Bibliographie

[1] J. M. Fontaine, *Groupes de ramification et représentations d'Artin*, Ann. Scient. Ec. Norm. Sup., t. 4, fac. 3, 1971, p. 337–392.
[2] H. Hasse, *Zahlentheorie*, Akademie Verlag, Berlin 1949.
[3] E. Maus, *Existenz p-adischer Zahlkörper zu vorgegebenem Verzweigungsverhalten*, Dissertation, Hamburg 1965.
[4] Shankar Sen, *Ramification in p-adic Lie extensions*, Inventiones Math. 50 (1972), p. 44–50.
[5] J. P. Serre, *Corps locaux*, Hermann, Paris 1968.
[6] B. Wyman, *Wildly ramified gamma extensions*, Amer. J. Math., Janv. 1969, p. 135–152.

Reçu le 9. 12. 1974 (655)

ACTA ARITHMETICA
XXX (1976)

On two conjectures of Kátai

by

P. D. T. A. Elliott (Boulder, Colo.)

1. An arithmetical function $f(n)$ is said to be additive if $f(ab) = f(a)+ +f(b)$ whenever a and b are coprime integers, and completely additive if this relation holds whether they are coprime or not.

In this paper I establish two conjectures of Kátai, the one a particular case of the other.

Let $f(n)$ be an additive arithmetic function. For each real number $x \geqslant 1$ define

$$M(x) = \max_{n\leqslant x}|f(n)|, \qquad E(x) = \max_{p\leqslant x}|f(p+1)|$$

where in the definition of $E(x)$, p runs over prime numbers only.

Theorem. *There are positive absolute constants A, B and c, so that*

$$M(x) \leqslant AE(x^B)+AM\big((\log x)^B\big) \qquad (x \geqslant c).$$

Corollary 1. *Let $f(n)$ be a completely additive arithmetic function. Then there are positive constants A, B and c (possibly different from those in the theorem) so that*

$$M(x) \leqslant AE(x^B) \qquad (x\geqslant c).$$

Corollary 2. *Let $f(n)$ be a completely additive arithmetic function and let*

$$|f(p+1)| \leqslant A\log(p+1)$$

hold for every prime p. Then there is a positive absolute constant B so that

$$|f(n)| \leqslant AB\log n \qquad (n\geqslant 1).$$

Corollary 3. *Let $f(n)$ be completely additive and satisfy*

$$\lim_{p\to\infty}\frac{f(p+1)}{\log p} = 0 \qquad (p\ \textit{prime}).$$

Then $f(n)$ is identically zero.

The Corollary 2 of this theorem was conjectured in 1969 by Kátai [10]. He established the inequality $|f(n)| \leqslant k \log n \log\log 10n$, where k depends weakly upon f, subject to the validity of the Riemann–Piltz conjecture for L-series. He conjectured the validity of Corollary 1 in another paper [11], question 5.

To deduce Corollary 1 set $y = (\log x)^B$. Then if x is sufficiently large, $y^{A+1} \leqslant x$ and

$$(A+1)\, M(y) \leqslant M(y^{A+1}) \leqslant M(x) \leqslant AE(x^B) + AM(y).$$

Hence $M(y) \leqslant AE(x^B)$ and so $M(x) \leqslant 2AE(x^B)$.

To deduce Corollary 2, let n be a positive integer, $n \geqslant 2$. Choose an integer $k \geqslant 1$ so large that $n^k > c$, where the constant c is that which appears in Corollary 2. Then

$$|f(n)| = \frac{1}{k}|f(n^k)| \leqslant \frac{1}{k} A \log n^{kB} = AB \log n,$$

and Corollary 2 is established.

Corollary 3 can be similarly proved.

Remark. We shall use c_0, c_1, c_2 to denote constants. These will generally be absolute. From time to time it will be convenient to renumber them.

This ends the remark.

2. We need three results concerning the distribution of prime numbers in arithmetic progressions with large moduli.

For each real number $x \geqslant 1$ and pair of integers D $(\geqslant 1)$ and l, let $\pi(x, D, l)$ denote the number of primes p not exceeding x which satisfy the congruence $p \equiv l \pmod{D}$.

Lemma 1. *Let ε be a real number, $0 < \varepsilon < 1$. Then there is a positive real number $c = c(\varepsilon)$ so that the estimate*

$$\pi(x, D, l) = (1+\theta\varepsilon)\frac{x}{\varphi(D)\log x} \qquad (x \geqslant 2,\ |\theta| \leqslant 1),$$

holds uniformly for all l prime to D, for all moduli D not exceeding x^c with the possible exception of certain moduli, all of which are multiples of a particular D_0.

Remark. Although D_0 may depend upon c and x, it will satisfy $D_0 > (\log x)^{C_1}$ for any fixed $C_1 > 0$ and all sufficiently large values of x.

Proof. We recall the explicit formula

$$(1) \qquad \sum_{n \leqslant x} \chi(n)\Lambda(n) = E_0 x - E_1 \frac{x^{\beta_1}}{\beta_1} - \sum_{|\gamma| \leqslant T} \frac{x^\varrho}{\varrho} + O\left(\frac{x}{T}\log^2 Dx + E_1 x^{1/4}\log Dx\right)$$

where χ is a Dirichlet character mod D, $x \geqslant T \geqslant 2$, $E_0 = 1$ if χ is principal, and zero otherwise, and $E_1 = 1$ if an exceptional zero β_1 exists, and $= 0$ otherwise. The sum which appears on the right hand side of this estimate runs over the zeros $\varrho = \beta + i\gamma$ of the associated Dirichlet L-series

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s},$$

except for the possible exceptional zero β_1 and its companion $1-\beta_1$. (See, for example, Prachar [14], VII, § 4, p. 233.)

We shall need an estimate involving $N(\alpha, T, \chi)$ the number of zeros of $L(s, \chi)$ which lie in the rectangle $\alpha \leqslant \sigma \leqslant 1$, $|t| \leqslant T$ $(s = \sigma + it)$. We count here the possible exceptional zeros as well. To be precise we need the result (Fogels [8]) that there is a positive absolute constant C_2 so that

$$(2) \qquad \sum_{\chi \pmod{D}} N(\alpha, T, \chi) \leqslant T^{C_2(1-\alpha)}$$

uniformly for all $T \geqslant 2D$, $0 \leqslant \alpha \leqslant 1$.

Suppose now that no L-series defined $\pmod{D}$ vanishes in the rectangle $1-\eta \leqslant \sigma \leqslant 1$, $|t| \leqslant T$, where $\eta > 0$. Then

$$\sum_{\chi \pmod{D}} \left|\sum_{|\gamma| \leqslant T}' \frac{x^\varrho}{\varrho}\right| \leqslant \sum_{\chi \pmod{D}} \left(x^{1/2}\eta^{-1}T^{C_2} + \sum_{1/2 \leqslant \beta < 1} x^\beta\right)$$

$$\leqslant \eta^{-1}x^{1/2}T^{C_2}D + \sum_{\chi \pmod{D}} \left(x^{1/2}\log x + \int_{1/2}^{1-\eta} x^\sigma \log x\, d\sigma\right)$$

$$\leqslant 2\eta^{-1}x^{1/2}T^{C_2}D\log x + \int_{1/2}^{1-\eta} \sum_{\chi \pmod{D}} N(\sigma, T, \chi) x^\sigma \log x\, d\sigma.$$

We can estimate the size of this last integral, by means of (2), not to exceed

$$T^{C_2}\log x \int_{1/2}^{1-\eta} \left(\frac{x}{T^{C_2}}\right)^\sigma d\sigma < 2x(xT^{-C_2})^{-\eta}$$

provided that $x^{1/2} \geqslant T^{C_2}$, say.

Multiplying (1) by $\bar{\chi}(l)/\varphi(D)$ and summing over all characters $\chi \pmod{D}$, we deduce that

$$(3) \qquad \left|\sum_{\substack{n \equiv l \pmod{D} \\ n \leqslant x}} \Lambda(n) - \frac{x}{\varphi(D)}\right|$$

$$\leqslant \frac{E_1}{\varphi(D)} \cdot \frac{x^{\beta_1}}{\beta_1} + \frac{2\eta^{-1}x^{1/2}T^{C_2}D\log x + 2x(xT^{-C_2})^{-\eta}}{\varphi(D)} +$$

$$+ O\left(\frac{x\log^2 Dx}{T} + E_1 \frac{x^{1/4}\log Dx}{\varphi(D)}\right).$$

We recall two further results from the theory of L-series.

Let N be a real number, $N \geqslant 2$. Then the L-series formed with real characters $\chi \pmod D$ have no zeros in the region

$$\sigma \geqslant 1 - \frac{c_3}{\log N}, \quad t = 0 \quad (c_3 > 0),$$

for all moduli D not exceeding N, with the possible exception of certain moduli, all of which are multiples of a particular modulus D_1. If $c_4 > 0$ and N is sufficiently large then we may assume that $D_1 > (\log N)^{c_4}$. (See Prachar [14], Satz 6.6, p. 129, and Satz 8.1 (Siegel's theorem), p. 143, respectively.)

There is a positive constant c_5 so that there are no zeros of any L-series $\pmod D$ for any character $\pmod D$, which lie in the region

$$(4) \qquad \frac{3}{4} \leqslant 1 - \frac{c_5}{\log D(|t|+2)} \leqslant \sigma \leqslant 1 \quad (t \text{ arbitrary})$$

with the possible exception of at most one real zero β_1 (the so-called exceptional zero). For this result we refer to Prachar [14], Satz 6.9, p. 130.

We apply the first of these two remarks with $N = x^c$, $0 < c < 1$. Then if D is not an exceptional modulus, and χ is real, $L(s, \chi)$ does not vanish on the line segment $t = 0$, $\sigma \geqslant 1 - c_3(c\log x)^{-1}$. Moreover, from (4) with $T = D^\delta \log^3 Dx$ ($\delta > 0$ to be chosen presently), there are no zeros of any L-series $\pmod D$ (other than possibly β_1), in the rectangle

$$1 \geqslant \sigma \geqslant 1 - c_5\big(3(1+\delta)\log D + 5\log\log x\big)^{-1}, \qquad |t| \leqslant T,$$

say. This certainly holds if x is sufficiently large in terms of δ.

Since $D \leqslant x^c$ we conclude that there is a positive absolute constant c_6, so that for $x \geqslant x_0(\delta, c)$ no L-series $\pmod D$ vanishes in the rectangle

$$1 - \frac{c_6}{c(1+\delta)\log x} \leqslant \sigma \leqslant 1, \qquad |t| \leqslant D^\delta \log^3 Dx.$$

For such a modulus D we can set $\eta = c_6\{c(1+\delta)\log x\}^{-1}$ in (3). Then if c is sufficiently small (but fixed)

$$\eta^{-1} x^{1/2} T^{C_2} D \log x = O\big(x^{1/2}(D^\delta \log^3 Dx)^{C_2} D (\log x)^2\big) = O\big(x\{\varphi(D)\log x\}^{-1}\big).$$

Since $E_1 = 0$ for the modulus under consideration

$$\Big|\sum_{\substack{n \equiv l \pmod D\\ n \leqslant x}} \Lambda(n) - \frac{x}{\varphi(D)}\Big|$$

$$\leqslant \frac{2x}{\varphi(D)} \exp\Big(-\frac{c_6}{2(1+\delta)c}\Big) + O\Big(\frac{x}{\varphi(D)\log x}\Big) + O\Big(\frac{x}{D^\delta \log Dx}\Big).$$

The estimate of Lemma 1 may now be deduced by removing the contribution of the prime powers p^m, with $m \geqslant 2$, and integrating by parts. This will be justifiable if the size of c is slightly decreased.

We are left with the exceptional moduli. They are all multiples of D_1, and this modulus satisfies $D_1 > (c\log x)^{c_4}$, for each fixed $c_4 > 0$ and all sufficiently large values of x.

This completes the proof of Lemma 1.

A value for the constant c (in terms of ε) could be computed if desired.

Lemma 2. *Let D be a prime power, $D = q^m$. Then there is a positive constant c_7 so that there is a prime p, not exceeding D^{c_7}, which satisfies*

$$p \equiv -1 \pmod D, \qquad p \not\equiv -1 \pmod{qD}.$$

Remark. In our application of Lemma 2 essential use will be made of the fact that q divides $p+1$ exactly to the mth power. The condition $p \equiv -1 \pmod D$ could of course be replaced by $p \equiv l \pmod D$ for any l prime to D.

Proof. We first remark that the reduced residue-class groups $\pmod D$ and $\pmod{qD}$ are cyclic, so there will be exactly one real character $\chi_1 \pmod D$, and one real character $\chi_2 \pmod{qD}$. Moreover, χ_2 will be induced by χ_1. Since no L series vanishes on the line $\sigma = 1$, the L series formed with χ_1 and χ_2 will have the same zeros in the half-plane $\sigma > 0$. In particular they can only have (or not have) the same exceptional zero β_1.

If neither χ_1 nor χ_2 has an exceptional zero then the result of Lemma 2 follows from Lemma 1 by choosing $\varepsilon = 1/8$ say, and c sufficiently small that

$$\sum_{\substack{p \leqslant x\\ p \equiv -1 \pmod D}} 1 - \sum_{\substack{p \leqslant x\\ p \equiv -1 \pmod{qD}}} 1 \geqslant \frac{7x}{8\varphi(D)\log x} - \frac{9x}{8\varphi(qD)\log x}$$

$$= \frac{7x}{8\varphi(D)\log x}\Big(1 - \frac{9}{7q}\Big) > 0.$$

Suppose therefore that χ_1 and χ_2 both possess an exceptional zero β_1. According to a result of Linnik (Prachar [14], Satz 3.1, p. 349) if

$$1 - \frac{A_1}{\log D} \leqslant \beta_1 < 1 \qquad (t = 0)$$

and if we set $\delta_1 = 1 - \beta_1$, then in the region

$$(5) \quad \sigma \geqslant 1 - \frac{A_2}{\log D(|t|+1)} \log \frac{eA_1}{\delta_1 \log D(|t|+1)}, \qquad \delta_1 \log D(|t|+1) \leqslant A_1$$

there is no zero ($\neq \beta_1$) of any of the L-series (mod D). Here the constants A_1 and A_2 have positive absolute (prescribed) values. Moreover, since β_1 is also a zero of χ_2 (mod qD), if

$$1-\frac{A_1}{\log qD} \leqslant \beta_1 < 1,$$

then we can assert a similar result concerning the zero-free region of the L-series (mod qD) provided only that in (5) we replace D at every occurence by qD.

We now follow the proof of Lemma 1. We set $T = D^{10}$, and include the contribution of the exceptional zero into the main term, thus if $(l, D)=1$

$$\Big|\sum_{\substack{n\equiv l \,(\mathrm{mod}\, D)\\ n\leqslant x}} \Lambda(n) - \frac{x}{\varphi(D)} + \frac{\bar{\chi}(l)}{\varphi(D)}\frac{x^{\beta_1}}{\beta_1}\Big|$$
$$\leqslant \frac{2x^{1-(\eta/2)}}{\varphi(D)} + O\left(\frac{\eta^{-1}x^{1/2}T^{C_2}\log x}{\varphi(D)} + \frac{x\log^2 Dx}{T}\right) + O\left(\frac{x^{1/4}\log Dx}{T}\right)$$

where η is chosen to be the largest value consistent with (5), using Dq in place of D, and $T = D^{10}$.

Subtracting a similar expression concerning those integers n which satisfy $n \equiv l$ (mod qD) we deduce that

$$(6)\qquad \Big|\sum_{\substack{n\leqslant x\\ n\equiv l \,(\mathrm{mod}\, D)\ \mathrm{exactly}}} \Lambda(n) - \frac{1}{\varphi(D)}\left\{1-\frac{1}{q}\right\}\left(x - \bar{\chi}(l)\frac{x^{\beta_1}}{\beta_1}\right)\Big|$$
$$\leqslant \frac{5x^{1-(\eta/2)}}{\varphi(D)} + O\left(\frac{\eta^{-1}x^{1/2}T^{C_2}\log x}{\varphi(D)} + \frac{x\log^2 Dx}{T}\right) + O\left(\frac{x^{1/4}\log Dx}{T}\right).$$

Here we note that it follows from Dirichlet's class number formula (see Davenport [4], Chapter 6, and Prachar [14], p. 145) that there is an effectively (computable) constant $c_0 > 0$ so that $\delta_1 > c_0 D^{-1}$. In particular $\eta^{-1} < Dc_0^{-1}$, and for $D \leqslant x^c$ and c sufficiently small the last three error terms in the above estimate are

$$O\left(\frac{x\log^2 x}{D^{10}}\right).$$

If now $\delta_1 > (\log x)^{-1/2}$, then arguing as in the proof of Lemma 1 we can deduce Lemma 2 at once (essentially, both D and qD will be non-exceptional). Thus without loss of generality $\delta_1 \leqslant (\log x)^{-1/2}$ and this in combination with the above lower bound for δ_1 shows that $D > c_0(\log x)^{1/2}$. Hence the size of the last three error terms in (6) is

$$O\left(\frac{x}{D^2\log^2 x}\right).$$

Moreover, since $qD \leqslant D^2$

$$x^{-(\eta/2)} = \exp\left(-\frac{A_2\log x}{2\log D^2(D^{10}+1)}\log\left(\frac{eA_1}{\delta_1\log D^2(D^{10}+1)}\right)\right)$$

provided that $\delta_1\log D^2(D+1) \leqslant A_1$. Let $\delta_1\log D \leqslant \varepsilon_0$, where ε_0 will presently be chosen to have a positive absolute value. Then for c sufficiently small this last error term will not exceed $c_1(\delta_1\log D)^2/\varphi(D)$.

Altogether, therefore,

$$\sum_{\substack{n\leqslant x\\ n\equiv l \,(\mathrm{mod}\, D)\ \mathrm{exactly}}} \Lambda(n) \geqslant \frac{1}{2\varphi(D)}\left(x - \frac{x^{\beta_1}}{\beta_1}\right) - O\left(x(\delta_1\log D)^2/\varphi(D) + x/(D^2\log^2 x)\right).$$

Here the main term (in an obvious notation), exceeds

$$\frac{x\delta_1\log x}{4\varphi(D)}$$

provided $\delta_1\log x \leqslant \varepsilon_0$, ε_0 is sufficiently small, and x sufficiently (absolutely) large. If we set $x = D^\mu$ with a large enough value of μ ($> 1/c$) then all of these conditions will be satisfied when $\delta_1\log D \leqslant \varepsilon_0/\mu$. In this case

$$\sum_{\substack{p\leqslant x\\ p\equiv l \,(\mathrm{mod}\, D)\ \mathrm{exactly}}} \log p \geqslant \frac{x\delta_1\log x}{4\varphi(D)} - O\left(\frac{x(\delta_1\log D)^2}{\varphi(D)} + \frac{x}{D^2\log^2 x}\right) - \sum_{\substack{n=p^m\leqslant x\\ m\geqslant 2}} \Lambda(n)$$
$$\geqslant \frac{x\delta_1\log D}{4\varphi(D)}\{1-O(\delta_1\log D)\} + O\left(\frac{x}{D^2\log^2 x} + x^{1/2}\right)$$
$$\geqslant \frac{c_2 x\log D}{D^2} + O\left(\frac{x}{D^2\log^2 x}\right)$$
$$= \frac{c_2 x\log D}{D^2}\left\{1 + O\left(\frac{1}{(\log x)^2\log D}\right)\right\} > 0$$

provided μ is sufficiently large (which ensures that x is sufficiently large).

This proves the lemma if $\delta_1\log D \leqslant \varepsilon_0\mu^{-1}$, where ε_0 and μ are certain absolute constants. If this last inequality fails then

$$\beta_1 < 1 - \varepsilon_0\mu^{-1}(\log D)^{-1},$$

and we can apply the method of Lemma 1.

This completes the proof of Lemma 2.

Our outline of this proof was a little longwinded, but it was desirable to show that all the constants involved could be computed.

Our next lemma concerns the average distribution of prime numbers in residue classes to large moduli. Here a direct application of the Bombieri–

Vinogradov theorem (see for example Bombieri [3], Gallagher [9], Montgomery [13]) is not useful, since we shall be dealing with too few moduli. In such situations the Riemann–Piltz conjecture that no L-series formed with a Dirichlet character has a zero in the half-plane $\sigma > \frac{1}{2}$ has more powerful consequences. An example which shows the limitations is the following.

Let $\tau(n)$ denote the Dirichlet divisor function. Then an asymptotic formula can be given for the sum

$$S = \sum_{\substack{p \leqslant x \\ p \equiv 1 \pmod D}} \tau(p-1)$$

whenever D does not exceed a fixed power of $\log x$, but not if D is any larger, say in the range $x^{1/4} \leqslant D \leqslant x^{1/3}$. Indeed, not even a decent lower bound can be given (the so-called exceptional modulus may divide D). However, the Riemann–Piltz conjecture allows one at once to assert that if $D \leqslant x^{1/2-\varepsilon}$, $0 < \varepsilon < 1/2$, then for a suitable constant $c = c(\varepsilon) > 0$,

$$S > cx/D.$$

We shall need an estimate of the type

$$\sum_{m \leqslant Q} \sum_{\chi \pmod m}^{*} N(\alpha, T, \chi) \leqslant c_0 (T+2) Q^{A_0(1-\alpha)} (\log Q(T+2))^{B_0}, \qquad T \geqslant 0,$$

for certain positive constants A_0 and B_0. Here * denotes that summation is restricted to primitive characters (mod m), for each m. A result of this type may be found in Montgomery [13], p. 99, with $A_0 = 5$, $B_0 = 14$, and valid for $1/2 \leqslant \alpha \leqslant 1$. It is of course classical that if $\alpha = 1/2$ then we may set $A_0 = 4$, $B_0 = 1$. (See Prachar [14], Satz 3.4, p. 22.) We shall assume (as we may without loss of generality) that $A_0 \geqslant 4$.

As is usual we define

$$\psi(y, m, l) = \sum_{\substack{n \leqslant y \\ n \equiv l \pmod m}} \Lambda(n) \qquad (y \geqslant 1, \ m \geqslant 1).$$

Lemma 3. *Let x and Q be positive real numbers, and let D be a positive integer. Then*

$$\sum_{d \leqslant Q}{}' \max_{y \leqslant x} \max_{(l, dD)=1} \left| \psi(y, Dd, l) - \frac{\psi(y, D, l)}{\varphi(d)} \right| \ll \frac{x (\log x)^{B_0+3-A}}{\varphi(D)}$$

provided that $2(QD)^{A_0} \leqslant x$, where $'$ indicates that every prime divisor q of each integer d satisfies $q > (\log x)^{A}$.

Similarly

$$\sum_{d \leqslant Q}{}' \max_{y \leqslant x} \max_{(l, dD)=1} \left| \pi(y, Dd, l) - \frac{1}{\varphi(d)} \pi(y, D, l) \right| \ll \frac{x (\log x)^{B_0+3-A}}{\varphi(D)}$$

with the same restriction on the d, but with

$$(QD)^{A_0} \leqslant x (\log x)^{B_0+3-A}.$$

Remarks. The results of this lemma are valid with no restriction upon the prime divisors of d provided that D does not exceed a fixed power of $\log x$. This will follow immediately from the Bombieri–Vinogradov theorem. In Lemma 3 we gain control over large moduli D at the expense of the small moduli d.

In our applications Lemma 3 will be combined with Lemma 1. This ends the remarks.

Proof. Consider first those moduli d for which $(D, d) > 1$. Let the prime q divide both D and d. Then $q > (\log x)^{A}$, and if $QD \leqslant x^{3/4}$, then

$$\sum_{\substack{d \leqslant Q \\ d \equiv 0 \pmod q}} \pi(x, Dd, l) \ll \sum_{m \leqslant Q} \frac{x}{Dqm} \ll \frac{x}{D} (\log x)^{-A} \log x \ll \frac{x (\log x)^{1-A}}{\varphi(D)}.$$

It is clear from this remark that the contribution towards the sum(s) in Lemma 3 which arises from these moduli d is at most

$$O\left(\frac{x (\log x)^{1-A}}{\varphi(D)} \sum_{q | D} 1 \right) = O\left(\frac{x (\log x)^{2-A}}{\varphi(D)} \right).$$

Consider now a modulus d which is prime to D. Then every character $\chi \pmod{Dd}$ can be written in the form $\chi_1 \chi_2$, where χ_1 is induced by a character (mod D), and χ_2 is induced by a character (mod d). If χ_2 is the principal character (mod d), then we say that χ is a *Karacter* (mod D) and write "χ is K". We note that even in this case χ need not be primitive (mod D).

Let now the character $\chi \pmod{Dd}$ be induced by the character $\chi_3 \pmod w$, say. If $w_1 = w(D, w)^{-1} = 1$, then it is easy to see that χ is actually a Karacter (mod D).

When χ is not a Karacter, $w_1 > (\log x)^{A}$ is satisfied.

We are now ready to complete the proof of Lemma 3.

From the explicit formula (1),

$$N_d(l, y) = \left| \sum_{\substack{n \equiv l \pmod{Dd} \\ n \leqslant y}} \Lambda(n) - \frac{1}{\varphi(D)\varphi(d)} \sum_{\chi \text{ is K}} \bar{\chi}(l) \sum_{n \leqslant y} \chi(n) \Lambda(n) \right|$$

$$\leqslant \frac{1}{\varphi(Dd)} \sum_{\chi \pmod{Dd}}{}'' \sum_{|\gamma| \leqslant T} \frac{y^{\beta}}{|\gamma|} + O\left(\frac{y \log^2 dDy}{T} + \frac{y^{1/4} \log y}{\varphi(dD)} \right)$$

where $\varrho^{-1} \ll \log dD(T+2)$ for every zero of the L-series (mod Dd), the contribution of the image of the exceptional zero (if it exists) being absorbed into the term involving $y^{1/4}$. Here $''$ denotes that χ is not a Karacter.

Let E_d denote

$$\max_{y\leqslant x}\max_{(l,D)=1} N_d(l, y).$$

Then summing over the (special) d not exceeding Q and prime to D, we have

$$(7)\qquad \sum_{\substack{d\leqslant Q\\(d,D)=1}} E_d \leqslant \frac{1}{\varphi(D)}\sum_{\substack{d\leqslant Q\\(d,D)=1}}\frac{1}{\varphi(d)}\sum_{\chi(\mathrm{mod}\,Dd)}{}''\sum_{|\gamma|\leqslant T}\frac{x^\beta}{|\gamma|}+ O\left(\frac{Qx\log^2 x}{T}+\frac{x^{1/4}(\log x)^2}{\varphi(D)}\right).$$

Let us first estimate the contribution of those zeros with $\beta\leqslant 1/2$. We shall assume for simplicity that T will ultimately be chosen so as not to exceed a fixed power of x. Then if $\beta\leqslant 1/4$, we have $|\varrho|^{-1}\ll \log x\{1+|\gamma|\}^{-1}$, whilst if $\beta > 1/4$, $|\varrho|^{-1}\ll (1+|\gamma|)^{-1}$ so that the desired contribution is

$$(8)\qquad \ll \frac{1}{\varphi(D)}\sum_{\substack{d\leqslant Q\\(d,D)=1}}\frac{1}{\varphi(d)}\sum_{\chi(\mathrm{mod}\,Dd)}{}''\sum_{|\gamma|\leqslant T}\frac{x^{1/2}}{1+|\gamma|}$$

$$\ll \frac{x^{1/2}}{\varphi(D)}\sum_{\substack{d\leqslant Q\\(d,D)=1}}\frac{1}{\varphi(d)}\sum_{\chi(\mathrm{mod}\,Dd)}{}''\sum_{|\gamma|\leqslant T}\left(\frac{1}{1+T}+\int_{|\gamma|}^{T}\frac{du}{(1+u)^2}\right)$$

$$\ll \frac{x^{1/2}\log x}{\varphi(D)}\sum_{\substack{d\leqslant Q\\(d,D)=1}}\frac{1}{\varphi(d)}\sum_{\chi(\mathrm{mod}\,Dd)}{}''\left(1+\int_0^T\frac{du}{1+u}\right)\ll Qx^{1/2}(\log x)^2.$$

Consider now the contribution towards (7) which arises from those zeros with $1/2<\beta<1$. From the identity

$$(9)\qquad x^\beta = x^{1/2}+\int_{1/2}^{\beta} x^\sigma\log x\,d\sigma$$

it is clear that at the expense of an error of at most (8) we can replace each x^β by the appropriate integral in (9). Next,

$$\frac{1}{1+|\gamma|}=\frac{1}{1+T}+\int_{|\gamma|}^{T}\frac{du}{(1+u)^2}$$

so that the sum which we wish to estimate does not exceed

$$(10)\qquad \frac{1}{\varphi(D)}\sum_{\substack{d\leqslant Q\\(d,D)=1}}\frac{1}{\varphi(d)}\sum_{\chi(\mathrm{mod}\,Dd)}{}''\int_{1/2}^{1}x^\sigma\log x\left\{\frac{N(\sigma,T,\chi)}{1+T}+\int_0^T\frac{N(\sigma,u,\chi)}{(1+u)^2}du\right\}d\sigma.$$

Let χ be a primitive character (mod w). Let (as before) $w_1 = w(w,D)^{-1}>1$. Then the zeros of the L-series formed with characters which are induced by χ, and which lie in the half-plane $\sigma>0$, are those of $L(s,\chi(\mathrm{mod}\,w))$ itself. Therefore each will be counted in the (appropriate) sums (7), (10), with a multiplicity of at most

$$\frac{1}{\varphi(D)}\sum_{\substack{d\leqslant Q\\d\equiv 0(\mathrm{mod}\,w_1)}}\frac{1}{\varphi(d)}\leqslant\frac{1}{\varphi(D)}\cdot\frac{1}{\varphi(w_1)}\sum_{m\leqslant Q}\frac{1}{\varphi(m)}\ll\frac{\log Q}{\varphi(D)(\log x)^A}.$$

Therefore the sum (10) does not exceed

$$\frac{c_1\log Q}{\varphi(D)(\log x)^A}\sum_{w\leqslant QD}\sum_{\chi(\mathrm{mod}\,w)}{}^{*}\int_{1/2}^{1}x^\sigma\log x\left\{\frac{N(\sigma,T,\chi)}{1+T}+\int_0^T\frac{N(\sigma,u,\chi)}{(1+u)^2}du\right\}d\sigma.$$

Making use of the estimate given immediately preceeding the statement of Lemma 3 we see that this is not more than

$$\frac{c_0c_1\log Q}{\varphi(D)(\log x)^A}\int_{1/2}^{1}x^\sigma\log x(QD)^{A_0(1-\sigma)}\big(\log QD(T+2)\big)^{B_0}\left\{\frac{T+2}{T+1}+\int_0^T\frac{u+2}{(u+1)^2}du\right\}d\sigma$$

$$\ll\frac{(\log Qx)^{B_0+3-A}}{\varphi(D)}\int_{1/2}^{1}(QD)^{A_0(1-\sigma)}x^\sigma\,d\sigma\ll\frac{x(\log x)^{B_0+3-A}}{\varphi(D)}$$

this last step being valid if $2(QD)^{A_0}\leqslant x$. Since $A_0\geqslant 4$ holds, this condition will ensure that the earlier condition $QD\leqslant x^{3/4}$ is amply satisfied. Altogether we have proved that

$$\sum_{d\leqslant Q}E_d\ll\frac{x(\log x)^{B_0+3-A}}{\varphi(D)}+Qx^{1/2}(\log x)^2+\frac{Qx\log^2x}{T}+\frac{x^{1/4}(\log x)^2}{\varphi(D)}.$$

This will be

$$\ll\frac{x(\log x)^{B_0+3-A}}{\varphi(D)}$$

if we set $T=x^{1/2}$ (not exceeding a fixed power of x, as was assumed earlier), and note that $QD\leqslant x^{1/4}$.

We now examine a typical term $N_d(l,y)$. We have

$$N_d(l,y)=\Big|\sum_{\substack{n\equiv l(\mathrm{mod}\,Dd)\\ n\leqslant y}}\Lambda(n)-M_d(l,y)\Big|$$

where

$$M_d(l,y)=\frac{1}{\varphi(d)}\frac{1}{\varphi(D)}\sum_{\chi\ is\ \mathrm{K}}\bar\chi(l)\sum_{n\leqslant y}\chi(n)\Lambda(n)$$

$$=\frac{1}{\varphi(d)}\sum_{n\leqslant y}\Lambda(n)\chi_0(n)\sum_{\chi(\mathrm{mod}\,D)}\frac{\bar\chi(l)\chi(n)}{\varphi(D)}.$$

Here the inner sum runs over all character $\chi\ (\mathrm{mod}\ D)$, and $\chi_0(n)$ denotes the principal character (mod d). This last sum is therefore

$$\frac{1}{\varphi(d)}\sum_{\substack{n\leqslant y\\ n\equiv l(\mathrm{mod}\,D)}}\Lambda(n)-\frac{1}{\varphi(d)}\sum_{\substack{n\leqslant y\\ n\equiv l(\mathrm{mod}\,D)\\ (n,d)>1}}\Lambda(n).$$

The second of these two sums does not exceed

$$\frac{\log x}{\varphi(d)}\sum_{p|d}\Big(\sum_{p\leqslant x^{1/2}}1+\sum_{p\leqslant x^{1/3}}1+\ldots\Big)\ll\frac{x^{1/2}\log x}{\varphi(d)}\sum_{p|d}1\ll\frac{x^{1/2}(\log x)^2}{\varphi(d)}$$

so that

$$\sum_{d\leqslant Q}\max_{y\leqslant x}\max_{(l,Dd)=1}\Big|\psi(y,Dd,l)-\frac{1}{\varphi(d)}\psi(y,D,l)\Big|\ll\frac{x(\log x)^{B_0+3-A}}{\varphi(D)}$$

as was asserted in Lemma 3, the only proviso being that $2(QD)^{A_0}\leqslant x$ and each d has no prime factor $q<(\log x)^A$.

To prove the second assertion of Lemma 3 we first remove from ψ the contribution of those prime power p^m with $m\geqslant 2$. This will not exceed

$$\sum_{d\leqslant Q}\Big\{\Big(\sum_{\substack{p\leqslant x^{1/2}\\ p^2\equiv l(\mathrm{mod}\,Dd)}}\log p+\sum_{\substack{p\leqslant x^{1/3}\\ p^3\equiv l(\mathrm{mod}\,Dd)}}\log p+\ldots\Big)+O\Big(\frac{x^{1/2}}{\varphi(D)}\Big)\Big\}$$

$$\ll\sum_{d\leqslant Q}x^{1/2}\ll\frac{x^{3/4}}{\varphi(D)}.$$

In the usual way we set

$$\theta(y,m,l)=\sum_{\substack{p\leqslant y\\ p\equiv l(\mathrm{mod}\,m)}}\log p$$

and integrate by parts,

$$\pi(y,m,l)=(\log y)^{-1}\theta(y,m,l)+\int_{2-}^{y}\theta(u,m,l)\frac{du}{u(\log u)^2}.$$

Therefore

$$F_d\overset{\mathrm{def}}{=}\Big|\pi(y,Dd,l)-\frac{1}{\varphi(d)}\pi(y,D,l)\Big|$$

$$\leqslant\frac{\Big|\theta(y,Dd,l)-\frac{1}{\varphi(d)}\theta(y,D,l)\Big|}{\log y}+\int_{2-}^{y}\frac{\Big|\theta(u,Dd,l)-\frac{1}{\varphi(d)}\theta(u,D,l)\Big|}{u\log^2u}\,du$$

and

$$\sum_{d\leqslant Q}\max_{y\leqslant x}\max_{(l,Dd)=1}F_d\ll\frac{x^{3/4}}{\varphi(D)}+\sum_{d\leqslant Q}E_d+\int_{2-}^{x}\sum_{d\leqslant Q}\max_{(l,Dd)=1}\frac{|\theta(\ldots)\ldots|}{u\log^2u}\,du.$$

The first two of these majorising terms are $O\big(x(\log x)^{B_0+3-A}/\varphi(D)\big)$ provided that $2(QD)^{A_0}\leqslant x$. Set $z=2(QD)^{A_0}$. Then for the range $z\leqslant u\leqslant x$ in the integral we obtain

$$\ll\int_z^x\frac{u(\log u)^{B_0+3-A}}{\varphi(D)u\log^2u}\,du\ll\frac{x}{\varphi(D)}(\log x)^{B_0+1-A}.$$

For the range $2-\leqslant u\leqslant z$ we use the crude estimate

$$\Big|\theta(u,Dd,l)-\frac{1}{\varphi(d)}\theta(u,D,l)\Big|\ll\frac{u}{Dd}+1+\Big(\frac{u}{D}+1\Big)\frac{1}{\varphi(d)}\ll\frac{u}{D\varphi(d)}+1$$

and obtain the upper bound

$$\ll\int_{2-}^{z}\sum_{d\leqslant Q}\Big\{\frac{u}{D\varphi(d)}+1\Big\}\frac{du}{u\log^2u}\ll\frac{z\log Q}{D\log^2z}+Q\ll\frac{z}{D}+Q.$$

Here $Q\leqslant x^{1/4}D^{-1}$ and the whole integral, and therefore the sum involving F_d, will be $O\big(x(\log x)^{B_0+3-A}/\varphi(D)\big)$ provided that $2(QD)^{A_0}=z\leqslant 2x(\log x)^{B_0+3-A}$.

This completes the proof of Lemma 3.

3. We next prove some results concerning the possible solution of certain equations in integers and prime numbers.

LEMMA 4. *Let x be a real number, $x\geqslant 2$. Let $d_1<d_2<\ldots<d_k$ denote those (squarefree) integers d, not exceeding x, for which the equation*

$$p+1=d(q+1)$$

is soluble with primes p and q not exceeding x, $p+1$ squarefree.

Then there are positive absolute constants c_1 and c_2 so that

$$\sum_{i=1}^{k}\frac{1}{d_i}\geqslant c_1\log x \quad (x\geqslant c_2).$$

Proof. Except for the condition that the $p+1$ be squarefree this lemma is established in the author's paper [5]. It is straightforward to modify the proof which is given there in order to add this extra condition.

LEMMA 5. *Let $x\geqslant 2$. Let $m_1<m_2<\ldots<m_r$ be a sequence of squarefree integers not exceeding x. Let $q_1<q_2<\ldots<q_w$ be a sequence of primes not exceeding x. Suppose that there are no solutions to the equation $m_i=m_j\lambda$ where the integer λ is composed entirely of the primes q_i.*

Let

$$S=\sum_{i=1}^{w}\frac{1}{q_i}.$$

Then there is an absolute constant c_3 so that

$$S^{1/2}\sum_{j=1}^{r}\frac{1}{m_j}\leqslant c_3\log x \quad (x\geqslant 2).$$

Remark. This lemma implicitly forms the fundament of Behrend's treatment of primitive sequences of integers [2], and Erdös' treatment of certain distributional problems concerning additive functions ([6], see also Erdös and Wintner [7]).

Proof. For each positive integer n not exceeding x let $a(n)$ denote the number of divisors of n which can be found amongst the m_i. Clearly

$$A\overset{\text{def}}{=}\sum_{n\leqslant x}a(n)=\sum_{n\leqslant x}\left[\frac{x}{m_i}\right]\geqslant x\sum_{i=1}^{r}\frac{1}{m_i}-x.$$

We shall now obtain an upper bound for the sum A.

Let g denote a positive real number. Consider those n which are divisible by at most g of the primes q_i. Then

$$a(n)\leqslant 2^g\sum_{d_1|n}\mu^2(d_1)$$

where d_1 denotes that the divisor d_1 has no prime factor q_i. The contribution towards A that arises from these integers n therefore does not exceed

$$2^g\sum_{d_1\leqslant x}\mu^2(d_1)\left[\frac{x}{d_1}\right]\leqslant\frac{2^g x\prod_{p\leqslant x}(1+1/p)}{\prod_{i=1}^{w}(1+1/q_i)}\leqslant c_4 x\log x\, 2^g e^{-S}.$$

If now n is an integer which has more than g prime factors q_i then we can write $n=n_0n_1$, where no q_i divides n_1. Each divisor d of w has a corresponding decomposition $d=d_0d_1$. Consider those divisors d with a particular (temporarily fixed) value of d_1. Clearly we cannot have $d_0|d_0'$ for any two divisors d, d' (taken from amongst the m_i) otherwise

$$d'=\lambda d,$$

contradicting a hypothesis of the lemma.

Let $l_1,\ldots,l_k$ denote the prime divisors of the (squarefree) integer d_0. Then to each divisor d_1 of n_1 there corresponds a set of suffices, no one of which sets is contained in another. By a theorem of Sperner [15] any such collection can contain at most

$$\binom{k}{\left[\frac{k}{2}\right]}$$

members. Here the symbol denotes the appropriate binomial coefficient. From Stirling's approximation this expression does not exceed $O(2^k k^{-1/2})$ so that

$$a(n)\leqslant\frac{c_5 2^k}{\sqrt{g}}\,2^{\nu(n)-k}$$

where $\nu(n)$ denotes the total number of distinct prime divisors of the integer n. Hence these integers n contribute towards the sum A at most

$$\frac{c_5}{\sqrt{g}}\sum_{n\leqslant x}\tau(n)\leqslant\frac{c_6 x\log x}{\sqrt{g}}.$$

Altogether therefore

$$\sum_{i=1}^{r}\frac{1}{m_i}\leqslant\log x\cdot\left\{\frac{c_6}{\sqrt{g}}+c_4 2^g e^{-S}+\frac{1}{\log x}\right\} \quad (x\geqslant 2).$$

Choosing $g=\max(S,1)$ we complete the proof of Lemma 5.

For $x\geqslant 2$ define

$$E(x)=\max_{p\leqslant x}|f(p+1)|.$$

Let the sequence $m_1<\ldots<m_r$ in Lemma 5 be the integers d_i of Lemma 4. Let the primes $q_1<\ldots<q_w$ in Lemma 5 be those primes q for which

$$f(q)>2E(x).$$

Then any equation $m_i=m_j\lambda$, with q_t dividing λ (say), would lead to the contradiction

$$2E(x)\geqslant f(m_i)-f(m_j)\geqslant f(q_t)>2E(x).$$

From the results of Lemmas 4 and 5 we deduce that

$$\sum_{i=1}^{w} \frac{1}{q_i} \leqslant \left(\frac{c_3}{c_1}\right)^2 < \infty.$$

Arguing similarly with those primes q for which $f(q) < -2E(x)$ leads to the result:

LEMMA 6. *For a certain absolute constant c_4,*

$$\sum_{\substack{q \leqslant x\\ |f(q)|>2E(x)}} \frac{1}{q} \leqslant c_4 \quad (x \geqslant 2).$$

LEMMA 7. *Let a and ε be positive real numbers, $0 < a < 1$, $\varepsilon > 0$. Then there is a further positive real number β, $0 < \beta \leqslant 1$, such that if x exceeds a certain value depending upon a and ε, then for some number $y \geqslant x^{\beta}$ we have*

$$\sum_{\substack{y^{a} < q \leqslant y\\ |f(q)|>2E(x)}} \frac{1}{q} < \varepsilon \quad (x \geqslant 2).$$

Proof. Let M be a positive integer. From Lemma 3

$$\sum_{m=1}^{M} \sum_{x^{a^m} < q \leqslant x^{a^{m-1}}}' \frac{1}{q} \leqslant \sum_{q \leqslant x}' \frac{1}{q} \leqslant c_4,$$

where $'$ indicates that summation is restricted to those primes for which $|f(q)| > 2E(x)$. We choose a fixed value of $M > c_4/\varepsilon$, and set $\beta = a^M$. Then at least one of the innermost sums with $y = x^{a^{m-1}}$ does what is required.

Our last preliminary result concerns a form of the Selberg sieve. The result which we need can be proved along any of the standard lines, and is of the type known as a 'fundamental lemma'. (See Kubilius [12], Chapter one, and Barban [1].)

LEMMA 8. *Let $a_1 < a_2 < \ldots < a_k$ denote a sequence of positive integers. Let $q_1 < q_2 < \ldots < q_w \leqslant r$ denote a sequence of primes not exceeding the number r. Denote their product by Q. Let $g(d)$ be multiplicative on the divisors of Q and satisfy $0 \leqslant g(p) < 1$ and*

$$\sum_{x/2 < p \leqslant x} \frac{g(p)}{1-g(p)} = O\left(\frac{1}{\log x}\right)$$

uniformly for all $x \geqslant 2$. Let X be a real number, and define

$$\sum_{a_i \equiv 0 \,(\mathrm{mod}\, d)} 1 = g(d)X + R(x, d) \quad (d|Q).$$

Let $I(k, Q)$ denote the number of members of the sequence of the a_i which are prime to Q, that is to say, not divisible by any of the q_i. Let z be a further real number $z \geqslant r \geqslant 2$.

Then there are positive absolute constants c_1 and c_2 so that

$$I(k, Q) = \left\{1 + \theta_1 c_1 \exp\left(-c_2 \frac{\log z}{\log r}\right)\right\} X \prod_{i=1}^{w} \left(1 - g(q_i)\right) +$$
$$+ \theta_2 c_1 \sum_{\substack{d|Q\\ d \leqslant z^3}} 4^{\nu(d)} \prod_{p|d} \left(1 - g(p)\right)^{-2} |R(X, d)|,$$

where $|\theta_j| \leqslant 1$ $(j = 1, 2)$.

4. Proof of the theorem. Let a, $0 < 4a < 1$, be a real number, to be chosen presently. Let $x \geqslant 2$ and ε be further real numbers, and let y be a number determined by Lemma 7. Let D be an integer which satisfies $D \leqslant y^{1/8}$.

Our first step is to estimate the number of solutions to the equation

$$(11) \qquad p \equiv -1 \,(\mathrm{mod}\, D),$$

$$p \text{ prime } y^{a} < p \leqslant y, \qquad q|(p+1)D^{-1} \Rightarrow q \leqslant (\log y)^{A} \quad \text{or} \quad q > y^{a}.$$

Here (and in what follows) p and q will be generic symbols for prime numbers.

We apply Lemma 8 taking the a_i to be those primes not exceeding y for which $p \equiv -1 \,(\mathrm{mod}\, D)$. Set

$$X = \frac{1}{\varphi(D)} \pi(y, D, -1), \qquad g(d) = \frac{1}{\varphi(d)}$$

and let Q be the product of all primes l in the range $(\log y)^{A} \leqslant l \leqslant y^{a}$. Then

$$I(k, Q) = \left\{1 + \theta_1 c_1 \exp\left(-c_2 \frac{\log z}{a \log y}\right)\right\} \pi(y, D, -1) \prod_{(\log y)^{A} < l \leqslant y^{a}} \left(1 - \frac{1}{l-1}\right) +$$
$$+ \text{error term } R,$$

where

$$|R| \leqslant c_1 \sum_{\substack{d|Q\\ d \leqslant z^3}} 4^{\nu(d)} \prod_{p|d} \left(1 - \frac{1}{\varphi(p)}\right)^{-2} |R(x, d)|.$$

By means of an application of the Cauchy–Schwarz inequality this error term is seen not to exceed

$$c_1\Big\{\sum_{d|Q,\,d\leqslant z^3} 8^{\nu(d)}\prod_{p|d}\Big(1-\frac{1}{\varphi(p)}\Big)^{-4}\frac{1}{\varphi(d)}\Big\}^{1/2}\Big\{\sum_{d|Q,\,d\leqslant z^3}\varphi(d)R^2(x,d)\Big\}^{1/2}.$$

The first sum in curly brackets does not exceed

$$\prod_{p\leqslant y^{\alpha}}\bigg(1+\frac{8}{l-1}\Big(1-\frac{1}{l-1}\Big)^{-8}\bigg)\ll(\log y)^8.$$

As for the second, if z is sufficiently small, say $z^3\leqslant y^{1/4}$, then by the Brun–Titchmarsh theorem (Prachar [14], Satz 4.1, p. 44),

$$\varphi(d)R(x,d)\ll y\{\varphi(D)\log y\}^{-1}+Dy^{\alpha}.$$

Applying Lemma 3 with $A_0=5$, $B_0=14$, the second sum in curly brackets is therefore

$$\ll\frac{y}{\varphi(D)\log y}\cdot\frac{y(\log y)^{17-A}}{\varphi(D)}$$

provided that $2(z^3D)^5\leqslant y(\log y)^{17-A}$.

We set $A=28$, so that $R\ll y\{\varphi(D)\log^2 y\}^{-1}$.

As for the main term, it will be as large as

$$\frac{\log\log y}{2\alpha\log y}\cdot\pi(y,D,-1)\quad(y\geqslant y_0),$$

provided that $\log z=C\alpha\log y$, and C is chosen so large (but fixed) that $c_1\exp(-C)<\frac{1}{2}$. The number y_0 may depend upon α, but not D. In order to satisfy all of our conditions upon z and D it will be enough that $D\leqslant y^{1/8}$ and $40C\alpha<1$.

Let us assume for the moment that $D\leqslant y^{c}$, where $c\leqslant 1/8$ and is a sufficiently small positive constant so that the result of Lemma 1 applies with an $\varepsilon\leqslant\frac{1}{2}$. Let us assume further that D is not a multiple of the (possible) exceptional modulus D_0. Then the number of solutions to the equation (11) is at least

$$\text{(12)}\qquad \frac{y\log\log y}{3\alpha\varphi(D)(\log y)^2}$$

provided only that $y\geqslant y_0(\alpha)$.

We can write the solutions of equation (11) in the form

$$\text{(13)}\qquad p+1=Dmr$$

where if $q|r$ then $q>y^{\alpha}$, whilst m is made up of the primes q not exceeding $(\log y)^{28}$. Let us show that there are few solutions where r is divisible by the square of a prime. In fact their number does not exceed

$$\sum_{q>y^{\alpha}}\sum_{\substack{n\leqslant y+1\\ n\equiv 0\,(\mathrm{mod}\,Dq^2)}}1\leqslant\sum_{q>y^{\alpha}}\frac{2y}{Dq^2}\leqslant\frac{2y^{1-\alpha}}{D}\qquad(y\geqslant 1),$$

and for a fixed $\alpha>0$ and large values of y this last expression is much smaller than that in (12).

Consider next the solutions where r is divisible by a prime q for which $Dq>y^{1-(\alpha/2)}$. Let

$$p+1=Dmqr_0$$

say. Here any prime divisor l of r_0 would satisfy $l>y^{\alpha}$, so that for all values $y\geqslant 1$

$$2y\geqslant p+1>my^{1-(\alpha/2)}y^{\alpha}\geqslant y^{1+(\alpha/2)},$$

which is impossible. Therefore $r_0=1$. We note that if α is sufficiently small, $Dq>y^{1/2}$, so that $Dm\leqslant 2y/q=2Dy/Dq\leqslant 2Dy^{1/2}\leqslant 2y^{3/4}$.

For a fixed value of Dm, the number of solutions to the equation

$$\text{(14)}\qquad p+1=Dmq\qquad(p,q\leqslant y)$$

is at most

$$c_3\frac{y}{\varphi(Dm)\log^2 y}\qquad(c_3\text{ absolute, }y\geqslant y_0).$$

The number of solutions to (14) (and (13)) with q free to vary over the primes $q>D^{-1}y^{1-(\alpha/2)}$ is therefore

$$\leqslant c_3\frac{y}{\varphi(D)\log^2 y}\sum_{m\leqslant(\log y)^{28}}\frac{1}{\varphi(m)}\leqslant\frac{c_4 y\log\log y}{\varphi(D)\log^2 y}.$$

If α is sufficiently small this will not exceed one thirteenth of the amount (12). We consider α now to be so chosen.

After removing these, so-to-speak, unwanted solutions, we are left with at least

$$\frac{1}{4\alpha}\cdot\frac{y\log\log y}{\varphi(D)(\log y)^2}$$

solutions to the equation (11), now with certain additional restrictions.

Our next step is to remove solutions to (13) for which some q_i divides $(p+1)$, $y^{\alpha}<q_i\leqslant y^{1-(\alpha/2)}D^{-1}$, where q_i is defined as in Lemmas 6, 7.

We need an upper bound for the number N of solutions to the equation

$$p+1 = Dq_i r; \quad p \leqslant y, \; Dq_i \leqslant y^{1-(a/2)} \tag{15}$$

where if q is a prime dividing r, then $q \leqslant (\log y)^{28}$, or $q > y^a$.

Let a_0 be a positive number, $0 < a_0 < a$. The number N does not exceed the number N_1 of solutions to the equation (15) where the condition on q is weakened to $q > y^{a_0}$. We estimate N_1 by Selberg's sieve method. We apply Lemma 8 with the a_i chosen to be the integers $n+1$ in the range $y^{a_0} < n \leqslant y$, which satisfy

$$n+1 \equiv 0 \pmod{q_i D}; \quad n \not\equiv 0 \pmod{q} \quad \text{if} \quad q \leqslant y^{a_0}, \quad q \nmid (q_i D);$$
$$(n+1)D^{-1}q_i^{-1} \not\equiv 0 \pmod{q} \quad \text{if} \quad (\log y)^{28} < q \leqslant y^{a_0}.$$

In this case $X = y/(Dq_i)$ and

$$pg(p) = \begin{cases} 2 & \text{if} \quad (\log y)^{28} < q \leqslant y^{a_0}, \; q \nmid (q_i D), \\ 1 & \text{if} \quad (\log y)^{28} < q \leqslant y^{a_0}, \; q | (q_i D), \text{ or } q \leqslant (\log y)^{28}. \end{cases}$$

For each $d|Q$, where now $Q = \prod q$, $2 \leqslant q \leqslant y^{a_0}$, we have

$$R(X, d) \ll \left(\frac{y^a}{dDq_i} + 1\right) g(d)$$

so that

$$\sum_{\substack{d \leqslant z^3 \\ d|Q}} 4^{\nu(d)} \prod_{p|D} (1-g(p))^{-2} |R(x, d)| \ll \sum_{\substack{d \leqslant z^3 \\ d|Q}} 24^{\nu(d)} \left(\frac{y^a}{dDq_i} + 1\right)$$
$$\ll \frac{y(\log y)}{Dq_i} + y^{3a_0+\varepsilon_1} \ll \frac{y(\log y)^{-3}}{q_i \varphi(D)}$$

provided that a_0 is sufficiently small, but fixed. Hence

$$N \leqslant y^{a_0} + N_1$$
$$\ll y^{a_0} + \frac{y(\log y)^{-3}}{q_i\varphi(D)} + \frac{y}{Dq_i} \prod_{q \leqslant (\log y)^{28}} \left(1 - \frac{1}{q}\right)^{-1} \prod_{q \leqslant y^{a_0}} \left(1 - \frac{2}{q}\right) \prod_{q|q_i D} \left(1 - \frac{1}{q}\right)^{-1}$$
$$\ll y^{a_0} + \frac{y(\log y)^{-3}}{q_i\varphi(D)} + \frac{y \log\log y}{Dq_i (\log y)^2} \cdot \frac{q_i D}{\varphi(q_i D)} \leqslant \frac{G y \log\log y}{q_i \varphi(D) (\log y)^2}$$

for a certain constant G which depends upon a_0 only. Choosing for a_0 the fixed value $a/12$, we see that for all absolutely large values of y:

$$y^{a_0} < \frac{y}{q_i D (\log y)^3}.$$

Here (as earlier) we make essential use of the fact that $q_i D < y^{1-(a/2)}$. The total number of solutions to the equation (15) for the various q_i is therefore not more than

$$\frac{y \log\log y}{\varphi(D) \log^2 y} \cdot G \sum_{y^a < q_i \leqslant y}' \frac{1}{q_i} < \frac{G\varepsilon y \log\log y}{\varphi(D) \log^2 y}.$$

We choose ε in Lemma 7 so small that $G\varepsilon < 1/(20a)$. This fixes a value for $\beta > 0$.

We have arrived at the existence of at least

$$\frac{1}{5a} \cdot \frac{y \log\log y}{\varphi(D) \log^2 y}$$

solutions to the equation

$$p+1 = Dmr, \quad p \leqslant y, \tag{16}$$

where r is squarefree, and every prime divisor q of r satisfies $y^a < q \leqslant y$. Moreover, no q_i (of Lemma 7) divides r. Every prime divisor q of m satisfies $q \leqslant (\log y)^{28}$. All this holds subject to D being non-exceptional (in a certain well defined sense) and satisfying $D \leqslant y^c$, for a sufficiently small constant c. If we restrict c by $c < a$ then $(D, r) = 1$ must hold. We denote the result so far by (16).

We can now exercise some control over the size of m. We do this in two ways, according to its absolute size, and according to a certain function of its prime divisors.

Define the function

$$\eta(m) = \sum_{\substack{q^w \| m \\ q \leqslant (\log y)^A}} w \log q.$$

Then

$$S = \sum_{p \leqslant y, p+1 = Dmr}' \eta(m) = \sum_{q \leqslant (\log y)^A} \log q^w \sum_{p+1 = Dq^w m, r}' 1$$

where $'$ indicates that we count primes p which appear in solutions of the equation (16). If $Dq^w \leqslant y^{1-(a/2)}$ then exactly as in the previous step, an inner sum will not exceed

$$G \frac{y \log\log y}{\varphi(Dq^w) \log^2 y} \leqslant \frac{2Gy \log\log y}{q^w \varphi(D) \log^2 y}.$$

The total contribution towards S which these prime-powers q^w make is therefore at most

$$\frac{2Gy \log\log y}{\varphi(D) \log^2 y} \sum_{q \leqslant (\log y)^A}' \frac{\log q^w}{q^w} \leqslant H \frac{y \log\log y}{\varphi(D) \log^2 y} \cdot \log\log y$$

for some absolute constant H $(y \geqslant y_0)$.

If $Dq^w > y^{1-(a/2)}$ then $y^{1/4}q^w > y^{1/2}$ and

$$w > \frac{\log y}{4\log q} \geqslant \frac{\log y}{4A\log\log y} = w_0,$$

say. The contribution towards S which arises from the corresponding prime powers q^w is at most

$$\sum_{w>w_0} \log q^w \frac{2y}{Dq_1^w} \ll \frac{y}{D} 2^{-w_0/2} \ll \frac{y(\log y)^{-3}}{\varphi(D)}.$$

Thus for all $y \geqslant 2$, and a suitable constant H,

$$\sum_{\substack{p+1=Dmr\\ p\leqslant y}} \eta(m) \leqslant \frac{2Hy\log\log y}{\varphi(D)\log^2 y}\cdot\log\log y.$$

It follows that if we set $A = 28$ and replace $5a$ in (16) by $10a$, then we may assert that m satisfies

$$m = \exp(\eta(m)) \leqslant (\log y)^{20aH}.$$

Here $20aH$ is an absolute constant.

Define the function

$$\Delta(m) = \sum_{q|m} \frac{1}{q}.$$

An argument similar to that given above shows that for a certain absolute constant J,

$$\sum_{\substack{p+1=Dmr\\ p\leqslant y}}' \Delta(m) \leqslant J\frac{y\log\log y}{\varphi(D)\log^2 y} \quad (y \geqslant y_0),$$

so that replacing the (now) $10a$ in (16) by $20a$ we can add the further condition that

$$\Delta(m) \leqslant 40aJ.$$

We summarise what we have proved so far in two results.

Let q_0 be a prime divisor of the exceptional modulus D_0. Let D be an integer not exceeding x^c, and which is not divisible by q_0. Then D is non-exceptional. If every prime divisor q of D satisfies $q > (\log y)^{28}$ we have

$$f(D) = f(p+1) - f(m) - f(r)$$

since $(D, m) = 1$. Here $m \leqslant (\log y)^{20aH}$ and r is squarefree, consisting of at most a^{-1} distinct prime factors q, for each of which $|f(q)| \leqslant 2E(x)$.

Hence

$$(17) \qquad |f(D)| \leqslant M((\log x)^{20aH}) + (2/a)E(x).$$

Suppose now that D is still non-exceptional, but does have prime factors $q \leqslant (\log y)^{28}$. Consider a solution to (16). Write $D = D_1D_2$, where D_1 is prime to m, and D_2 is made up from those primes which appear in m, but possibly with exponents different to those in m. Define $m = m_1m_2$ where m_1 is made up of primes which appear in D_2, and m_2 is prime to m_1. Then we have

$$(18) \qquad f(D_1) = f(p+1) - f(D_2m_1) - f(m_2) - f(r).$$

Everything now goes as before, save for the introduction of the integer D_2m_1. Let us call this integer D_3. It satisfies $D_3 \leqslant D(\log y)^{20aH}$, moreover $\Delta(D_3) \leqslant \Delta(m) \leqslant 40aJ$. We notice also that D_3 is non-exceptional, since it is not a multiple of q_0.

5. Non-exceptional moduli with $\Delta(D)$ bounded. Let N be an integer, $N \leqslant x^\delta$, which is not divisible by q_0, and so is non-exceptional in the sense of Lemma 1, assuming for the moment that ε is chosen, $0 < \varepsilon < 1$. Here δ may need to be sufficiently small. Let q be a prime divisor of N. Then if we strengthen the condition upon the size of N to $N \leqslant x^{\delta/2}$ we shall have $qN \leqslant N^2 \leqslant x^\delta$, and qN will also be non-exceptional.

Let $\Delta(N) \leqslant 40aJ$. We apply Lemma 8, with a_j running through the primes not exceeding x which satisfy $p \equiv -1 \pmod D$. Let Q be the product of the distinct prime divisors of N. We set

$$X = x\{\varphi(N)\log x\}^{-1} \quad \text{and} \quad g(q) = q^{-1}$$

for each prime q which divides N. Then the number N_2 of solutions to the equation

$$p+1 = Dt, \quad (D, t) = 1, \quad p \leqslant x,$$

is at least

$$\left\{1 - c_1\exp\left(-c_2\frac{\log z}{\log r}\right)\right\}\frac{x}{\varphi(D)\log x}\prod_{q|D}\left(1-\frac{1}{q}\right) - \text{error term } R,$$

where

$$|R| \leqslant \sum_{\substack{d|Q\\ d\leqslant z^3}} 4^{\nu(d)}\prod_{q|d}\left(1-\frac{1}{q}\right)^{-2}\left|\pi(x, Nd, -1) - \frac{x}{\varphi(Nd)\log x}\right|$$

$$\leqslant \varepsilon\frac{x}{\varphi(N)\log x}\sum_{d|Q} 4^{\nu(d)}\prod_{q|d}\left(1-\frac{1}{q}\right)^{-2}\cdot\frac{1}{d}$$

$$\leqslant \frac{\varepsilon x}{\varphi(N)\log x}\prod_{q|N}\left(1+\frac{4}{q}\left(1-\frac{1}{q}\right)^{-2}\right) \leqslant \frac{c_3\varepsilon x}{\varphi(N)\log x}\exp(4\Delta(N)).$$

We set $r = N$, and then $z = N^{\mu}$, where μ is chosen so large, but fixed, that $c_1 \exp(-c_2\mu) < \frac{1}{2}$. With this value $z^3 = N^{3\mu} \leqslant x^{3\mu\delta}$ and if ε is chosen sufficiently small (here we make use of the fact that $\Delta(N)$ is absolutely bounded), then

$$N_2 \geqslant c_4 \frac{x}{\varphi(N)\log x}\left\{\exp(-\Delta(N)) - \frac{c_3}{c_4}\varepsilon \exp(4\Delta(N))\right\} \geqslant \frac{c_5 x}{\varphi(N)\log x},$$

with a certain positive absolute constant c_5 $(x \geqslant x_0)$.

In a manner exactly similar to the treatment of equation (11) we prove that there exist many solutions to the equation

$$p+1 = NmM; \quad (N, mM) = 1,$$

where the prime divisors q of m satisfy $q \leqslant (\log x)^{29}$, and $m \leqslant (\log x)^L$ for a certain absolute constant L. Moreover, every prime divisor of M exceeds $(\log x)^{29}$. Hence

$$f(N) = f(p+1) - f(m) - f(M)$$

so that

$$|f(N)| \leqslant E(x) + M\big((\log x)^L\big) + M\big((\log x)^{20\alpha H}\big) + \left(\frac{2}{\alpha}+1\right)E(x),$$

since M falls within the scope of inequality (17).

Making use of this last inequality in (18) with $N = D_2 m_1$, we see that provided δ is chosen sufficiently small, and $D \leqslant x_i^{\delta/3}$, then

$$|f(D)| \leqslant \left(\frac{4}{\alpha}+4\right)E(x) + M\big((\log x)^K\big) \quad (q_0 \nmid D), \tag{19}$$

where $K = \max(L, 20\alpha H)$.

6. Completion of the proof. Let $\gamma = \delta/(3c_7)$, where c_7 is the constant which appears in Lemma 2, assumed, without loss of generality, to satisfy $c_7 \geqslant 1$. Let $t = q_0^w$ be a power of the exceptional prime q_0, $t \leqslant x^{\gamma}$. Then by Lemma 2 there is a prime p, not exceeding $x^{\delta/3}$, so that

$$p+1 = tD \quad (q_0 \nmid D). \tag{20}$$

Here the integer D is one to which inequality (19) will apply, and so $|f(t)|$ will satisfy an inequality exactly similar to (19), save that the coefficient of $E(x)$ is increased to $(4\alpha^{-1}+5)$.

Finally, let D be any integer not exceeding x^{γ}. Write $D = q_0^s D'$ where $q_0 \nmid D'$. Then from (19) and (20)

$$|f(D)| \leqslant |f(q_0^s)| + |f(D')| \leqslant \left(\frac{8}{\alpha}+9\right)E(x) + 2M\big((\log x)^K\big) \quad (x \geqslant x_0).$$

We replace x by $x^{1/\gamma}$ and have then proved the theorem for all sufficiently (absolutely) large values of x with

$$A = \left(\frac{8}{\alpha}+9\right), \quad B = \max\left(\frac{1}{\gamma}, K+1\right).$$

References

[1] M. B. Barban, *On a theorem of I. P. Kubilius*, Izv. Akad. Nauk UzSSR, Ser. Mat. 5 (1961), pp. 3–9.
[2] F. Behrend, *On sequences of integers not divisible one by another*, Journ. London Math. Soc. 10 (1935), pp. 42–44.
[3] E. Bombieri, *On the "Large Sieve"*, Mathematika 12 (1965), pp. 201–225.
[4] H. Davenport, *Multiplicative Number Theory*, Markham, Chicago 1967.
[5] P. D. T. A. Elliott, *A conjecture of Kátai*, Acta Arith. 26 (1974), pp. 11–20.
[6] P. Erdös, *On the density of some sequences of numbers: III*, Journ. London Math. Soc. 13 (1938), pp. 119–127.
[7] P. Erdös and A. Wintner, *Additive arithmetical functions and statistical independence*, Amer. Journ. Math. 61 (1939), pp. 713–721.
[8] E. Fogels, *On the zeros of L-functions*, Acta Arith. 11 (1965), pp. 67–96.
[9] P. X. Gallagher, *Bombieri's mean value theorem*, Mathematika 15 (1968), pp. 1–6.
[10] I. Kátai, *Some remarks on additive arithmetic functions*, Litovsk. Mat. Sb. IX (1969), pp. 515–518.
[11] — *Számelméleti Problémak I*, Mat. Lapok. 19 (1968), pp. 317–325.
[12] J. Kubilius, *Probabilistic Methods in the Theory of Numbers*, Amer. Math. Soc. Translations, vol. 11.
[13] H. Montgomery, *Topics in Multiplicative Number Theory*, Springer Lecture Notes, no. 227, Berlin, New York 1971.
[14] K. Prachar, *Primzahlverteilung*, Springer, Berlin 1957.
[15] E. Sperner, *Ein Satz über Untermengen einer endlichen Menge*, Math. Zeitschr. 27 (1928), pp. 544–548.

Received on 13. 12. 1974 (654)