to discover 13 numbers with $\tau(n) \geqslant 135$. By a separate check it was determined that $\tau(n) = \chi(n)$ holds for $3 \leqslant n \leqslant 868$. By the jump value table one then concludes that $\tau(n) = \chi(n)$ holds for $3 \leqslant n \leqslant 250000$. This new information and the jump value table allows us to conclude that $\tau(n) = \chi(n)$ when $\tau(n) \leqslant 2592$. Since points with $\tau(n) > 2592$ are extremely rare, it is rather likely that our conjecture concerning τ and χ is actually true.

Among the hypothetical numbers n for which $\chi(n)$ may be infinite there may exist some larger than 2 which have the pleasant property that $T^k n = n$ for some positive k. Such numbers will be said to generate *loops*. For a loop point the smallest k such that $T^k n = n$ will be called the *period* of the loop. One notes that a consequence of (1) is that $\tau(n) \leqslant k$ holds for any loop point with period k. At the same time we may assume without loss of generality that $n \leqslant T^i n$ holds for all indices and thus $\chi(n) = \infty$. Thus if $\tau(n) = \chi(n)$ holds for all n with $\tau(n)$ finite then there can not exist any loops. The jump value table thus excludes loops up to period 2592.

The author was able to determine some details on the history of the problem after the submission of the manuscript. In its original form the problem was devised by Lothar Collatz in 1931. Subsequently the problem attracted the attention of Kakutani and Hasse. The name "Syracuse Problem" was devised by Hasse during a visit to Syracuse.

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The diophantine equations $(x^2-c)^2 = (t^2 \pm 2)y^2 + 1$

by

J. H. E. Cohn (London)

It is the object of this note to demonstrate that the two equations of the title have only finitely many solutions in positive integers x, y, t for any given integer $c \neq \pm 1$ and to provide a method for finding all the solutions by reducing the problem to finitely many diophantine equations in two variables, each of which will have only finitely many solutions in integers. The cases $c = \pm 1$ are in principle similar, except that there may be rather trivial infinite families of solutions. Compare also [1].

The results are somewhat exceptional in that for every fixed $k \neq \pm 2k_1^2$, there are infinitely many values of c for which the equation $(x^2-c)^2 = (t^2+k)y^2+1$ has infinitely many solutions in positive integers x, y, t.

In the first place any solutions with $x^2 - c \leqslant 0$ and/or $t^2 - 2 < 0$ are finite in number and can be found by simple enumeration. Secondly if $t^2 - 2 = 2$, i.e. $t = 2$, we find since $x^2 - c > 0$ that

$$(x^2-c)^2 - 2y^2 = 1,$$

$$(x^2-c) + y\sqrt{2} = (1+\sqrt{2})^{2n}, \quad n \geqslant 1.$$

Thus

$$x^2 - c = \frac{(1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n}}{2},$$

$$x^2 - c + (-1)^{n-1} = \left\{\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{\sqrt{2}}\right\}^2 = z^2, \text{ say},$$

where z is a rational integer. Thus $x^2 - z^2 = c \pm 1$, which can be solved immediately, giving only finitely many possible values for x, if $c \neq \pm 1$, and hence only finitely many possible corresponding values for y.

We therefore suppose from now on that $x^2 - c > 0$ and that $D = t^2 \pm 2 \geqslant 3$.

Consider first the case $D = t^2-2$, where the equation $u^2-Dv^2 = 2$ has solutions, with fundamental solution α, say. Then the fundamental solution of $u^2-Dv^2 = 1$ is $\beta = \frac{1}{2}\alpha^2$. If now $(x^2-c)^2-Dy^2 = 1$, then

$$(x^2-c)+y\sqrt{D} = \beta^n = (\tfrac{1}{2}\alpha^2)^n,$$

i.e.

$$x^2-c = \frac{\alpha^{2n}+\alpha'^{2n}}{2^{n+1}}.$$

Then

$$x^2-c+1 = \frac{(\alpha^n+\alpha'^n)^2}{2^{n+1}},$$

since $\alpha\alpha' = 2$. If n is odd, this yields $x^2-c+1 = z^2$ where z is a rational integer, and this is easily solved. If $n = 2m$ is even, then

$$x^2-c+1 = 2z^2,$$

where

$$z = \frac{\alpha^{2m}+\alpha'^{2m}}{2^{m+1}} = -1+\frac{(\alpha^m+\alpha'^m)^2}{2^{m+1}} = v^2-1 \text{ or } 2v^2-1,$$

where v is a rational integer, according as m is odd or even.

Thus we obtain either

$$x^2 = 2v^4-4v^2+(c+1)$$

or

$$x^2 = 8v^4-8v^2+(c+1),$$

and each of these equations has but a finite number of solutions in integers for each given $c \neq -1$. Thus for each given c there are but finitely many possible values of x, and hence of corresponding y and t.

The case $D = t^2+2$ is entirely similar, except that now α is the fundamental solution of $u^2-Dv^2 = -2$, $\alpha\alpha' = -2$ and $\beta = \frac{1}{2}\alpha^2$. Then

$$x^2-c+y\sqrt{D} = \beta^n = (\tfrac{1}{2}\alpha^2)^n,$$

$$x^2-c = \frac{\alpha^{2n}+\alpha'^{2n}}{2^{n+1}},$$

$$x^2-c+(-1)^n = \frac{(\alpha^n+\alpha'^n)^2}{2^{n+1}}.$$

If n is odd then $x^2-c-1 = z^2$, etc., as before. If $n = 2m$ is even, then

$$x^2-c+1 = 2z^2,$$

where

$$z = \frac{\alpha^{2m}+\alpha'^{2m}}{2^{m+1}} = (-1)^{m+1}+\frac{(\alpha^m+\alpha'^m)^2}{2^{m+1}} = v^2+1 \text{ or } 2v^2-1.$$

Thus we obtain in this case, either

$$x^2 = 2v^4+4v^2+(c+1)$$

or

$$x^2 = 8v^4-8v^2+(c+1),$$

and the result follows as before. This concludes the proof of the main result of the paper.

However, if $k \neq \pm 2k_1^2$, then the equation

$$(x^2-c)^2 = (t^2+k)y^2+1$$

is satisfied by integers x, y, t where $y = 2u$, $t = |k|u$ provided

$$(x^2-c)^2 = (k^2u^2+k)\cdot 4u^2+1 = (2ku^2+1)^2,$$

i.e. provided that either

$$x^2-2ku^2 = c+1$$

or

$$x^2+2ku^2 = c-1,$$

and since $k \neq \pm 2k_1^2$, one of these equations has infinitely many solutions for suitable values of c.

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ROYAL HOLLOWAY COLLEGE
Egham, Surrey, England

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