türliche Zahl m, die durch eine nicht zum Hauptgeschlecht gehörende Form darstellbar ist, einen Primfaktor p mit $(D/p) = 1$. Nach [1] ist m und damit auch p höchstens gleich $c(\varepsilon)|D|^{3/8+\varepsilon}$.

5. Beispiele zu Lemma 5.

$-D = 15$ ($p = 2$), 35 (3), 84 (5), 91 (5), 187 (7), 195 (7), 403 (11), 420 (11), 435 (11), 483 (11), 532 (13), 555 (13), 595 (13), 627 (13), 660 (13), 1012 (17), 1092 (17), 1155 (17), 1380 (19), 1428 (19), 1435 (19), 1995 (23), 3003 (29, 31), 3315 (29, 31), 5460 (37).

Literaturverzeichnis

[1] A. Baker and A. Schinzel, *On the least integers represented by the genera of binary quadratic forms*, Acta Arith. 18 (1971), S. 137–144.

[2] Z. J. Borewicz und I. R. Šafarevič, *Zahlentheorie*, Basel und Stuttgart 1966.

[3] E. Grosswald, *Negative discriminants of binary quadratic forms with one class in each genus*, Acta Arith. 8 (1963), S. 295–306.

[4] C. S. Herz, *Construction of class fields*. In: *Seminar on complex multiplication*, Lecture Notes in Mathematics 21 (1966), VII, S. 1–21.

[5] Yu. V. Linnik and A. I. Vinogradov, *Hyperelliptic curves and the least prime quadratic residue*, Doklady Akad. Nauk SSSR 166 (1966), S. 259–261; = Soviet Math. Dokl. 7 (1966), S. 612–614.

[6] J. D. Swift, *Note on discriminants of binary quadratic forms with a single class in each genus*, Bull. Amer. Math. Soc. 54 (1948), S. 560–561.

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The Fourier expansion of Epstein's zeta function for totally real algebraic number fields and some consequences for Dedekind's zeta function

by

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1. Introduction. Let K be a totally real algebraic number field of odd degree $m>1$ (over $\boldsymbol{Q}$). And suppose that the class number of K is one. We shall need the following concepts from algebraic number theory. For definitions, etc., one should refer to Lang [7]. Let O_K denote the ring of integers of K and U_K the unit group of O_K. Suppose the m embeddings of K into $\boldsymbol{R}$ over $\boldsymbol{Q}$ are denoted $x\mapsto x^{(j)}$ for $j=1,2,\ldots,m$. Let $Nx=\prod_{j=1}^{m} x^{(j)}$, for $x\in K$. Let d_K be the absolute value of the discriminant of K and δ_K be the different of K. The Dedekind zeta function of K is

$$\zeta_K(s) = \sum_{a\in O_K^*/U_K} Na^{-s}, \quad \text{for} \quad \operatorname{Re} s>1. \tag{1.1}$$

Here the sum is over non-zero integers of K ($O_K^* = O_K-\{0\}$) which are not equivalent under multiplication by units. We use here the assumption that the class number of K is one so that all ideals are principal in O_K.

We shall prove that

$$\begin{aligned}\zeta_K(2s)s^m + d_K^{1-2s}\big(\pi^{2s-1}\Gamma(1-s)\Gamma(s)^{-1}\big)^m\zeta_K(2-2s)(1-s)^m \\ = -d_K^{-1/2}\big(\pi^s\Gamma(s)^{-1}\big)^m \sum_{a\in O_K^*/U_K;\, b\in\delta_K^{-1*}} \left|\frac{Na}{Nb}\right|^{\frac{1-2s}{2}} \prod_{j=1}^{m}\{K_{1/2-s}(2\pi|a^{(j)}b^{(j)}|)+ \\ +4\pi|a^{(j)}b^{(j)}|K'_{1/2-s}(2\pi|a^{(j)}b^{(j)}|)\}.\end{aligned} \tag{1.2}$$

$K_\nu(z)$ is the modified Bessel function of the second kind defined by (2.1); $K'_\nu(z) = \dfrac{dK_\nu(z)}{dz}$. Here a runs over non-zero integers of K non-equivalent

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under multiplication by units and b runs over non-zero elements of the inverse different. It follows from (1.2) by setting $s = 1$ that

$$(1.3)\qquad \zeta_K(2) = d_K^{-1/2}(2\pi^2)^m \sum_{a\in O_K^*/U_K;\, b\in\delta_K^{-1*}} |Nb| \exp\Big\{-2\pi\sum_{j=1}^{m} |a^{(j)}b^{(j)}|\Big\},$$

for example. This generalizes the following result of Schlömilch and Ramanujan for the Riemann zeta function ($\zeta(s) = \zeta_Q(s)$):

$$\zeta(2) = \frac{\pi}{2} + 4\pi^2 \sum_{n\geq 1} \sigma_1(n)\exp\{-2\pi n\},$$

where $\sigma_1(n) = \sum_{0<d|n} d$. One could also use (1.2) to derive formulas similar to but more complicated than Ramanujan's other results of this type. A good reference for such formulas is Berndt [2], which has a long list of papers on the subject. Another reference is Grosswald [4]. The restriction that K be totally real and the restriction that $m = [K:Q]$ be odd are unnecessary except to simplify the results. It is a little more complicated to drop the class number one assumption. It is possible that (1.3) could have interesting consequences for totally real fields of odd degree and class number one.

Another possible use of (1.2) is the evaluation of ζ_K in the interval $(0, 1)$. One might have thought that the natural formula to use in order to study ζ_K in $(0, 1)$ would be Hecke's integral formula ([5], page 205) — a result which expresses the Dedekind zeta function as a certain integral over a cube, the integrand being an Epstein zeta function whose quadratic form argument depends on the units and the integral basis of the field as well as the variable of integration. If one substitutes the Fourier expansion ([12], formula (1.2), page 477) of the Epstein zeta function into the integral, it is very difficult to see what is happening, because the series expansion converges very slowly, even for small discriminants. Thus (1.2) may be of some use in such questions — the lack of integration being its main virtue.

The proof of (1.2) is to be found in § 3 and results from the Fourier expansion of the Epstein zeta function for the totally real algebraic number field K. If $K = Q$ the Epstein zeta function has two arguments, the first being a positive definite symmetric $n\times n$ matrix P and the second a complex variable s with $\operatorname{Re} s > n/2$. The Epstein zeta function over Q is defined by:

$$(1.4)\qquad Z_n^Q(P, s) = \tfrac{1}{2}\sum_{a\in Z^{n*}} P[a]^{-s}$$

where $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n\end{pmatrix}$ runs over non-zero column vectors with integer entries and $P[a] = {}^t aPa$. The space of positive definite symmetric $n\times n$ matrices P is the symmetric space (see [6] for definition) $\mathrm{GL}(n, \boldsymbol{R})/O(n)$ with $\mathrm{GL}(n, \boldsymbol{R})$ = the general linear group of $n\times n$ invertible real matrices and $O(n)$ = the orthogonal group. If $G\in \mathrm{GL}(n, \boldsymbol{R})$, let $P = G^tG$ to obtain the above identification. Then $Z_n(P, s)$ is an "automorphic form" — specifically an Eisenstein series — in the sense of Borel ([3], page 200 and example 2, page 209). This means that Z_n satisfies certain differential equations in P and that Z_n is invariant under the map $P\mapsto P[A] = {}^tAPA$, for any $n\times n$ matrix A with integer entries and determinant ± 1.

Motivated by the simple algebraic fact that $K\otimes_Q \boldsymbol{R} \cong \boldsymbol{R}\oplus\ldots\oplus\boldsymbol{R}$ ($m = [K:Q]$ copies) if K is totally real, one sees that the Epstein zeta function over K ought to be a function on

$$\big(\mathrm{GL}(n, \boldsymbol{R})/O(n)\big)\times\ldots\times\big(\mathrm{GL}(n, \boldsymbol{R})/O(n)\big), \qquad m \text{ copies}.$$

This is analogous to the case of Hilbert modular forms (Siegel [10], pages 273–274). So let $P = (P^{(1)}, \ldots, P^{(m)})$ be a vector of positive definite symmetric $n\times n$ matrices and let s be a complex variable with $\operatorname{Re} s > n/2$. Then define the Epstein zeta function over K by:

$$(1.5)\qquad Z_n^K(P, s) = Z_n(P, s) = \sum_{g\in O_K^{n*}/U_K} \prod_{j=1}^{m} \big(P^{(j)}[g^{(j)}]\big)^{-s}.$$

The sum is over a complete set of representatives for n-tuples from O_K (not all of whose entries are zero) with respect to the equivalence relation from scalar multiplication by units. Tamagawa [11] indicates that the usual theta function methods yield the analytic continuation and functional equation of (1.5). We have again an "automorphic form", whose invariance property is:

$$(1.6)\qquad Z_n(P, s) = Z_n(P[A], s),$$

for A in the discrete group of matrices with entries in O_K and determinant in U_K. Here

$$(1.7)\qquad P[A] = (P^{(1)}[A^{(1)}], \ldots, P^{(m)}[A^{(m)}]),$$

with $A^{(j)}$ denoting the matrix obtained by replacing every element of the matrix A by its jth conjugate.

The main result of § 2 is Theorem 1, which gives the $n-1$ Fourier expansions of $Z_n^K(P, s)$, generalizing the result in the case $K = Q$ obtained in [12], Theorem 1, page 480. The result also generalizes Asai ([1], formula 13, pages 203–204). The latter is seen to be the case $n = 2$ of our result. However Asai does not restrict himself to totally real fields.

There are many open questions. One would like to generalize Asai's results on $\log|\eta(z)|$ for algebraic number fields ([1], Theorem 4, pages 207–208, and Theorem 5, page 210). Asai defines a function $h_K(z)$ which has three properties of $\log|\eta(z)|$, i.e., h_K appears in a Kronecker limit formula for the Epstein zeta function ($n = 2$) over K. Also $h_K(z)$ is harmonic in the upper half plane. Finally the Mellin transform of $h_K(z)$ is $\zeta_K(s, \lambda)\zeta_K(s+1, \lambda)$, where $\zeta_K(s, \lambda)$ is a Hecke zeta function with Grössencharacter. The generalization to higher dimensions ($n \geqslant 3$) appears difficult. For example there are more invariant differential operators, so it is more difficult to be harmonic (see [6], page 432 or [9], page 64).

Another open problem is the extension of these results to other Eisenstein series for the general linear group. Such series are considered by Langlands in [8], where he obtains the functional equation and analytic continuation. One should at least be able to generalize the Fourier expansion obtained in [14] to Eisenstein series of one complex variable over algebraic number fields.

A final question concerns the shape of the fundamental domain in $GL(n, \boldsymbol{R})/O(n) \times \ldots \times GL(n, \boldsymbol{R})/O(n)$ for the discontinuous group of invertible matrices with integer entries from the number field and unit determinant. One ought to consider the question of the number of cusps and the Fourier expansion at each cusp (cf. [10], Chapter III, § 2).

2. The Fourier expansions of Epstein's zeta function over a totally real algebraic number field. We need a few definitions before stating the result on the Fourier expansions of $Z_n(P, s)$. The modified Bessel function of the second kind is defined by

(2.1) $$K_\nu(z) = \frac{1}{2}\int_0^\infty \exp\left\{-\frac{z}{2}\left(u+u^{-1}\right)\right\}u^{\nu-1}\,du,$$

for $|\arg z| < \pi/2$. As usual $\Gamma(z)$ is the gamma function.

For any integer r with $1 \leqslant r \leqslant n-1$, one obtains a decomposition of $P^{(j)}$ in block form as in [12], formula (2.1), page 479:

(2.2) $$P^{(j)} = \begin{pmatrix} T^{(j)} & 0 \\ 0 & P_2^{(j)} \end{pmatrix}\begin{bmatrix} I & 0 \\ Q^{(j)} & I \end{bmatrix} = {}^t\begin{pmatrix} I & 0 \\ Q^{(j)} & I \end{pmatrix}\begin{pmatrix} T^{(j)} & 0 \\ 0 & P_2^{(j)} \end{pmatrix}\begin{pmatrix} I & 0 \\ Q^{(j)} & I \end{pmatrix},$$

where $P_2^{(j)}$ is the $r \times r$ block in the lower right-hand corner of $P^{(j)}$, I is the identity matrix, 0 is a matrix of zeros, and where $j = 1, 2, \ldots, m = [K:\boldsymbol{Q}]$. Note that here $P^{(j)}$ denotes the jth matrix component $P = (P^{(1)}, \ldots, P^{(m)})$ rather than a $j \times j$ matrix, as it did in [12]. If $P = (P^{(1)}, \ldots, P^{(m)})$, define

(2.3) $$|P| = \prod_{j=1}^m \det(P^{(j)}).$$

And for any O_K-module M in K^m define $M^* = M - \{0\}$.

It is now possible to state the Fourier expansion of $Z_n(P, s)$ as a periodic function of $Q = (Q^{(1)}, \ldots, Q^{(m)})$. The result generalizes Theorem 1 of [12], page 480.

THEOREM 1. *For any totally real algebraic number field K,*

$$Z_n(P, s) = Z_r(P_2, s) + d_K^{-r/2}\left(\pi^{r/2}\Gamma\left(s-\frac{r}{2}\right)\Gamma(s)^{-1}\right)^m |P_2|^{-1/2} Z_{n-r}\left(T, s-\frac{r}{2}\right) + $$
$$+ d_K^{-r/2}\left(2\pi^s\Gamma(s)^{-1}\right)^m H_r(P, s).$$

The notation is that of (2.2) and (2.3). Here

$$H_r(P, s) = |P_2|^{-1/2} \sum_{a \in O_K^{n-r*}/U_K;\, b \in (\delta_K^{-1})^{r*}} \prod_{j=1}^m \left(\frac{T^{(j)}[a^{(j)}]}{P_2^{(j)-1}[b^{(j)}]}\right)^{(r-2s)/4} \times$$
$$\times \exp\{2\pi i\, {}^t b^{(j)} Q^{(j)} a^{(j)}\} K_{r/2-s}\left(2\pi\sqrt{T^{(j)}[a^{(j)}] P_2^{(j)-1}[b^{(j)}]}\right),$$

where the sum is over non-equivalent non-zero vectors a in O_K^{n-r} with equivalence defined by componentwise multiplication by units in U_K, and over non-zero vectors b with components in the inverse different. Finally $H_r(P, s)$ is an entire function of s.

Proof. The proof is an exact analogue of that of Theorem 1 of [12], page 480. We split the summation over the variable g in (1.5) into two parts $g = \begin{pmatrix} a \\ b \end{pmatrix}$, with a in O_K^{n-r} and b in O_K^r. Then from (2.2) it follows that

$$\prod_{j=1}^m \left(P^{(j)}\begin{bmatrix} a^{(j)} \\ b^{(j)} \end{bmatrix}\right) = \prod_{j=1}^m \left(T^{(j)}[a^{(j)}] + P_2^{(j)}[Q^{(j)}a^{(j)} + b^{(j)}]\right).$$

Thus

$$Z_n(P, s) = \sum_{a=0;\, b \in O_K^{r*}/U_K} + \sum_{a \in O_K^{n-r*}/U_K;\, b \in O_K^r}.$$

So

(2.4) $$Z_n(P, s) = Z_r(P, s) + \sum_{a \in O_K^{n-r*}/U_K;\, b \in O_K^r} \prod_{j=1}^m \left(T^{(j)}[a^{(j)}] + P_2^{(j)}[Q^{(j)}a^{(j)} + b^{(j)}]\right)^{-s}.$$

Now let $w_1, \ldots, w_m$ be a $\boldsymbol{Z}$-basis of O_K. And let $w_1^*, \ldots, w_m^*$ be a dual basis with respect to the trace. Then

(2.5) $$(w_k^{(j)})_{j,k=1,\ldots,m} \cdot {}^t(w_k^{*(j)})_{j,k=1,\ldots,m} = I.$$

Next apply Poisson's summation formula to the sum over b in $O_K^r = (\sum_{j=1}^{m}{}^{\oplus} \mathbf{Z}w_j)^r$. It follows that the sum over b in (2.4) is $\sum_{c\in \mathbf{Z}^{rm}} f(a, c)$, where

$$f(a, c) = \int_{x_1\in \mathbf{R}^r} \cdots \int_{x_m\in \mathbf{R}^r} \prod_{j=1}^{m} \Big(T^{(j)}[a^{(j)}] + P_2^{(j)}\Big[Q^{(j)}a^{(j)} + \sum_{k=1}^{m} x_k w_k^{(j)}\Big]\Big)^{-s} \times$$
$$\times \exp\Big\{2\pi i \sum_{k=1}^{m} {}^t c_k x_k\Big\} dx_1 \ldots dx_m.$$

The natural course of action is to separate variables in the integral. So we make the change of variables

$$v_j = \sum_{k=1}^{m} x_k w_k^{(j)}, \quad j = 1, \ldots, m,$$

where v_j and x_j are in $\mathbf{R}^r$ for $j = 1, \ldots, m$. If we think of $v = {}^t(v_1 \ldots v_m) = (v^1 \ldots v^r)$ and $x = {}^t(x_1 \ldots x_m) = (x^1 \ldots x^r)$ as $m\times r$ matrices, then $v = (Mx^1 \ldots Mx^r)$, where $M = (w_k^{(j)})_{j,k=1,\ldots,m}$. We already have the inverse for this transformation thanks to (2.5), namely

$$M^{-1} = {}^t(w_k^{*(j)})_{j,k=1,\ldots,m}.$$

It follows that, as in [12], pages 480–481:

$$f(a, 0) = d_K^{-r/2} |P_2|^{-1/2} \left(\pi^{r/2} \Gamma\left(s - \frac{r}{2}\right) \Gamma(s)^{-1}\right)^m \prod_{j=1}^{m} (T^{(j)}[a^{(j)}])^{r/2-s},$$

and if c is a non-zero $m\times r$ matrix of integers, then:

$$f(a, c) = d^{-r/2} |P_2|^{-1/2} (2\pi^s \Gamma(s)^{-1})^m \prod_{j=1}^{m} \left(\frac{T^{(j)}[a^{(j)}]}{P_2^{(j)-1}[b^{(j)}]}\right)^{(r-2s)/4} \times$$
$$\times \exp\{2\pi i\, {}^t b^{(j)} Q^{(j)} a^{(j)}\} K_{r/2-s}\big(2\pi \sqrt{T^{(j)}[a^{(j)}] P_2^{(j)-1}[b^{(j)}]}\big),$$

with ${}^t(b^{(1)} \ldots b^{(m)}) = (M^* c^1 \ldots M^* c^r)$ if $M^* = (w_k^{*(j)})_{j,k=1,\ldots,m}$ and $c = (c^1 \ldots c^r) \neq 0$ in $\mathbf{Z}^{mr}$. Thus $b^{(j)}$ is the jth conjugate of an arbitrary element b of $(\delta_K^{-1})^{r*}$.

This completes the proof of the theorem except for the statement that $H_r(P, s)$ is entire in s. For this one needs the exponential decay of the modified Bessel function of the second kind.

We leave to the reader the task of generalizing the other results of [12] to totally real algebraic number fields. It will be convenient however to state the results of Theorem 1 in the special case $n = 2$ separately. This is Asai's result ([1], formula 13, pages 203–204) essentially. One has to translate from binary quadratic forms $P > 0$ to complex numbers z

in the upper half plane in order to go from our result to Asai's. This is achieved by looking at solutions z in the upper half plane of $P\begin{bmatrix} z \\ 1 \end{bmatrix} = 0$.

We make the simplifying assumption that K is a totally real algebraic number field with class number one. Then the formula (1.1) is a correct expression for the Dedekind zeta function ζ_K of K. Thus, setting $n = 2$ and $r = 1$ in Theorem 1 yields:

$$(2.6)\quad Z_2(P, s) = |P_2|^{-s} \zeta_K(2s) + d_K^{-1/2} (\pi^{1/2} \Gamma(s - \tfrac{1}{2}) \Gamma(s)^{-1})^m |P_2|^{-1/2} \times$$
$$\times |T|^{1/2-s} \zeta_K(2s-1) + d_K^{-1/2} (2\pi^s \Gamma(s)^{-1})^m H_1(P, s),$$

where

$$H_1(P, s) = |P_2|^{-1/2} |P|^{(1-2s)/4} \sum_{a\in O_K^*/U_K;\, b\in \delta_K^{-1*}} \left|\frac{Na}{Nb}\right|^{(1-2s)/2} \times$$
$$\times \prod_{j=1}^{m} \exp\{2\pi i\, {}^t b^{(j)} Q^{(j)} a^{(j)}\} K_{1/2-s}\big(2\pi \sqrt{T^{(j)}[a^{(j)}] P_2^{(j)-1}[b^{(j)}]}\big).$$

To study the situation when the class number is bigger than one, Epstein zeta functions associated with ideals as in Tamagawa [11] will no doubt be necessary. We leave this question open.

3. Some formulas relating $\zeta_K(2s)$ and $\zeta_K(2s-1)$. We first prove generalizations of Proposition 2 of [13] to totally real algebraic number fields of class number one. The result appears more complicated than (1.2) because we do not assume $m = [K : \mathbf{Q}]$ is odd.

THEOREM 2. *Let ζ_K be the Dedekind zeta function of a totally real algebraic number field K with class number one. Then*

$$s^m \zeta_K(2s) + (1 - (2s-1)^m) d_K^{-1/2} \left(\frac{\sqrt{\pi}}{2} \Gamma(s - \tfrac{1}{2}) \Gamma(s)^{-1}\right)^m \zeta_K(2s-1)$$
$$= d_K^{-1/2} (2\pi^s \Gamma(s)^{-1})^m \sum_{a\in O_K^*/U_K;\, b\in \delta_K^{-1*}} \left|\frac{Na}{Nb}\right|^{(1-2s)/2} \times$$
$$\times \Big\{\prod_{j=1}^{m} \Big[\Big(\frac{s}{2} - \frac{1}{4}\Big) K_{1/2-s}(2\pi |a^{(j)} b^{(j)}|) - \pi |a^{(j)} b^{(j)}| K'_{1/2-s}(2\pi |a^{(j)} b^{(j)}|)\Big] -$$
$$- \prod_{j=1}^{m} \Big[\Big(\frac{s}{2} + \frac{1}{4}\Big) K_{1/2-s}(2\pi |a^{(j)} b^{(j)}|) + \pi |a^{(j)} b^{(j)}| K'_{1/2-s}(2\pi |a^{(j)} b^{(j)}|)\Big]\Big\}.$$

Here $K'_\nu(z) = \dfrac{d}{dz} K_\nu(z)$.

Proof. There are many possibilities for deriving such formulas. We choose the following modifications of the proof of Proposition 2 of [13]. Let $x_1, \ldots, x_m$ be positive real numbers and set

$$U^{(j)} = \begin{pmatrix} x_j & 0 \\ 0 & 1 \end{pmatrix}, \quad V^{(j)} = \begin{pmatrix} 1 & 0 \\ 0 & x_j \end{pmatrix} = U^{(j)}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad j = 1, \ldots, m.$$

Then $Z_2(U, s) = Z_2(V, s)$ if $U = (U^{(1)}, \ldots, U^{(m)})$ and $V = (V^{(1)}, \ldots, V^{(m)})$. Now equating the expressions for $Z_2(U, s)$ and $Z_2(V, s)$ given by (2.6) yields:

$$(3.1) \quad \zeta_K(2s)\,(1-|x|^{-s}) + \\ + d_K^{-1/2}\big(\pi^{1/2}\Gamma(s-\tfrac{1}{2})\Gamma(s)^{-1}\big)^m \zeta_K(2s-1)\,(|x|^{1/2-s}-|x|^{-1/2}) \\ = d_K^{-1/2}\big(2\pi^s\Gamma(s)^{-1}\big)^m \sum_{a\in O_K^*/U_K;\, b\in\delta_K^{-1*}} \left|\frac{Na}{Nb}\right|^{(1-2s)/2} \times \\ \times\Big\{\prod_{j=1}^m x_j^{-1/4-s/2} K_{1/2-s}(2\pi x_j^{-1/2}|a^{(j)}b^{(j)}|) - \prod_{j=1}^m x_j^{1/4-s/2} K_{1/2-s}(2\pi x_j^{1/2}|a^{(j)}b^{(j)}|)\Big\},$$

with $|x| = \prod_{j=1}^m x_j$. Then apply $\dfrac{\partial^m}{\partial x_1 \ldots \partial x_m}\Big|_{x_1=\ldots=x_m=1}$ to both sides of (3.1) to complete the proof.

Corollary 1. *If ζ_K is the Dedekind zeta function of a totally real algebraic number field K with class number one, then*

$$s^m\zeta_K(2s) + \big(1-(2s-1)^m\big) d_K^{1-2s}\big(\tfrac{1}{2}\pi^{2s-1}\Gamma(1-s)\Gamma(s)^{-1}\big)^m \zeta_K(2-2s) \\ = d_K^{-1/2}\big(2\pi^s\Gamma(s)^{-1}\big)^m \sum_{a\in O_K^*/U_K;\, b\in\delta_K^{-1*}} \left|\frac{Na}{Nb}\right|^{(1-2s)/2} \times \\ \times\Big\{\prod_{j=1}^m \Big[\Big(\frac{s}{2}-\frac{1}{4}\Big)K_{1/2-s}(2\pi|a^{(j)}b^{(j)}|) - \pi|a^{(j)}b^{(j)}|\,K'_{1/2-s}(2\pi|a^{(j)}b^{(j)}|)\Big] - \\ - \prod_{j=1}^m \Big[\Big(\frac{s}{2}+\frac{1}{4}\Big)K_{1/2-s}(2\pi|a^{(j)}b^{(j)}|) + \pi|a^{(j)}b^{(j)}|\,K'_{1/2-s}(2\pi|a^{(j)}b^{(j)}|)\Big]\Big\}.$$

Proof. Combine Theorem 2 and the functional equation of ζ_K ([7], Theorem 2, page 259):

$$F_K(s) \equiv (d_K^{1/2}\pi^{-m/2})^s\,\Gamma\Big(\frac{s}{2}\Big)^m \zeta_K(s) = F_K(1-s).$$

The next corollary generalizes the formula of Schlömilch and Ramanujan for $\zeta_Q(2) = \zeta(2)$ referred to in the introduction after (1.3) to zeta functions of totally real fields with class number one. The result appears more complicated than (1.3) because we are not assuming $m = [K:Q]$ is odd.

Corollary 2. *If ζ_K is the Dedekind zeta function for the totally real algebraic number field K of class number one, then*

$$\zeta_K(2) = \frac{m}{4}(2\pi)^m\frac{R_K}{d_K} + d_K^{-1/2}\pi^{2m} \sum_{a\in O_K^*/U_K;\, b\in\delta_K^{-1*}} |Nb| \exp\Big\{-2\pi\sum_{j=1}^m |a^{(j)}b^{(j)}|\Big\} \times \\ \times\Big\{\prod_{j=1}^m [(2\pi|a^{(j)}b^{(j)}|)^{-1}+1] - \prod_{j=1}^m [(2\pi|a^{(j)}b^{(j)}|)^{-1}-1]\Big\},$$

where R_K is the regulator of K.

Proof. Substitute $s = 1$ into the preceeding Corollary, using the following formula from Lang [7], Theorem 2, page 259:

$$(3.2) \qquad \lim_{s\to 0}\Big\{s\Gamma\Big(\frac{s}{2}\Big)^m\zeta_K(s)\Big\} = -2^{m-1}R,$$

since K is totally real and thus has only two roots of unity.

One obtains

$$\zeta_K(2) + d_K^{-1}\Big(\frac{\pi}{2}\Big)^m \lim_{s\to 1}\frac{1-(2s-1)^m}{2-2s}\{(2-2s)\Gamma(1-s)^m\zeta_K(2-2s)\} \\ = \zeta_K(2) - \frac{m}{4}(2\pi)^m\frac{R_K}{d_K}.$$

And the preceeding Corollary implies that

$$\zeta_K(2) - \frac{m}{4}(2\pi)^m\frac{R_K}{d_K} = d_K^{-1/2}(2\pi)^m \sum_{a\in O_K^*/U_K;\, b\in\delta_K^{-1*}} \left|\frac{Na}{Nb}\right|^{-1/2} \times \\ \times\Big\{\prod_{j=1}^m [\tfrac{1}{4}K_{1/2}(2\pi|a^{(j)}b^{(j)}|) - \pi|a^{(j)}b^{(j)}|\,K'_{1/2}(2\pi|a^{(j)}b^{(j)}|)] - \\ - \prod_{j=1}^m [\tfrac{3}{4}K_{1/2}(2\pi|a^{(j)}b^{(j)}|) + \pi|a^{(j)}b^{(j)}|\,K'_{1/2}(2\pi|a^{(j)}b^{(j)}|)]\Big\}.$$

From $K_{1/2}(z) = \sqrt{\dfrac{\pi}{2z}}\,e^{-z}$, it follows that

$$K'_{1/2}(z) = -\sqrt{\frac{\pi}{2z}}\,e^{-z}\Big\{1+\frac{1}{2z}\Big\}.$$

Thus

$$K_{1/2}(z) - 2zK'_{1/2}(z) = \sqrt{2\pi z}\,e^{-z}(z^{-1}+1), \\ 3K_{1/2}(z) + 2zK'_{1/2}(z) = \sqrt{2\pi z}\,e^{-z}(z^{-1}-1).$$

Substitution of these results into the term in braces completes the proof of the corollary.

Next we prove another generalization of the Schlömilch–Ramanujan formula. This result is equivalent to (1.3).

COROLLARY 3. *If K is a totally real algebraic number field of class number one and odd degree $m = [K:Q] > 1$, then*

$$\zeta_K(2) = d_K^{-3/2}(2\pi^2)^m \sum_{a\in O_K^*/U_K;\, b\in\delta_K^{-1*}} |Na| \exp\Big\{-2\pi\sum_{j=1}^m |a^{(j)}b^{(j)}|\Big\}.$$

Proof. Multiply both sides of the formula of Corollary 1 to Theorem 2 by $\Gamma(s)^m$ and let s approach zero to obtain:

$$2d_K(2\pi)^{-m}\zeta_K(2) = -2d_K^{-1/2}2^m \sum_{a\in O_K^*/U_K;\, b\in\delta_K^{-1*}} \left|\frac{Na}{Nb}\right|^{1/2} \times$$
$$\times \prod_{j=1}^m [\tfrac{1}{4}K_{1/2}(2\pi|a^{(j)}b^{(j)}|) + \pi|a^{(j)}b^{(j)}|K'_{1/2}(2\pi|a^{(j)}b^{(j)}|)].$$

Since $K_{1/2}(z) + 2zK'_{1/2}(z) = -\sqrt{2\pi z}e^{-z}$, the proof of Corollary 3 is complete.

Next we prove formula (1.2).

THEOREM 3. *If ζ_K is the Dedekind zeta function of a totally real algebraic number field K of odd degree $m = [K:Q]$ and having class number one, then:*

$$\zeta_K(2s)s^m + d_K^{-1/2}(\sqrt{\pi}\,\Gamma(s-\tfrac{1}{2})\Gamma(s)^{-1})^m\zeta_K(2s-1)(1-s)^m$$
$$= -d_K^{-1/2}(\pi^s\Gamma(s)^{-1})^m \sum_{a\in O_K^*/U_K;\, b\in\delta_K^{-1*}} \left|\frac{Na}{Nb}\right|^{(1-2s)/2} \times$$
$$\times \prod_{j=1}^m [K_{1/2-s}(2\pi|a^{(j)}b^{(j)}|) + 4\pi|a^{(j)}b^{(j)}|K'_{1/2-s}(2\pi|a^{(j)}b^{(j)}|)].$$

Proof. Proceed as in the proof of Theorem 2, except multiply (3.1) by $|x|^{s/2}$ before applying $\left.\dfrac{\partial^m}{\partial x_1\ldots\partial x_m}\right|_{x_1=\ldots=x_m=1}$. Thus (3.1) is replaced by

(3.3) $$\zeta_K(2s)(|x|^{s/2} - |x|^{-s/2}) + $$
$$+ d_K^{-1/2}(\pi^{1/2}\Gamma(s-\tfrac{1}{2})\Gamma(s)^{-1})^m\zeta_K(2s-1)(|x|^{1/2-s/2} - |x|^{s/2-1/2})$$
$$= d_K^{-1/2}(2\pi^s\Gamma(s)^{-1})^m \sum_{a\in O_K^*/U_K;\, b\in\delta_K^{-1*}} \left|\frac{Na}{Nb}\right|^{(1-2s)/2} \times$$
$$\times\Big\{\prod_{j=1}^m x_j^{-1/4}K_{1/2-s}(2\pi x_j^{-1/2}|a^{(j)}b^{(j)}|) - \prod_{j=1}^m x_j^{1/4}K_{1/2-s}(2\pi x_j^{1/2}|a^{(j)}b^{(j)}|)\Big\},$$

with $|x| = \prod_{j=1}^m x_j$.

COROLLARY 1. *If ζ_K is the Dedekind zeta function of the totally real algebraic number field K of odd degree $m = [K:Q]$ and class number one, then*

$$s^m\zeta_K(2s) + (1-s)^m d_K^{1-2s}(\pi^{2s-1}\Gamma(1-s)\Gamma(s)^{-1})^m\zeta_K(2-2s)$$
$$= -d_K^{-1/2}(\pi^s\Gamma(s)^{-1})^m \sum_{a\in O_K^*/U_K;\, b\in\delta_K^{-1*}} \left|\frac{Na}{Nb}\right|^{(1-2s)/2} \times$$
$$\times \prod_{j=1}^m \{K_{1/2-s}(2\pi|a^{(j)}b^{(j)}|) + 4\pi|a^{(j)}b^{(j)}|K'_{1/2-s}(2\pi|a^{(j)}b^{(j)}|)\}.$$

Proof. Proceed exactly as in the proof of Corollary 1 of Theorem 2.

Next we give another proof of the formula of Corollary 3 of Theorem 2.

COROLLARY 2. *Let ζ_K be the Dedekind zeta function of K, a totally real field of odd degree $m = [K:Q] > 1$, with class number one. Then*

$$\zeta_K(2) = d_K^{-1/2}(2\pi^2)^m \sum_{a\in O_K^*/U_K;\, b\in\delta_K^{-1*}} |Nb| \exp\Big\{-2\pi\sum_{j=1}^m |a^{(j)}b^{(j)}|\Big\}.$$

References

[1] T. Asai, *On a certain function analogous to* $\log|\eta(z)|$, Nagoya Math. J. 40 (1970), pp. 193–211.

[2] B. Berndt, *Ramanujan's formula for* $\zeta(2n+1)$, Ramanujan memorial volume.

[3] A. Borel, *Introduction to automorphic forms*, Proceedings of Symposia in Pure Math., Vol. 9, Amer. Math. Soc., Providence, Rhode Island, 1966.

[4] E. Grosswald, *Relations between the values at integral arguments of Dirichlet series that satisfy functional equations*, Proceedings of Symposia in Pure Math., Vol. 24, Amer. Math. Soc., Providence, Rhode Island, 1973.

[5] E. Hecke, *Mathematische Werke*, Göttingen 1959.

[6] S. Helgason, *Differential geometry and symmetric spaces*, New York 1962.

[7] S. Lang, *Algebraic number theory*, Reading, Massachusetts, 1970.

[8] R. P. Langlands, *Dirichlet series associated with quadratic forms*, unpublished manuscript, 1961.

[9] H. Maass, *Siegel's modular forms and Dirichlet series*, Lecture Notes in Math., Vol. 216, New York 1971.

[10] C. L. Siegel, *Lectures on advanced analytic number theory*, Bombay 1957.

[11] T. Tamagawa, *On some extensions of Epstein's Z-series*, Proc. Internatl. Symp. on Alg. Number Theory, Tokyo-Nikko(1955), pp. 259–261.

[12] A. Terras, *Bessel series expansions of the Epstein zeta function and the functional equation*, Trans. Amer. Math. Soc. 183 (1973), pp. 477–486.

[13] — *Some formulas for the Riemann zeta function at odd integer argument resulting from Fourier expansions of the Epstein zeta function*, Acta Arith. 29 (1976), pp. 181–189.

[14] — *Fourier coefficients of Eisenstein series of one complex variable for the special linear group*, Trans. Amer. Math. Soc. 205 (1975), pp. 97–114.

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