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On the greatest prime factor of 2^p-1 for a prime p and other expressions

by

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1. For a natural number a, denote by $P(a)$ the greatest prime factor of a. Stewart [10] proved that there exists an effectively computable constant $c > 0$ such that

$$\frac{P(2^p-1)}{p} \geqslant \tfrac{1}{2}(\log p)^{1/4} \tag{1}$$

for all primes $p > c$. In § 2, we shall prove that $P(2^p-1)/p$ exceeds constant times $\log p$ for all primes. In § 5, we shall prove that for 'almost all' primes p,

$$\frac{P(2^p-1)}{p} \geqslant \frac{(\log p)^2}{(\log\log p)^3}. \tag{2}$$

For the definition of 'almost all', see § 5. Let $u > 3$ and $k \geqslant 2$ be integers and denote by $P(u, k)$ the greatest prime factor of $(u+1)\ldots(u+k)$. It follows from Mahler's work [6a] that $P(u, k) \gg \log\log u$. See also [6] and [8]. In § 4, we shall show that for $u \geqslant k^{3/2}$

$$P(u, k) > c_1 k \log\log u$$

where $c_1 > 0$ is a constant independent of u and k. It follows from well-known results on differences between consecutive primes that $P(u, k) \geqslant u+1$ whenever $k \leqslant u \leqslant k^{3/2}$. Let $a < b$ be positive integers which are composed of the same primes. Then, in § 3, we shall show that there exist positive constants c_2 and c_3 such that

$$b-a \geqslant c_2(\log a)^{c_3}.$$

Erdös and Selfridge [5] conjectured that there exists a prime between a and b.

The proof of all these theorems depend on the following recent result on linear forms in the logarithms of algebraic numbers.

Let $n > 1$ be an integer. Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers of heights less than or equal to $A_1, \ldots, A_n$ respectively, where each $A_i \geqslant 27$.

Let $\beta_1, \ldots, \beta_{n-1}$ denote algebraic numbers of heights less than or equal to B ($\geqslant 27$). Suppose that $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_{n-1}$ all lie in a field of degree D over the rationals. Set

$$A = \log A_1 \ldots \log A_n, \qquad E = (\log A + \log\log B).$$

LEMMA 1. *Given $\varepsilon > 0$, there exists an effectively computable number $C > 0$ depending only on ε such that*

$$|\beta_1 \log\alpha_1 + \ldots + \beta_{n-1}\log\alpha_{n-1} - \log\alpha_n|$$

exceeds

$$\exp\left(-(nD)^{Cn} A (\log A)^2 \left(\log(AB)\right)^2 E^{2n+2+\varepsilon}\right)$$

provided that the above linear forms does not vanish.

This was proved by the second author in [9]. It has been assumed that the logarithms have their principal values but the result would hold for any choice of logarithms if C were allowed to depend on their determinations.

The earlier results in the direction of Lemma 1 (i.e. lower bound for the linear form with every parameter explicit) are due to Baker [1] and Ramachandra [8]. Stewart applied the result of [1] to obtain (1). We remark that the result of [8] gives the inequality (1) with constant times $(\log p)^{1/2}/(\log\log p)$. The theorems on linear forms of [1] and [8] also give (weaker) results in the direction of the inequality (2) and the other results of this paper.

2. For a natural number a, denote by $\omega(a)$ the number of distinct prime factors of a.

LEMMA 2. *Let p (> 27) be a prime. Assume that*

$$P(2^p - 1) \leqslant p^2.$$

Then there exists an effectively computable constant $c_4 > 0$ such that

$$\omega(2^p - 1) \geqslant c_4 \log p/\log\log p.$$

We mention a consequence of Lemma 2.

THEOREM 1. *There exists an effectively computable constant $c_5 > 0$ such that*

$$P(2^p - 1) \geqslant c_5 p \log p$$

for all primes p.

Proof. Assume that

$$P(2^p - 1) < p \log p.$$

Without loss of generality, we can assume that $p > 27$. Then $P(2^p - 1) \leqslant p^2$. By Lemma 2, we have

$$\omega(2^p - 1) \geqslant c_4 \log p/\log\log p.$$

By using Brun–Titchmarsh theorem ([7], p. 44) and the fact that the prime factors of $2^p - 1$ are congruent to $1 \bmod p$, we obtain

$$P(2^p - 1) \geqslant c_6 p \log p$$

for some constant $c_6 > 0$. Set $c_5 = \min(1, c_6)$. Thus

$$P(2^p - 1) \geqslant c_5 p \log p.$$

This completes the proof of Theorem 2.

Proof of Lemma 2. Let $1 > \varepsilon_1 > 0$ be a small constant to be suitably chosen later. Set

$$r = [\varepsilon_1 \log p/\log\log p] + 1.$$

We shall assume that

$$\omega(2^p - 1) \leqslant r$$

and arrive at a contradiction. Write

$$2^p - 1 = q_1^{u_1} \ldots q_r^{u_r}$$

where for $i = 1, \ldots, r$, $q_i \leqslant p^2$ are primes and $u_i < p$ are non-negative integers. We have

$$2^{-p} = |(2^p - 1)2^{-p} - 1| = |q_1^{u_1} \ldots q_r^{u_r} 2^{-p} - 1|.$$

From here, it follows that

$$0 < |u_1 \log q_1 + \ldots + u_r \log q_r - p\log 2| < 2^{-p+1}. \tag{3}$$

By Lemma 1, it is easy to check that

$$|u_1 \log q_1 + \ldots + u_r \log q_r - p\log 2| > \exp(-p^{\varepsilon_1 D}) \tag{4}$$

where $D > 0$ is a certain large constant independent of ε_1. If we take $\varepsilon_1 = 1/4D$, the inequalities (3) and (4) clearly contradict each other. This completes the proof of Lemma 2.

For any integer $n > 0$ and relatively prime integers a, b with $a > b > 0$, we denote $\Phi_n(a, b)$ the nth cyclotomic polynomial, that is

$$\Phi_n(a, b) = \prod_{\substack{i=1 \\ (i,n)=1}}^{n} (a - \zeta^i b)$$

where ζ is a primitive nth root of unity. We write

$$P_n = P\left(\Phi_n(a, b)\right).$$

Stewart [10] proved the following theorem.

THEOREM 2. *For any K with $0 < K < 1/\log 2$ and any integer n (> 2) with at most $K\log\log n$ distinct prime factors, we have*

$$P_n/n > f(n)$$

where f is a function, strictly increasing and unbounded, which can be specified explicitly in terms of a, b and K.

The proof of Theorem 3 depends on Baker's result [3] on linear forms in the logarithms of algebraic numbers. If that is replaced by Lemma 1 in Stewart's paper [10], then the method of Stewart [10] gives the following result for the size of f.

THEOREM 3. *We have*

$$f(n) = c_7(\log n)^\lambda/\log\log n$$

where $\lambda = 1-K\log 2$ and $c_7>0$ is an effectively computable number depending only on a, b and K.

3. Let $b>a\geqslant 2$ be integers. We recall that a and b are composed of the same primes if

$$a = p_1^{u_1}\dots p_s^{u_s}, \qquad b = p_1^{v_1}\dots p_s^{v_s} \tag{5}$$

where $p_1,\dots,p_s$ are positive primes and $u_1,\dots,u_s,v_1,\dots,v_s$ are positive integers. We prove the following

THEOREM 4. *Let $b>a\geqslant 2$ be integers that are composed of the same primes. Then there exist effectively computable positive constants c_8 and c_9 such that*

$$b-a\geqslant c_8(\log a)^{c_9}.$$

Proof. Let $0<\varepsilon_2<1$ be a small constant which we shall choose later. Without loss of generality, we can assume that $a\geqslant a_0$ where a_0 is a large positive constant depending only on ε_2, since

$$b-a\geqslant 2 = (2/\log a_0)\log a_0\geqslant(2/\log a_0)\log a$$

whenever $a\leqslant a_0$. We shall assume that

$$b-a<(\log a)^{\varepsilon_2}$$

and arrive at a contradiction. Recall the expressions (5) for a and b. Notice that

$$p_1\cdots p_s\leqslant b-a<(\log a)^{\varepsilon_2}.$$

From here, it follows that

$$s\leqslant\frac{8\varepsilon_2\log\log a}{\log\log\log a}.$$

Further observe that $P(a) = P(b)<(\log a)^{\varepsilon_2}$ and the integers u_i and v_i do not exceed $8\log a$. Now

$$\left(\frac{b}{a}-1\right) = \frac{1}{a}(b-a)<\frac{\log a}{a}<a^{-1/2}.$$

Further

$$a^{-1/2}>\left(\frac{b}{a}-1\right) = |p_1^{u_1-v_1}\dots p_s^{u_s-v_s}-1|$$
$$>\tfrac{1}{2}|(u_1-v_1)\log p_1+\dots+(u_s-v_s)\log p_s|>0.$$

From these inequalities, we obtain

$$0<|(u_1-v_1)\log p_1+\dots+(u_s-v_s)\log p_s|<a^{-1/4}. \tag{6}$$

By Lemma 1, it is easy to check that

$$|(u_1-v_1)\log p_1+\dots+(u_s-v_s)\log p_s|\underset{\varepsilon_2}{\geqslant}\exp\left(-(\log a)^{E\varepsilon_2}\right) \tag{7}$$

where $E>0$ is a certain large constant independent of ε_2. If we take $\varepsilon_2 = 1/4E$, then the inequalities (6) and (7) clearly contradict each other. This completes the proof of Theorem 4.

Let $b>a\geqslant 2$ be integers such that $P(a) = P(b)$. Then Tijdeman [11] proved that

THEOREM 5.

$$b-a\geqslant 10^{-6}\log\log a.$$

The proof of Tijdeman [11] for this theorem depends on Baker's work [2] on $y^2 = x^3+k$. We remark that Theorem 5 follows easily from Lemma 1. The details for its proof are similar to those of Theorem 4.

By using Baker's work [2] on $y^2 = x^3+k$, Keates [6] and Ramachandra [8] proved

THEOREM 6. *Let u (>3) be an integer. Then*

$$P\big((u+1)(u+2)\big)>c_{10}\log\log u.$$

Theorem 6 also follows immediately from Lemma 1. The details for its proof are similar to those of Theorem 4. We shall use Theorem 6 for the proof of Theorem 7.

4. In this section, we shall prove the following

THEOREM 7. *Let $u>3$ and $k\geqslant 2$ be integers. Assume that*

$$u\geqslant k^{3/2}. \tag{8}$$

Then there exists an effectively computable constant $c_{11}>0$ independent of u and k such that

$$P(u,k)>c_{11}k\log\log u.$$

Proof. In view of Theorem 6, we can assume that $k\geqslant k_0$ where k_0 is a large constant. Erdös [4] proved that $P(u,k)>c_{12}k\log k$ for some constant $c_{12}>0$. So it is sufficient to prove the theorem when

$$\log k<\log\log u. \tag{9}$$

We write, for brevity,

$$P = P(u, k), \quad r = [2\pi(P)/k]+2.$$

Let us write $n = m'm''$ where $u < n \leqslant u+k$ and m' is the product of all powers of primes not exceeding k and m'' consists of powers of primes exceeding k. Observe that

$$\sum_n \omega(m'') \leqslant \pi(P).$$

Hence the number of integers n with $\omega(m'') \geqslant r$ does not exceed $k/2$. Hence there exist at least $[k/2]$ integers n with $\omega(m'') < r$. For each prime $q \leqslant k$, we omit amongst these n, one n for which q divides n to a maximal power. If star denotes omission of these n, then it follows, by an argument of Erdös, that

$$\prod_n{}^* m' \leqslant k^k.$$

The number of n's counted in this product is at least

$$[k/2]-\pi(k) \geqslant k/4.$$

So there exist, among these n, the integers n_1, n_2 $(n_1 \neq n_2)$ whose m' do not exceed k^{20}. Write

$$n_1 = m_1' p_1^{u_1} \ldots p_r^{u_r}, \quad n_2 = m_2' q_1^{v_1} \ldots q_r^{v_r}$$

where $m_1', m_2' < k^{20}$, $p_1, \ldots, p_r, q_1, \ldots, q_r$ are primes greater than k and not exceeding P. Observe that for $i = 1, \ldots, r$, u_i and v_i are non-negative integers not exceeding $8\log u$. Using (8), we get

$$(10) \qquad 0 < \left|\sum_{i=1}^{r} u_i \log p_i - \sum_{i=1}^{r} v_i \log p_i + \log\frac{m_1'}{m_2'}\right| < u^{-1/6}.$$

By Lemma 1 and (9), the left-hand side of this inequality exceeds

$$(11) \qquad \exp\left(-(r\log P\log\log u)^{c_{13}r}\right).$$

Now the theorem follows immediately from (9), (10) and (11).

The following theorem follows from the work of Baker and Sprindžuk.

THEOREM 8. *Let $f(x)$ be a polynomial with rational integers as coefficients. Assume that $f(x)$ has at least two distinct roots. Then for every integer $X > 3$,*

$$P(f(X)) > c_{14}\log\log X$$

where $c_{14} > 0$ is an effectively computable constant depending only on f.

By using a result of Baker on diophantine equations, Keates [6] rpoved Theorem 8 for polynomials of degree two and three. The proof of Baker and Sprindžuk for Theorem 8 depends on p-adic versions of inequalities on linear forms in logarithms. We remark that it is easy to deduce Theorem 8 from Lemma 1.

5. A property U holds for 'almost all' primes if given $\varepsilon > 0$, there exists $x_0 > 0$ depending only on ε such that for every $x \geqslant x_0$, the number of primes $p \leqslant x$ for which the property U does not hold is at most $\varepsilon x/\log x$. We shall prove that for almost all primes p,

$$(12) \qquad \frac{P(2^p-1)}{p} \geqslant \frac{(\log p)^2}{(\log\log p)^3}.$$

In fact we shall prove that

THEOREM 9. *Given $\varepsilon > 0$, there exist positive constants n_0 and c_{15} depending only on ε such that for every $n \geqslant n_0$, the number of primes p between n and $2n$ for which*

$$(13) \qquad \frac{P(2^p-1)}{p} < c_{15}\left(\frac{\log p}{\log\log p}\right)^2,$$

is at most $\varepsilon n/\log n$.

It is easy to see that the inequality (12) for 'almost all' primes p follows from Theorem 9.

Proof of Theorem 9. We shall assume that n_0 is a large positive constant depending only on ε. Set

$$r = [\varepsilon n/\log n]+1.$$

Assume that there are r primes $p_1, \ldots, p_r$ between n and $2n$ satisfying

$$(14) \qquad \frac{P(2^{p_i}-1)}{p_i} < \left(\frac{\log p_i}{\log\log p_i}\right)^2 \quad (i = 1, \ldots, r).$$

By Lemma 2,

$$\omega(2^{p_i}-1) \geqslant c_4\frac{\log p_i}{\log\log p_i} > c_4\frac{\log n}{\log\log n}$$

for every $i = 1, \ldots, r$. Observe that for distinct i, j $(1 \leqslant i, j \leqslant r)$, the prime factors of $2^{p_i}-1$ and $2^{p_j}-1$ are distinct. This is because if q is a prime number and q divides both $2^{p_i}-1$ and $2^{p_j}-1$, then $q \equiv 1 \pmod{p_i}$ and $q \equiv 1 \pmod{p_j}$. Therefore $q \equiv 1 \pmod{p_ip_j}$. Since $p_ip_j > n^2$, the inequality (14) is contradicted. Hence

$$(15) \qquad \sum_{i=1}^{r}\omega(2^{p_i}-1) \geqslant c_4 r\frac{\log n}{\log\log n} > c_4\varepsilon\frac{n}{\log\log n}.$$

Denote by

$$P = \max_{1\leqslant i\leqslant r} P(2^{p_i}-1).$$

If a prime number q divides $2^{p_i}-1$ for some $i=1,\ldots,r$, then

(i) $q \leqslant P$.
(ii) $q-1 = ap_i$ with an integer a.
(iii) $1 \leqslant a \leqslant (\log n)^2$.

By Brun's Sieve method, we get

$$\sum_{i=1}^{r} \omega(2^{p_i}-1) \leqslant c_{16} P \frac{\log\log n}{(\log n)^2} \tag{16}$$

for some constant $c_{16} > 0$. (For this, see page 207 of a paper of P. Erdös: *On the normal number of prime factors of $p-1$ and some related problems concerning Euler φ-function*, The Quaterly Journ. of Math. 6 (1935), pp. 203–213.) Comparing (15) and (16), we obtain

$$P \geqslant c_{17} n \left(\frac{\log n}{\log\log n}\right)^2,$$

for some positive constant c_{17} depending only on ε. Observe that the primes $p_1, \ldots, p_r$ lie between n and $2n$. Now the theorem follows immediately.

Remark. In fact the inequality (16) with $c_{16} P \dfrac{\log\log\log n}{(\log n)^2}$ is valid. For this, one can refer to the above mentioned paper of Erdös. In view of this, the Theorem 9 holds with

$$\frac{P(2^p-1)}{p} < c_{15} \frac{(\log p)^2}{(\log\log p)(\log\log\log p)}$$

in place of the inequality (13).

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