3) Il existe une constante réelle $C'$ telle que, pour tout $P$ s'appartenant
pas aux fibres dégénérées:

$$h(P) \leq \Phi(P) + 2h_{e}(P) + C'.$$

4) L'encadrement obtenu en 2) et 3) est le plus fin possible, en ce sens
que, dans le cas $q = 3$, s'il existe $r_{1}, r_{2}, C$ dans $\mathbb{R}$ tels que, sur le complémen-
taire $\Omega$ des fibres dégénérées, on ait:

$$\forall P \in \Omega \setminus \{Q\}, \quad \Phi(P) + r_{1}h_{e}(P) - C \leq h(P) \leq \Phi(P) + r_{2}h_{e}(P) + C$$

alors $r_{1} \leq 1$ et $r_{2} \geq 2$.

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**References**


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**On basis problem for Siegel modular forms of degree 2**

by

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1. Introduction. In the theory of modular forms of a complex
variable there is a famous problem so called as “basis problem” (for
the details see [3]). In [1] van der Blij treated the special case of the
above problem. His main result can be stated as “the space of modular
forms of level one and of weight $k$ is spanned by the theta-series attached
to positive definite even integral quadratic forms of determinant unity
if and only if the weight $k$ is a multiple of 4". In this paper we shall treat
the corresponding problem in the case of Siegel modular forms of degree
two. Our main result is the following:

**Theorem.** Let $M(2, k)$ be the linear space of Siegel modular forms
of degree 2 and of weight $k$ ($k$ is an even non-negative integer), then $M(2, k)$
is spanned by theta-series attached to positive definite even integral quadratic
forms of determinant unity if and only if $k$ is a multiple of 4.

The proof of this theorem rests partly on equipment and precise
observation of certain positive definite even integral quadratic lattices
of determinant unity and partly on the work of Igusa [5] which
determines the graded structure of Siegel modular forms of degree 2.

2. Some preliminaries. Let $\mathbb{S}_{2}$ be Siegel upper-half space of degree 2
and $\varphi(\tau)$ be a Siegel modular form of degree 2 and of weight $k$, then
$\varphi(\tau)$ can be expanded in a Fourier series

$$\varphi(\tau) = \sum_{T \in T} a(T)e^{2\pi i \text{Tr}(T \tau)},$$

where $T$ runs over the set $T$ of all positive semi-definite semi-integral
matrices of size 3 and $\text{Tr}(T \tau)$ means the trace of the matrix $T \tau$ ([11], [5]).

**Proposition 2.1.** Let

$$\varphi_{1}(\tau) = \sum_{T \in T} a(T)e^{2\pi i \text{Tr}(T \tau)} \quad \text{and} \quad \varphi_{2}(\tau) = \sum_{T \in T} b(T)e^{2\pi i \text{Tr}(T \tau)}$$

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be Fourier expansions of Siegel modular forms of degree 2 with weights \( k_1 \) and \( k_2 \) respectively, then the product \( \varphi_1(\tau)\varphi_2(\tau) \) is of weight \( k_1 + k_2 \) and its Fourier expansion is given by:

\[
\varphi_1(\tau)\varphi_2(\tau) = \sum_{\mathcal{T}} c(\mathcal{T}) e^{2\pi i \mathcal{T} \tau},
\]

where \( c(\mathcal{T}) = \sum_{\Gamma_1, \Gamma_2} a(\mathcal{T}) b(\mathcal{T}) \) and \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) run over all possible pairs of solutions \( (\mathcal{T}_1, \mathcal{T}_2) \) of \( \mathcal{T}_1 + \mathcal{T}_2 = \mathcal{T} \) with \( \mathcal{T}_1, \mathcal{T}_2 \) and \( \mathcal{T} \) in \( \mathcal{L} \).

The proof of this proposition is clear and we omit it.

After Witt [12] and Richert [4] we shall consider positive definite integral quadratic forms in the language of lattices in the linear space over the field of rational numbers \( \mathbb{Q} \) with positive metric. We shall assume this settings throughout this paper. We shall say a lattice \( \mathcal{L} \) is integral if we have \( (x, y) \in \mathbb{Z} \) for any pair \( x \) and \( y \) in \( \mathcal{L} \), where \( (\ , \) \) means the positive metric of \( \mathcal{L} \) and \( \mathbb{Z} \) is the ring of rational integers. If \( \mathcal{L} \) is an integral lattice, then \( (x, x) \) is a positive integer for any \( x \in \mathcal{L} \) other than zero vector and we shall call \( x \in \mathcal{L} \) as \( m \)-vector when \( x \) satisfies \( (x, x) = m \) with some positive integer \( m \). We shall denote by \( \text{Aut}(\mathcal{L}) \) the group of all automorphisms of the lattice \( \mathcal{L} \). An integral lattice \( \mathcal{L} \) is called even integral if we have \( (x, x) \equiv 0 \pmod{2} \) for any \( x \in \mathcal{L} \). The determinant of a lattice \( \mathcal{L} \) is defined by the determinant of the quadratic form corresponding to \( \mathcal{L} \). It is known that if \( \mathcal{L} \) is an even integral quadratic lattice with determinant unity then the rank of \( \mathcal{L} \) is necessarily divisible by 8.

Now we define theta-series \( \vartheta(2, L) \) of degree 2 associated with even integral lattice \( L \) which is essentially the same thing as theta-series of degree 2 for positive definite even integral quadratic form. Let \( x \) and \( y \) be in \( \mathcal{L} \), then we denote by \([x, y] \) the matrix

\[
\begin{pmatrix}
(x, x) & (x, y) \\
(x, y) & (y, y)
\end{pmatrix}
\]

and theta-series \( \vartheta(2, L) \) is defined by:

\[
\vartheta(2, L) = \sum_{\alpha \in \mathcal{L}} e^{2\pi i \alpha^t \cdot \beta},
\]

where \( \alpha \) and \( \beta \) run over \( \mathcal{L} \) independently and \( \tau \) is a variable on \( \mathbb{H} \).

By Satz D of Witt [12] \( \vartheta(2, L) \) becomes a Siegel modular form of degree 2 with weight equal to the half of the rank of \( L \) when \( L \) is an even integral lattice of determinant unity. Since \( L \) is even integral lattice with positive metric, \([x, y] \) is always positive semi-definite even integral matrix of degree 2 for any \( x \) and \( y \) in \( L \). Let \( \mathcal{T} \) be in \( \mathbb{Z} \) and \( a(\mathcal{T}, L) \) be the number of solutions of pairs \( x \) and \( y \) in \( L \) such that \( [x, y] = 2\mathcal{T} \), then \( \vartheta(2, L) \)

can be expanded as:

\[
\vartheta(2, L) = \sum_{\mathcal{T}} a(\mathcal{T}, L) e^{2\pi i \mathcal{T} \tau}.
\]

It is clear that \( a(\mathcal{T}, L) \) is a finite non-negative integer for each \( \mathcal{T} \) and \( L \) and (4) is Fourier expansion of \( \vartheta(2, L) \). From now on we shall restrict ourselves to the case where \( L \) is even integral.

We shall equip some suitable lattices. Let \( \mathcal{A}_n, \mathcal{D}_n \) and \( \mathcal{E}_n \) be even integral lattices given in [6] or [8]. We shall conventionally use the following notations:

\[
\begin{align*}
\mathcal{A}_n &= [e_1 - e_2, e_3 - e_4, \ldots, e_{n-1} - e_n, e_{n+1}]_Z, \\
\mathcal{D}_n &= [e_1 - e_2, e_3 - e_4, \ldots, e_{n-1} - e_n, e_n + e_{n+1}]_Z, \\
\mathcal{E}_n &= [e_1 - e_2, \ldots, e_n - e_1, e_{n+1} + e_1, \frac{1}{2} \sum_{i=1}^{n} e_i]_Z
\end{align*}
\]

where \( e_1, \ldots, e_n \) are orthonormal vectors and \([e_1 - e_2, e_3 - e_4, \ldots, e_{n+1}]_Z \) means the lattice spanned by \( e_1 - e_2, \ldots, e_n - e_{n+1} \) over \( Z \) and so on.

**Proposition 2.2.** When and only when \( n+1 \) is a square integer congruent to one modulo 8 there exists an even integral lattice \( \mathcal{A}_n \) of rank \( n \) and of determinant unity containing \( \mathcal{A}_n \).

**Proof.** After Niemeier [8] we use the dual lattice \( \mathcal{A}_n^\perp \) of \( \mathcal{A}_n \). Since \( \mathcal{A}_n^\perp / \mathcal{A}_n \) is a cyclic group of order \( n+1 \) with its generator \( u = \frac{1}{n+1} \sum_{i=1}^{n+1} e_i - e_{n+1} \) modulo \( \mathcal{A}_n \) (8), it is easy to see that \( n+1 \) be a square (say \( r^2 \)) if \( \mathcal{A}_n \) exists at all because such \( \mathcal{A}_n \) must satisfy the isomorphism condition \( \mathcal{A}_n^\perp / \mathcal{A}_n \cong \mathcal{A}_n / \mathcal{A}_n \). In this case \( \mathcal{A}_n / \mathcal{A}_n \) is a cyclic group of order \( r \) and the representative of \( \mathcal{A}_n / \mathcal{A}_n \) must be of the form:

\[
(ru, ru) \quad \text{with} \quad 1 < j < r \quad \text{and} \quad (j, r) = 1.
\]

But there holds \( \mathcal{A}_n + Z \cdot ru = \mathcal{A}_n + Z \cdot ru \) for each pair of integers \( j \) and \( j_r \) such that \( (j, r) = (j_r, r) = 1 \). So we can take \( \mathcal{A}_n \) as \( \mathcal{A}_n + Z \cdot ru \). Since \( \mathcal{A}_n \) is even integral, the necessary and sufficient condition that \( \mathcal{A}_n \) is even integral is \((ru, ru)\) is a positive even integer and we see that

\[
(ru, ru) = \frac{ru^2}{(n+1)^2} + \frac{ru^2}{(n+1)^2} = n = r^2 - 1.
\]

Thus \( r^2 - 1 \) is an even integer and we can say that \( n+1 \) is a square integer congruent to one modulo 8. Conversely, if \( n+1 \) is such a number,
then $\tilde{A}_{n+1} = A_n + Zw$ with $s^4 = n+1$ and
\[ w = \frac{1}{n+1} \sum_{i=1}^{n} e_i - \frac{n}{n+1} e_{n+1} \]
is an even integral lattice of rank $n$ and of determinant unity. □

Proposition 2.3. When and only when $n$ is a multiple of 8 ($n \geq 8$) there exists an even integral lattice $\tilde{B}_n$ of determinant unity containing $\tilde{D}_n$.

The content of this proposition is already discussed at pp. 330–331 in [12] and we omit its proof. We only give the basis of $\tilde{D}_n (n \equiv 0 \mod 8)$ by:
\[ \tilde{D}_n = \left\{ e_2 - e_4, \ldots, e_{n-2} - e_{n-4}, e_{n-1} - e_{n-3}, \frac{1}{2} \sum_{i=1}^{n} e_i \right\} \]
$\tilde{D}_n$ is nothing else but $\tilde{B}_n$.

3. Some auxiliary lemmas and propositions. Let $L$ be an even integral lattice, then we denote by $V(m, L)$ the set of $m$-vectors in $L$ for each even integer $m$. We shall use the symbol $\|S\|$ for the cardinality of a finite set $S$.

Lemma 3.1. (i) $V(2, \tilde{A}_{24}) = V(2, A_{24})$ and $\|V(2, \tilde{A}_{24})\| = 600$. (ii) $V(4, \tilde{A}_{24})$ consists of two transitive classes $C_1$ and $C_2$ under the action of $\text{Aut}(\tilde{A}_{24})$, where $C_1$ and $C_2$ are given by:
\[ C_1 = \{ e_4 + e_3 - e_2 - e_1 | 1 \leq i_1, i_2, i_3, i_4 \leq 25 \}, \]
\[ C_2 = \left\{ \left\{ \sum_{i=1}^{25} e_i - (e_4 + e_3 + e_2 + e_1 + e_i) \right\} | 1 \leq i_1 < i_2 < i_3 < i_4 \leq 25 \right\}. \]

Proof of (i). Using the fact that
\[ 5w = \frac{1}{5} \sum_{i=1}^{25} e_i - \frac{25}{5} e_5 = \frac{1}{5} \sum_{i=1}^{25} e_i - \frac{25}{5} e_i \mod A_{24}, \]
we see that $\tilde{A}_{24}$ is also expressed as $A_{24} + Zw$ with $w = \frac{1}{5} \sum_{i=1}^{25} e_i$. As Niemeier remarked p. 150 in [8] $w$ has the following property:
\begin{align*}
(\text{iv}) \quad (w, w) &\leq (w + v, w + v) \quad \text{for all } v \in A_{24}.
\end{align*}
$w$ has the order 5 modulo $A_{24}$, that is, it holds that we have $\lambda, w = \lambda w \mod A$ with integers $\lambda_1$ and $\lambda_2$ if and only if $\lambda_1 = \lambda_2 (\mod 5)$.

It can be observed that:
\[ 2w = \sum_{i=1}^{25} e_i - \sum_{i=1}^{25} e_i \mod A_{24}, \]
\[ 3w = \sum_{i=1}^{25} e_i - \sum_{i=1}^{25} e_i \mod A_{24}, \]
\[ 4w = \sum_{i=1}^{25} e_i - \sum_{i=1}^{25} e_i \mod A_{24}, \]
and
\[ 5w = 0 \mod A_{24}. \]
\[ w_2 = \sum_{i=1}^{25} e_i - \sum_{i=1}^{25} e_i, \quad w_3 = \sum_{i=1}^{25} e_i - \sum_{i=1}^{25} e_i, \quad w_4 = \sum_{i=1}^{25} e_i - \sum_{i=1}^{25} e_i \]
have the property (iv) (see also pp. 134–135 in [8]) and we see that $(w_2, w_3) = 4, (w_2, w_4) = 6, (w_3, w_4) = 6$ and $(w_2, w_4) = 4$. By the above discussion we can say that $V(2, \tilde{A}_{24}) = V(2, A_{24})$. By calculating combinatorially, we get $\|V(2, \tilde{A}_{24})\| = 600$.

Proof of (ii). Since the series $\sum e_i w_2 w_3$, where $x$ is a complex variable with positive imaginary part, is a modular form of weight 12 of level 1, $\|V(4, \tilde{A}_{24})\|$, the number of 4-vectors in $\tilde{A}_{24}$, is given by the formula (11) of [9] with $n = 2$ and $1104 - 384 = 600$. Hence $\|V(4, \tilde{A}_{24})\|$ is 182160. Clearly $\tilde{A}_{24}$ contains the following two types of 4-vectors:
\[ C_1 = \{ e_4 + e_3 - e_2 - e_1 | 1 \leq i_1, i_2, i_3, i_4 \leq 25 \}, \]
\[ C_2 = \left\{ \sum_{i=1}^{25} e_i - (e_4 + e_3 + e_2 + e_1 + e_i) | 1 \leq i_1 < i_2 < i_3 < i_4 \leq 25 \right\}. \]

It is clear that $C_1$ and $C_2$ are disjoint sets. By calculating combinatorially we get $|C_1| = 759000$ and $|C_2| = 106360$ and $|C_1| + |C_2| = \|V(4, \tilde{A}_{24})\|$. This means that $V(4, \tilde{A}_{24}) = C_1 \cup C_2$ (disjoint union). Since $\text{Aut}(\tilde{A}_{24})$ contains $\text{Aut}(A_{24})$ as a subgroup it can be seen that $C_1$ (resp. $C_2$) is transitive under the action of $\text{Aut}(\tilde{A}_{24})$. If there exists an element $\varrho$ of $\text{Aut}(\tilde{A}_{24})$ such that $\varrho$ carries an element $\epsilon$ of $C_1$ into an element $\nu$ of $C_2$, then without loss of generality we can assume that
\[ \varrho = e_1 + e_5 - e_5 - e_4, \]
\[ \varrho = e_1 - e_5 + e_4 - e_3 \]
and we have $(\varrho, \varrho) = 2$. But $(\varrho, \varrho) = (\varrho, \varrho)$ is not equal to 2 because of the shape of $\varrho$ and because $e_1 - e_5$ is another 2-vector in $A_{24}$ and has the form $e_1 - e_5$ with $1 < j_2, j_2 < 25$ and $j_2 \neq j_2$. This means that such $\varrho \in \text{Aut}(\tilde{A}_{24})$ does not exist. □
Lemma 3.2. (i) $V(2, E_8)$ is transitive under the action of $\text{Aut}(E_8)$ and $\|V(2, E_8)\| = 240$. (ii) $V(4, E_8)$ is transitive under the action of $\text{Aut}(E_8)$ and $\|V(4, E_8)\| = 2160$.

The transitivity statement of this lemma is asserted by Hilfssätze (5.4) of [8] and $\|V(2, E_8)\| = 240$ and $\|V(4, E_8)\| = 2160$ are merely calculations and we omit those.

**Lemma 3.3.** (i) $V(2, \Phi_m) = V(2, D_m)$ ($n \geq 2$) is transitive under the action of $\text{Aut}(\Phi_m)$ and $\|V(2, \Phi_m)\| = 16n(8n-1)$; (ii) $V(4, \Phi_m)$ consists of three transitive classes $C_3$, $C_4$ and $C_5$ under the action of $\text{Aut}(\Phi_m)$, where $C_3$, $C_4$ and $C_5$ are given by:

$$
C_3 = \{ \pm 2e_i \mid 1 \leq i \leq 16 \},
$$

$$
C_4 = \{ \pm e_i \pm e_j \pm e_k \pm e_l \mid 1 \leq i < j < k < l \leq 16 \},
$$

$$
C_5 = \{ \sum_{i=1}^{16} x_i e_i \mid x_i = \pm 1, \sum_{i=1}^{16} x_i = 1 \}.
$$

(iii) $V(4, \Phi_m)$ ($n \geq 3$) consists of two transitive classes $C_6(8n)$ and $C_7(8n)$ under the action of $\text{Aut}(\Phi_m)$, where $C_6(8n)$ and $C_7(8n)$ are given by:

$$
C_6(8n) = \{ \pm 2e_i \mid 1 \leq i \leq 8n \},
$$

$$
C_7(8n) = \{ \pm e_i \pm e_j \pm e_k \pm e_l \mid 1 \leq i < j < k < l \leq 8n \}.
$$

Proof of (i). Since $V(2, \Phi_m)$ ($n \geq 2$) consists of $\pm e_i \pm e_j$ with $1 \leq i < j \leq 8n$, the transitivity is clear. By calculating we get $\|V(2, \Phi_m)\| = 16n(8n-1)$.

Proof of (ii). Since the series $\sum_{\Phi_m} e^{\alpha(\Phi_m)}$ is a modular form of weight 8 and of level 1 and the dimension of the space of modular forms of weight 8 and of level 1 is one, $\sum_{\Phi_m} e^{\alpha(\Phi_m)}$ must be equal to primitive Eisenstein series of weight 8, namely, to

$$
1 + 480 \sum_{n=1}^{\infty} \frac{\sigma_1(n)e^{2\pi in}}{n^2}, \quad \text{where} \quad \sigma_1(n) = \sum_{d \mid n} d.
$$

So we can say that $\|V(4, \Phi_m)\| = 480 \sigma_1(2) = 61920$. It can be seen that $C_3$, $C_4$ and $C_5$ are mutually disjoint subsets of $V(4, \Phi_m)$. An easy computation shows that $|C_3| = 32$, $|C_4| = 2560$, and $|C_5| = 32768$ and that $|C_3| + |C_4| + |C_5| = \|V(4, \Phi_m)\|$. This means that $V(4, \Phi_m) = C_3 \cup C_4 \cup C_5$ (disjoint union). Since $\text{Aut}(\Phi_m)$ is generated by reflections with respect to 2-vectors in $\Phi_m$, it can be observed that $C_3$, $C_4$ and $C_5$ are transitive classes under the action of $\text{Aut}(\Phi_m)$.

Proof of (iii). Since we know $\Phi_m = D_m = Z + Z \mathbf{u}$ with $\mathbf{u} = \frac{1}{2} \sum_{i=1}^{\infty} e_i$ and $\mathbf{u}$ has the property (see p. 150 of [8]):

$$\langle u, u \rangle \leq \langle u + v, u + v \rangle \quad \forall v \in D_m,
$$

we can say that $V(4, \Phi_m) = V(4, D_m)$ for $n \geq 3$. It is easy to verify that $V(4, \Phi_m)$ is a disjoint union of $C_6(8n)$ and $C_7(8n)$ and that they are transitive classes of $\text{Aut}(\Phi_m)$.

It should be remarked that

$$
\|C_6(8n)\| = 16n \quad \text{and} \quad \|C_7(8n)\| = 416n(8n-1)(8n-2)(8n-3).
$$

Now we shall describe the process of calculating Fourier coefficients $a(T, L)$ of $\theta(2, L)$ for some $T \in \mathbb{Z}$ and for some even integral lattice $L$. For the later convenience we set:

$$
T_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad T_7 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix},
$$

It is obvious that $a(T_0, L) = \|V(2, L)\|$ and $a(T_3, L) = \|V(4, L)\|$ for each even integral $L$. To calculate $a(T_3, L)$ we need the number of pairs of 2-vectors $x$ and $y$ in $L$ such that $\langle x, y \rangle = 0$. $a(T_3, L)$ is the number of pairs of 2-vectors $x$ and $y$ such that $\langle x, y \rangle = 1$. $a(T_5, L)$ is the number of pairs of 2-vector $x$ and 4-vector $y$ such that $\langle x, y \rangle = 0$ and so on. We shall number theta-series attached to special even integral lattices of determinant unity as follows:

$$
\theta_j = \theta(2, L) = \sum_{T \in T} a(T, L)e^{\alpha(\Phi_m)},
$$

where $L_4 = E_8$, $L_2 = \Phi_4$, $L_2 = E_8 \oplus \mathbb{Z}_2 \oplus E_8$ (orthogonal sum), $L_4 = \Phi_4$, $L_6 = E_8 \oplus \mathbb{Z}_2 \oplus E_8$, $L_9 = \Phi_4 \oplus \mathbb{Z}_3 \oplus E_8$, $L_9 = \Phi_4 \oplus \mathbb{Z}_3 \oplus E_8$, $L_11 = \Phi_4 \oplus \mathbb{Z}_5 \oplus E_8$, $L_{11} = \Phi_4 \oplus \mathbb{Z}_5 \oplus E_8$, $L_{13} = \Phi_4 \oplus \mathbb{Z}_7 \oplus E_8$, $L_{13} = \Phi_4 \oplus \mathbb{Z}_7 \oplus E_8$. We should keep in mind that $\theta_j$ is Siegel modular form of weight 4, $\theta_j$ is Siegel modular form of weight 8 (i.e., $\theta_j \in M(2, 8)$), $\theta_j$, $\theta_j$, and $\theta_j$ are in $M(2, 12)$, $\theta_j$, $\theta_j$, and $\theta_j$ are in $M(2, 16)$, $\theta_j$, $\theta_j$, and $\theta_j$ are in $M(2, 24)$. As typical calculations of $a(T, L)$ we show how $a_1(T_3)$, $a_2(T_3)$, $a_2(T_3)$, and $a_2(T_3)$ are calculated.
Taking 2-vector $e_1 - e_2 \in E_2$, then we must look for all 2-vectors $y \in E_6$ such that $(e_1 - e_2, y) = 0$. Solutions of such $y$'s are given by:

$$\pm (e_1 + e_2), \quad \pm e_1 \pm e_2, \quad 3 \leq i < j \leq 8$$

and

$$\pm \frac{1}{2} \left( e_1 + e_2 + \sum_{i=3}^{8} e_i q_i \right), \quad s_i = \pm 1, \quad \prod_{i=3}^{8} s_i = 1.$$

(It is understood henceforth that the indices $i, j, \ldots$ of $e_i, e_j, \ldots$ are mutually different in their range of running.) The number of such $y$'s is 126. By Lemma 3.3(iii) we can say that to any 2-vector $x \in E_6$ the number of 2-vectors $y$ such that $(x, y) = 0$ is equal to 126, so we have $a_2(T_3) = 240 \times 126$. Taking 2-vector $e_1 - e_2 \in E_6$ and looking for all 4-vectors $y \in E_8$ such that $(e_1 - e_2, y) = 1$, we get solutions $y$ as follows:

$$e_1 \pm e_2 \pm e_3 \pm e_4, \quad 3 \leq i < j < k \leq 8,$$

and

$$\pm \frac{1}{2} \left( 2e_1 + e_2 + \sum_{i=3}^{8} e_i q_i \right), \quad s_i = \pm 1, \quad \prod_{i=3}^{8} s_i = 1,$$

and

$$\pm \frac{1}{2} \left( e_1 - e_2 + \sum_{i=3}^{8} e_i q_i \right), \quad s_i = \pm 1, \quad \prod_{i=3}^{8} s_i = 1.$$

The number of such $y$'s is 576. By Lemma 3.3(iii), (iv) we can say that to any 2-vector $x \in E_6$ the number of 4-vectors $y$ such that $(x, y) = 1$ is equal to 576, so we have $a_4(T_4) = 240 \times 126$. Calculation of $a_4(T_4)$ is a little complicated. We pick up 2 vectors $2e_1 + e_2 + e_3 + e_4 + e_5 + e_6$ and $\frac{1}{2} \sum_{i=3}^{8} e_i \in C_6$ as the representatives of transitive classes $C_5$, $C_4$ and $C_6$ in Lemma 3.3(ii).

4-vectors $y \in \tilde{A}_{18}$ such that $(2e_1, y_3) = 0$ are given by:

$$\pm 2e_i, \quad 2 \leq i \leq 16$$

and

$$\pm e_1 \pm e_2 \pm e_3 \pm e_4, \quad 2 \leq i < j < k < l \leq 16.$$

The number of such $y$'s is 21870. 4-vectors $y \in \tilde{A}_{18}$ such that $(e_1 + e_2 + e_3 + e_4, y) = 0$ are given by:

$$\pm 2e_i, \quad 5 \leq i \leq 16,$$

and

$$\pm (e_1 - e_4) \pm e_1 \pm e_4, \quad 1 \leq i_1 < i_2 < i_3 < i_4 \leq 16,$$

$$\pm e_1 \pm e_2 \pm e_3 \pm e_4, \quad 5 \leq i_1 < i_2 < i_3 < i_4 \leq 16,$$

$$(e_1 + e_2 - e_3 - e_4), \quad \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}.$$
are given by:
\[-(e_1 + e_2) + e_3 + e_4, \quad 1 \leq i_1, i_2 \leq 5, \quad 6 \leq i_3, i_4 \leq 25 \]
and
\[
\frac{1}{2} \sum_{i=1}^{25} e_i = (e_1 + e_2 + e_3 + e_4 + e_6), \quad 1 \leq i_1, i_2, i_3 \leq 5, \quad 6 \leq i_4, i_5 \leq 25.
\]
The number of such \( y \)'s is 3800. We have
\[
a_q(T_i) = 4760 \times |C_1| + 3800 \times |C_2| = 765072000.
\]
Lemmas 3.1-3.3 will be sufficient for calculating \( a_i(T_j) \) for \( 0 \leq j \leq 9 \) and \( i = 1, 2, 4, 5, 9, 14 \). To calculate \( a_q(T_j) \) for \( 0 \leq j \leq 9 \) and \( i = 3, 6, 7, 8, 10, 11, 12, 13 \) we have only to utilize Proposition 2.1 and the following facts (I), (II) and (III).

(I) \( \vartheta_3 = \vartheta_1, \vartheta_4 = \vartheta_3, \vartheta_5 = \vartheta_6, \vartheta_6 = \vartheta_2, \vartheta_7 = \vartheta_3, \vartheta_8 = \vartheta_4, \vartheta_9 = \vartheta_5, \vartheta_{10} = \vartheta_6. \)

(II) The decompositions:
\[
T_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 1/4 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \quad T_7 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_8 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \quad T_9 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.
\]

(III) Let \( \varphi(v) = \sum_{T} a(T) e^{2\pi i m(T^t \cdot v)} \) be Fourier expansion of Siegel modular form \( \varphi(v) \) of degree 2, then it holds that \( a(T) = a(U^t T U) \) for any unitmodular matrix \( U \) of degree 2, where \( U^t \) denotes the transpose of \( U \) (formula (48) in [11]). As special case we have
\[
a \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad a \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad a \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad a \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\]
along the above way we get the following table of \( a_i(T_j) \).

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The blanks in the above table are not necessary for our purpose.
PROPOSITION 3.4. The dimension of $\mathcal{M}(2, 4)$ is one and $\mathcal{M}(2, 4)$ is spanned by $\vartheta_1$.

Proof. The former part of the statement is obtained by Corollary of p. 194 in [5] and the latter part is clear because $\vartheta_2$ is a Siegel modular form of weight $4$.

PROPOSITION 3.5. The dimension of $\mathcal{M}(2, 8)$ is one and $\mathcal{M}(2, 8)$ is spanned by $\vartheta_1$.

Proof. The former part of the statement is obtained by Corollary of p. 194 in [5] and the latter part is clear because $\vartheta_3$ is a Siegel modular form of weight $8$.

Remark. E. Witt [12] obtained the result $\vartheta_2^2 = \vartheta_2$.

PROPOSITION 3.6. The dimension of $\mathcal{M}(2, 12)$ is $3$ and $\mathcal{M}(2, 12)$ is spanned by $\vartheta_1, \vartheta_2, \vartheta_3$.

Proof. The former part of the statement is obtained by Corollary of p. 194 in [5] and we have only to show that $\vartheta_2, \vartheta_3$, and $\vartheta_4$ are linearly independent because they are Siegel modular forms of weight $12$. Since we can verify that the determinant of the matrix $(a_i(T_j))$, where $i = 3, 4, 5$ and $j = 0, 1, 3$, is different from zero, we can conclude $\vartheta_2, \vartheta_3$, and $\vartheta_4$ are linearly independent.

PROPOSITION 3.7. The dimension of $\mathcal{M}(2, 16)$ is $4$ and $\mathcal{M}(2, 16)$ is spanned by $\vartheta_1, \vartheta_2, \vartheta_3$, and $\vartheta_4$.

Proof. The former part of the statement is obtained by Corollary of p. 194 in [5] and we have only to show that $\vartheta_2, \vartheta_3$, $\vartheta_4$, and $\vartheta_5$ are linearly independent because they are Siegel modular forms of weight $16$. Since we can verify that the determinant of the matrix $(a_i(T_j))$, where $i = 6, 7, 8, 9$ and $j = 0, 1, 3, 4$, is different from zero, we can conclude $\vartheta_2, \vartheta_3$, $\vartheta_4$, and $\vartheta_5$ are linearly independent.

PROPOSITION 3.8. The dimension of $\mathcal{M}(2, 20)$ is $5$ and $\mathcal{M}(2, 20)$ is spanned by $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$, $\vartheta_5$, and $\vartheta_6$.

Proof. The former part of the statement is obtained by Corollary of p. 194 in [5] and we have only to show that $\vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5$, and $\vartheta_6$ are linearly independent because they are Siegel modular forms of weight $20$. Since we can verify that the determinant of the matrix $(a_i(T_j))$, where $i = 10, 11, 12, 13, 14$, and $j = 0, 1, 3, 4, 5$, is different from zero, we can conclude $\vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5$, and $\vartheta_6$ are linearly independent.

As the immediate consequence of Propositions 3.6, 3.7, and 3.8, we have the following:

LEMMA 3.9. Let $\varphi_k$ be Eisenstein series of degree $2$ and of weight $k$ (see p. 645 of [11] or p. 189 of [5]), then (i) $\varphi_k$ and $\varphi_{2k}$ are expressed as linear combinations of $\vartheta_1$, $\vartheta_2$, and $\vartheta_3$, (ii) $\varphi_k \varphi_{2k}$ is expressed as linear combinations of $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5$, and $\vartheta_6$, and (iii) $\varphi_4 \varphi_{10}$ is expressed as linear combinations of $\vartheta_4, \vartheta_5, \vartheta_6, \vartheta_7$, and $\vartheta_8$.

Remar. $\varphi_6 = \varphi_4 \varphi_2$ may be one of interesting Siegel modular forms of degree $2$. This is unique cusp form of weight $10$ (up to a constant factor) and is not expressed as linear combination of theta-series. But we have the following equations:

\[
\begin{align*}
2^{-30}3^{-18}5^{-7}7^{-3}23^{-1}3^{-3}25^{-1}43867 & (\varphi_6 - \varphi_4 \varphi_2)^2 \\
2^{-30}5^{-11}7^{-1}11^{-1}13^{-1}17^{-1}19^{-1}23^{-1}29^{-1}31^{-1} & (\varphi_6 - \varphi_4 \varphi_2)^2 \\
6 & = 6^{e^2 + \text{cubic}(T_0)} - 4e^{\text{cubic}(T_0)} + \ldots
\end{align*}
\]

LEMMA 3.10. Assume that $k$ is a positive integer divisible by $4$, then the non-negative integer solutions of the linear Diophantine equation

\[
k = 4p + 6q + 10r + 12s
\]

are exhausted by the following types:

(i) $q = r = 1$ (mod 2) or
(ii) $q = r = 0$ (mod 2).

Proof. Otherwise we get a contradiction.

4. Proof of Theorem. Since the rank of even integral lattice $L$ of determinant unity is divisible by $8$, $\Theta(2, L)$ must be of weight $k$ divisible by $4$. So the proof of "only if" part is clear. For the brevity of later descriptions we denote by $\Theta(2, k)$ the linear subspace of Siegel modular forms of weight $k$ which is spanned by all theta-series in case of $k = 0$ (mod 4). We shall prove $\Theta(2, k) = \Theta(2, k)$ for $k = 0$ (mod 4). By Corollary of p. 195 in [5] $\mathcal{M}(2, k)$ is spanned by $\varphi_k \varphi_6 \varphi_2$ and $\varphi_4 \varphi_{10} \varphi_{12}$, where the exponents $p, q, r$, and $s$ are non-negative integer solutions of the equation (6). Hence to prove that $\Theta(2, k) = \Theta(2, k)$ for $k = 0$ (mod 4) we have only to show that

(*) each $\varphi_k \varphi_6 \varphi_2 \varphi_{12}$ of weight $k$, where $k = 4p + 6q + 10r + 12s$, belongs to $\Theta(2, k)$.

We shall prove the statement (*) by induction on $k$. For $k \leq 20$ the statement (*) is proved in Propositions 3.6, 3.7, and 3.8 and Lemma 3.9. We can assume that $k > 20$ and that the statement (*) is proved for $20 \leq k \leq k$ with $k = 0$ (mod 4). Since $k$ is divisible by $4$, by Lemma 3.10 the exponents $q$ and $r$ in $\varphi_k \varphi_6 \varphi_2 \varphi_{12}$ are of the either type $q = r = 1$ (mod 2) or $q = r = 0$ (mod 2). In case of $q = r = 1$ (mod 2), we rewrite $\varphi_k \varphi_6 \varphi_2 \varphi_{12}$ as $\varphi_k \varphi_6 \varphi_2 \varphi_{12} - \varphi_k \varphi_6 \varphi_2 \varphi_{12}$, then by induction hypothesis we have $\varphi_k \varphi_6 \varphi_2 \varphi_{12}$ for $\Theta(2, k - 16)$ and by Lemma 3.9(2) $\varphi_k \varphi_6 \varphi_2 \varphi_{12}$ for $\Theta(2, k)$. Hence $\varphi_k \varphi_6 \varphi_2 \varphi_{12}$, where $q = r = 1$, belongs to $\Theta(2, k)$. In case of $q = r = 0$ (mod 2) with $q > 0$, we rewrite $\varphi_k \varphi_6 \varphi_2 \varphi_{12}$ as $\varphi_k \varphi_6 \varphi_2 \varphi_{12}$, then by
induction hypothesis we have $\psi^r \psi^{-r} \psi_1 \psi_3 e^{\Theta(2, k-12)}$ and by Lemma 3.9(i) $\psi_0 e^{\Theta(2, 12)}$. Hence in this case we have also $\psi^r \psi_0 \psi_1 \psi_3 e^{\Theta(2, k)}$.

In case of $r = q = 0 \pmod{2}$ with $r > 0$, we rewrite $\psi^r \psi_0 \psi_1 \psi_3$ as $\psi^{r'} \psi_1 \psi_3$, then by induction hypothesis we have $\psi^{r'} \psi_1 \psi_3 e^{\Theta(2, k-20)}$ and by Lemma 3.9(iii) $\psi_0 e^{\Theta(2, 20)}$. Hence this time we have $\psi^{r'} \psi_0 \psi_1 \psi_3 e^{\Theta(2, k)}$. In case of $q = r = 0$ and $p > 0$, we can easily see that $\psi^p \psi_0 e^{\Theta(2, k)}$ by using Lemma 3.9(i) and Proposition 3.4. We have thus proved our theorem.

References


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Ob одной сумме в теории зета-функции Римана

Ян Моллер (Иоаниславет)

Прежде чем сформулируем соответствующую теорему, введем нужные обозначения. Положим ([1], стр. 94)

$$Z(t) = e^{it\theta} \xi(\frac{1}{2} + it),$$

gде ([4], стр. 388)

(1)

$$\varphi(t) = \frac{1}{2} \ln t - \frac{1}{4} (\ln 2\pi + 1) - \frac{1}{4} \pi + O(1/t),$$

и ([4], стр. 260)

$$\theta'(t) = \frac{1}{2} \ln t - \frac{1}{4} \ln 2\pi + O(1/t).$$

Пусть $\{t_n\}$ обозначает последовательность определенной соответствием (так как, в силу (2), функция $\theta(t)$ — возрастающая)

(2)

$$\varphi(t_n) = \pi v,$$

gде $v$ — целое положительное (ср. [4], стр. 261).

Пусть, наконец,

(3)

$$S(a, b) = \sum_{0 < c < a, c \equiv b \pmod{a}} \varphi(a),$$

обозначает элементарную тригонометрическую сумму.

В этой работе покажем, что имеет место следующая Теорема. Если

(4)

$$|S(a, b)| < A \sqrt{at}, \quad 0 < A < \frac{1}{2},$$

tо

(5)

$$\left| \sum_{T \leq t \leq T + H} Z(t_n) \right| < A (A) T^{3/2 + A \ln T},$$

где

(6)

$$0 < H < \frac{1}{10} T.$$