The blanks in the above table are not neccessary for our purpose.

PROPOSITION 3.4. *The dimension of $M(2,4)$ is one and $M(2,4)$ is spanned by ϑ_1.*

Proof. The former part of the statement is obtained by Corollary of p. 194 in [5] and the latter part is clear because ϑ_1 is a Siegel modular form of weight 4. ■

PROPOSITION 3.5. *The dimension of $M(2,8)$ is one and $M(2,8)$ is spanned by ϑ_2.*

Proof. The former part of the statement is obtained by Corollary of p. 194 in [5] and the latter part is clear because ϑ_2 is a Siegel modular form of weight 8. ■

Remark. E. Witt [12] obtained the result $\vartheta_1^2 = \vartheta_2$.

PROPOSITION 3.6. *The dimension of $M(2,12)$ is 3 and $M(2,12)$ is spanned by ϑ_3, ϑ_4 and ϑ_5.*

Proof. The former part of the statement is obtained by Corollary of p. 194 in [5] and we have only to show that ϑ_3, ϑ_4 and ϑ_5 are linearly independent because they are Siegel modular forms of weight 12. Since we can verify that the determinant of the matrix $(a_i(T_j))$, where $i = 3, 4, 5$ and $j = 0, 1, 3$, is different from zero, we can conclude ϑ_3, ϑ_4 and ϑ_5 are linearly independent. ■

PROPOSITION 3.7. *The dimension of $M(2,16)$ is 4 and $M(2,16)$ is spanned by $\vartheta_6, \vartheta_7, \vartheta_8$ and ϑ_9.*

Proof. The former part of the statement is obtained by Corollary of p. 194 in [5] and we have only to show that $\vartheta_6, \vartheta_7, \vartheta_8$ and ϑ_9 are linearly independent because they are Siegel modular forms of weight 16. Since we can verify that the determinant of the matrix $(a_i(T_j))$, where $i = 6, 7, 8, 9$ and $j = 0, 1, 3, 4$, is different from zero, we can conclude $\vartheta_6, \vartheta_7, \vartheta_8$ and ϑ_9 are linearly independent. ■

PROPOSITION 3.8. *The dimension $M(2,20)$ is 5 and $M(2,20)$ is spanned by $\vartheta_{10}, \vartheta_{11}, \vartheta_{12}, \vartheta_{13}$ and ϑ_{14}.*

Proof. The former part of the statement is obtained by Corollary of p. 194 in [5] and we have only to show that $\vartheta_{10}, \vartheta_{11}, \vartheta_{12}, \vartheta_{13}$ and ϑ_{14} are linearly independent because they are Siegel modular forms of weight 20. Since we can verify that the determinant of the matrix $(a_i(T_j))$, where $i = 10, 11, 12, 13, 14$, and $j = 0, 1, 3, 7, 8$, is different from zero, we can conclude $\vartheta_{10}, \vartheta_{11}, \vartheta_{12}, \vartheta_{13}$, and ϑ_{14} are linearly independent. ■

As the immediate consequence of Propositions 3.6, 3.7, and 3.8 we have the following:

LEMMA 3.9. *Let ψ_k be Eisenstein series of degree 2 and of weight k (see p. 645 of [11] or p. 189 of [5]), then* (i) *ψ_6^2 and ψ_{12} are expressed as linear combinations of ϑ_3, ϑ_4 and ϑ_5,* (ii) *$\psi_6\psi_{10}$ is expressed as linear combination of $\vartheta_6, \vartheta_7, \vartheta_8$ and ϑ_9 and* (iii) *ψ_{10}^2 is expressed as linear combination of $\vartheta_{10}, \vartheta_{11}, \vartheta_{12}, \vartheta_{13}$ and ϑ_{14}.*

Remark. $\psi_{10} - \psi_4\psi_6$ may be one of interesting Siegel modular forms of degree 2. This is unique cusp form of weight 10 (up to a constant factor) and is not expressed as linear combination of theta-series. But we have the following equations:

$$
\begin{aligned}
&2^{-20}3^{-10}5^{-4}7^{-2}53^{-2}43867^2(\psi_{10} - \psi_4\psi_6)^2 \\
&= 2^{-25}3^{-2}5^{-1}7^{-1}41^{-1}(168\vartheta_{14} + 2048\vartheta_{12} + 2320\vartheta_{11} - 3591\vartheta_{10} - 945\vartheta_{13}) \\
&= 6e^{2\pi i\sigma(T_7\tau)} - 4e^{2\pi i\sigma(T_8\tau)} + e^{2\pi i\sigma(T_9\tau)} + \ldots
\end{aligned}
$$

LEMMA 3.10. *Assume that k is a positive integer divisible by 4, then the non-negative integer solutions of the linear Diophantine equation*

$$k = 4p + 6q + 10r + 12s \tag{6}$$

are exhausted by the following types:

(i) $q \equiv r \equiv 1 \pmod 2$ *or*
(ii) $q \equiv r \equiv 0 \pmod 2$.

Proof. Otherwise we get a contradiction. ■

4. Proof of theorem. Since the rank of even integral lattice L of determinant unity is divisible by 8, $\vartheta(2, L)$ must be of weight k divisible by 4. So the proof of "only if" part is clear. For the brevity of later descriptions we denote by $\Theta(2,k)$ the linear subspace of Siegel modular forms of weight k which is spanned by all theta-series in case of $k \equiv 0 \pmod 4$. We shall prove $M(2,k) = \Theta(2,k)$ for $k \equiv 0 \pmod 4$. By Corollary of p. 195 in [5] $M(2,k)$ is spanned by $\psi_4^p\psi_6^q\psi_{10}^r\psi_{12}^s$, where the exponents p, q, r and s are non-negative integer solutions of the equation (6). Hence to prove that $M(2,k) = \Theta(2,k)$ for $k \equiv 0 \pmod 4$ we have only to show that

(*) each $\psi_4^p\psi_6^q\psi_{10}^r\psi_{12}^s$ of weight k, where $k = 4p + 6q + 10r + 12s$, belongs to $\Theta(2,k)$.

We shall prove the statement (*) by induction on k. For $k \leq 20$ the statement (*) is proved in Propositions 3.6, 3.7 and 3.8 and Lemma 3.9. We can assume that $k > 20$ and that the statement (*) is proved for $k_1 < k$ with $k_1 \equiv 0 \pmod 4$. Since k is divisible by 4, by Lemma 3.10 the exponents q and r in $\psi_4^p\psi_6^q\psi_{10}^r\psi_{12}^s$ are of the either type $q \equiv r \equiv 1 \pmod 2$ or $q \equiv r \equiv 0 \pmod 2$. In case of $q \equiv r \equiv 1 \pmod 2$, we rewrite $\psi_4^p\psi_6^q\psi_{10}^r\psi_{12}^s$ as $\psi_4^p\psi_6^{q-1}\psi_{10}^{r-1}\psi_{12}^s\cdot\psi_6\psi_{10}$, then by induction hypothesis we have $\psi_4^p\psi_6^{q-1}\psi_{10}^{r-1}\psi_{12}^s \in \Theta(2, k-16)$ and by Lemma 3.9(ii) $\psi_6\psi_{10} \in \Theta(2,16)$. Hence $\psi_4^p\psi_6^q\psi_{10}^r\psi_{12}^s$, where $q \equiv r \equiv 1$, belongs to $\Theta(2,k)$. In case of $r \equiv q \equiv 0 \pmod 2$ with $q > 0$, we rewrite $\psi_4^p\psi_6^q\psi_{10}^r\psi_{12}^s$ as $\psi_4^p\psi_6^{q-2}\psi_{10}^r\psi_{12}^s\cdot\psi_6^2$, then by induction hypothesis we have $\psi_4^p \psi_6^{q-2} \psi_{10}^r \psi_{12}^s \in \Theta(2, k-12)$ and by Lemma 3.9(i) $\psi_6^2 \in \Theta(2, 12)$. Hence in this case we have also $\psi_4^p \psi_6^q \psi_{10}^r \psi_{12}^s \in \Theta(2, k)$. In case of $r \equiv q \equiv 0 \pmod 2$ with $r > 0$, we rewrite $\psi_4^p \psi_6^q \psi_{10}^r \psi_{12}^s$ as $\psi_4^q \psi_6^q \psi_{10}^{r-2} \psi_{12}^s \cdot \psi_{10}^2$, then by induction hypothesis we have $\psi_4^p \psi_6^q \psi_{10}^{r-2} \psi_{12}^s \in \Theta(2, k-20)$ and by Lemma 3.9(iii) $\psi_{10}^2 \in \Theta(2, 20)$. Hence this time we have $\psi_4^p \psi_6^q \psi_{10}^r \psi_{12}^s \in \Theta(2, k)$. In case of $q = r = 0$ and p or $s > 0$, we can easily see that $\psi_4^p \psi_{12}^s \in \Theta(2, k)$ by using Lemma 3.9(i) and Proposition 3.4. We have thus proved our theorem.

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Об одной сумме в теории дзета-функции Римана

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Прежде чем сформулируем соответствующую теорему, введем нужные обозначения. Положим ([4], стр. 94)

$$Z(t) = e^{i\vartheta(t)}\zeta(\tfrac{1}{2}+it), \tag{1}$$

где ([4], стр. 383)

$$\vartheta(t) = \tfrac{1}{2}t\ln t - \tfrac{1}{2}t(\ln 2\pi + 1) - \tfrac{1}{8}\pi + O(1/t), \tag{2}$$

и, ([4], стр. 260)

$$\vartheta'(t) = \tfrac{1}{2}\ln t - \tfrac{1}{2}\ln 2\pi + O(1/t). \tag{3}$$

Пусть $\{t_\nu\}$ обозначает последовательность определенную соотношением (так как, в силу (3), функция $\vartheta(t)$ — возрастающая)

$$\vartheta(t_\nu) = \pi\nu, \tag{4}$$

где ν — целое положительное (ср. [4], стр. 261).

Пусть, наконец,

$$S(a, b) = \sum_{0<a\leqslant n<b\leqslant 2a} e^{it\ln n}, \quad b \leqslant \sqrt{t/2\pi}, \tag{5}$$

обозначает элементарную тригонометрическую сумму.

В этой работе покажем, что имеет место следующая

Теорема. *Если*

$$|S(a, b)| < \Delta\sqrt{a}\,t^{\Delta}, \quad 0 < \Delta < \tfrac{1}{4}, \tag{6}$$

то

$$\Big|\sum_{T\leqslant t_\nu\leqslant T+H} Z(t_\nu)\Big| < A(\Delta)T^{1/8+\Delta/2}\ln T, \tag{7}$$

где

$$0 < H \leqslant \sqrt[4]{T}. \tag{8}$$