fürlische Zahl \( m \), die durch eine nicht zum Hauptgeschlecht gehörende Form darstellbar ist, einen Primfaktor \( p \) mit \( (D/p) = 1 \). Nach [1] ist \( m \) und damit auch \( p \) höchstens gleich \( o(s)|D|^{3/24+4} \).

5. Beispiele zu Lemma 5.
\(-D = 15 (p = 2), 35 (5), 84 (5), 91 (5), 137 (7), 195 (7), 403 (11), 420 (11), 485 (11), 483 (11), 533 (13), 655 (13), 595 (13), 627 (13), 660 (13), 1012 (17), 1092 (17), 1155 (17), 1380 (19), 1428 (19), 1435 (19), 1995 (23), 3068 (29, 31), 3315 (29, 31), 5460 (37).\)

**Literaturverzeichnis**


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The Fourier expansion of Epstein's zeta function for totally real algebraic number fields and some consequences for Dedekind's zeta function

by

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1. Introduction. Let \( K \) be a totally real algebraic number field of odd degree \( m > 1 \) (over \( \mathbb{Q} \)). And suppose that the class number of \( K \) is one. We shall need the following concepts from algebraic number theory. For definitions, etc., one should refer to Lang [7]. Let \( O_K \) denote the ring of integers of \( K \) and \( U_K \) the unit group of \( O_K \). Suppose the \( m \) embeddings of \( K \) into \( \mathbb{R} \) over \( \mathbb{Q} \) are denoted \( x \mapsto x^{(j)} \) for \( j = 1, 2, \ldots, m \).

Let \( N_x = \prod_{i=1}^m x^{(j)} \), for \( x \in K \). Let \( d_K \) be the absolute value of the discriminant of \( K \) and \( \delta_K \) the different of \( K \). The Dedekind zeta function of \( K \) is

\[
\zeta_K(s) = \sum_{\omega \in O_K^{*} / U_K} N \omega^{-s} \quad \text{for} \quad \text{Res} \, s > 1.
\]

Here the sum is over non-zero integers of \( K \) \((O_K^{*} = O_K \setminus \{0\})\) which are not equivalent under multiplication by units. We use here the assumption that the class number of \( K \) is one so that all ideals are principal in \( O_K \).

We shall prove that

\[
n_K(2s)^m + d_K^{-2s} \left( \pi^{s-1} \Gamma(1-s) \Gamma(s)^{-1} \right)^m \zeta_K(2-2s) (1-s)^m
\]

\[
= -d_K^{-1/2} \left( \pi^{s} \Gamma(s)^{-1} \right)^m \sum_{\omega \in O_K^{*} / U_K} \frac{N \omega}{N b} \left( k_{12}^{-1} \prod_{i=1}^m \left( K_{12} \setminus \{ \pi^i |a^{(i)}b^{(i)}\} \right) + 4\pi |a^{(i)}b^{(i)}| K_{12}^{-1}(2\pi |a^{(i)}b^{(i)}|) \right).
\]

\( K_*(s) \) is the modified Bessel function of the second kind defined by (2.1); \( K_*(s) = \frac{dK_*(s)}{ds} \). Here \( a \) runs over non-zero integers of \( K \) non-equivalent.

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under multiplication by units and \( b \) runs over non-zero elements of the inverse ring. It follows from (1.2) by setting \( s = 1 \) that

\[
\zeta_\mathcal{K}(2) = \sum_{a \in \mathcal{O}_K} \left\langle a \right\rangle \sum_{m=1}^{\infty} \left( \sum_{j=1}^{m} a(j)b(j) \right),
\]

for example. This generalizes the following result of Schlömilch and Ramanujan for the Riemann zeta function: \( \zeta(2) = \zeta(\mathcal{O}(2)) \):

\[
\zeta(2) = \frac{n^2}{3} + 4n^2 \sum \sigma_1(n) \exp(-2\pi in),
\]

where \( \sigma_1(n) = \sum_{d \mid n} d \). One could also use (1.2) to derive formulas similar to but more complicated than Ramanujan's other results of this type. A good reference for such formulas is Berndt [2], which has a long list of papers on the subject. Another reference is Grosswald [4]. The restriction that \( \mathcal{K} \) be totally real and the restriction that \( m = [K: \mathcal{O}] \) be odd are unnecessary except to simplify the results. It is a little more complicated to drop the class number one assumption. It is possible that (1.3) could have interesting consequences for totally real fields of odd degree and class number one.

Another possible use of (1.2) is the evaluation of \( \zeta_\mathcal{K} \) in the interval \((0, 1)\). One might have thought that the natural formula to use in order to study \( \zeta_\mathcal{K} \) in \((0, 1)\) would be Hecke's integral formula ([5], page 205) — a result which expresses the Dedekind zeta function as a certain integral over a cube, the integrand being an Epstein zeta function whose quadratic form argument depends on the units and the integral basis of the field as well as the variable of integration. If one substitutes the Fourier expansion ([12], formula (1.3), page 477) of the Epstein zeta function into the integral, it is very difficult to see what is happening, because the series expansion converges very slowly, even for small discriminants. Thus (1.2) may be of some use in such questions — the lack of integration being its main virtue.

The proof of (1.2) is to be found in § 3 and results from the Fourier expansion of the Epstein zeta function for the totally real algebraic number field \( \mathcal{K} \). If \( \mathcal{K} = \mathcal{O} \) the Epstein zeta function has two arguments, the first being a positive definite symmetric \( n \times n \) matrix \( P \) and the second a complex variable \( s \) with \( \text{Re} s > n/2 \). The Epstein zeta function over \( \mathcal{O} \) is defined by:

\[
\zeta_\mathcal{O}(P, s) = \sum_{a \in \mathcal{O}^n} P(a)^{-s}
\]

where \( a = (a_1, \ldots, a_n) \) runs over non-zero column vectors with integer entries and \( P[a] = a^TPa \). The space of positive definite symmetric \( n \times n \) matrices \( P \) is the symmetric space (see [6] for definition) \( \text{GL}(n, R)/O(n) \) with \( \text{GL}(n, R) \) the general linear group of \( n \times n \) invertible real matrices and \( O(n) \) the orthogonal group. If \( G \in \text{GL}(n, R) \), let \( P = G^T \bar{G} \) to obtain the above identification. Then \( Z_n(P, s) \) is an "automorphic form" — specifically an Eisenstein series — in the sense of Borel ([3], page 200 and example 2, page 209). This means that \( Z_n \) satisfies certain differential equations in \( P \) and that \( Z_n \) is invariant under the action \( P \mapsto P[A] = A^TPA \), for any \( n \times n \) matrix \( A \) with integer entries and determinant \( \pm 1 \).

Motivated by the simple algebraic fact that \( \mathcal{K} \mathcal{O} \mathcal{R} \cong \mathcal{R} \mathcal{O} \cdots \mathcal{R} \) (\( m = [\mathcal{K}: \mathcal{O}] \) copies) if \( \mathcal{K} \) is totally real, one sees that the Epstein zeta function over \( \mathcal{K} \) ought to be a function on \( \text{GL}(n, \mathcal{R})/O(n) \times \cdots \times \text{GL}(n, \mathcal{R})/O(n) \), \( m \) copies.

This is analogous to the case of Hilbert modular forms (Siegel [10], pages 273–274). So let \( P = (P^{(1)}, \ldots, P^{(m)}) \) be a vector of positive definite symmetric \( n \times n \) matrices and let \( s \) be a complex variable with \( \text{Re} s > n/2 \). Then define the Epstein zeta function over \( \mathcal{K} \) by:

\[
Z_n^\mathcal{K}(P, s) = \sum_{a \in \mathcal{O}^n_{\mathcal{K}}} \prod_{j=1}^{m} (P^{(j)}(a^{(j)})^{s}).
\]

The sum is over a complete set of representatives for \( n \)-tuples from \( O_{\mathcal{K}} \) (not all of whose entries are zero) with respect to the equivalence relation from scalar multiplication by units. Tamagawa [11] indicates that the usual theta function methods yield the analytic continuation and functional equation of (1.5). We have again an "automorphic form", whose invariance property is:

\[
Z_n^\mathcal{K}(P, s) = Z_n(P[A], s),
\]

for \( A \) in the discrete group of matrices with entries in \( O_{\mathcal{K}} \) and determinant in \( U_{\mathcal{K}} \). Here:

\[
P[A] = (P^{(1)}[A^{(1)}], \ldots, P^{(m)}[A^{(m)}]),
\]

with \( A^{(j)} \) denoting the matrix obtained by replacing every element of the matrix \( A \) by its \( j \)th conjugate.

The main result of § 2 is Theorem 1, which gives the \( n-1 \) Fourier expansions of \( Z_n^\mathcal{K}(P, s) \), generalizing the result in the case \( \mathcal{K} = \mathcal{O} \) obtained in [12], Theorem 1, page 480. The result also generalizes Asai [11], formula 13, pages 203–204. The latter is seen to be the case \( n = 2 \) of our result. However Asai does not restrict himself to totally real fields.

There are many open questions. One would like to generalize Asai's.
results on $\log|\eta(z)|$ for algebraic number fields ([1], Theorem 4, pages 207-208, and Theorem 5, page 310). Asai defines a function $h_K(z)$ which has three properties of $\log|\eta(z)|$, i.e., $h_K$ appears in a Kronecker limit formula for the Epstein zeta function ($n = 2$) over $K$. Also $h_K(z)$ is harmonic in the upper half plane. Finally the Mellin transform of $h_K(z)$ is

$$\zeta_K(s, \lambda) \zeta_K(s + 1, \lambda),$$

where $\zeta_K(s, \lambda)$ is a Hecke zeta function with Grössencharacter. The generalization to higher dimensions ($n \geq 3$) appears difficult. For example there are more invariant differential operators, so it is more difficult to be harmonic (see [9], page 432 or [9], page 64).

Another open problem is the extension of these results to other Eisenstein series for the general linear group. Such series are considered by Langlands in [8], where he obtains the functional equation and analytic continuation. One should at least be able to generalize the Fourier expansion obtained in [14] to Eisenstein series of one complex variable over algebraic number fields.

A final question concerns the shape of the fundamental domain in $GL(n, \mathbb{R})/O(n) \times \ldots \times GL(n, \mathbb{R})/O(n)$ for the discontinous group $G$ of invertible matrices with integer entries from the number field and unit determinant. One ought to consider the question of the number of cusps and the Fourier expansion at each cusp (cf. [10], Chapter III, § 2).

2. The Fourier expansions of Epstein's zeta function over a totally real algebraic number field. We need a few definitions before stating the result on the Fourier expansions of $Z_n(P, s)$. The modified Bessel function of the second kind is defined by

$$K_n(s) = \frac{1}{2} \int_0^{\infty} \exp \left\{ -x - \frac{s}{2} (u + u^{-1}) \right\} u^{-1} du,$$

for $|x| < \pi / 2$. As usual $\Gamma(s)$ is the gamma function.

For any integer $r$ with $1 \leq r \leq n - 1$, one obtains a decomposition of $P^{(0)}$ in block form as in [12], formula (2.1), page 479:

$$P^{(0)} = \begin{pmatrix} I^{(0)} & 0 \\ 0 & P^{(0)} \end{pmatrix} = \begin{pmatrix} I^{(0)} & 0 \\ 0 & P^{(0)} \end{pmatrix},$$

where $I^{(0)}$ is the $r \times r$ block in the lower right-hand corner of $I^{(0)}$, $I$ is the identity matrix, $0$ is a matrix of zeros, and where $j = 1, 2, \ldots, m = [K : Q]$. Note that here $P^{(0)}$ denotes the $j$th matrix component $P = P^{(0)}, \ldots, P^{(m)}$ rather than a $j \times j$ matrix, as it did in [12]. If $P = (P^{(0)}, \ldots, P^{(m)})$, define

$$|P| = \prod_{j=1}^{m} \det(P^{(0)}).$$

And for any $O_K$-module $M$ in $K^m$ define $M^* = M - \{0\}$.

It is now possible to state the Fourier expansion of $Z_n(P, s)$ as a periodic function of $Q = (Q^{(0)}, \ldots, Q^{(m)})$. The result generalizes Theorem 1 of [12], page 480.

**Theorem.** For any totally real algebraic number field $K$,

$$Z_n(P, s) = Z_n(P_0, s) + d_K^m \sum_{\gamma \in \mathcal{O}_K^*} (\frac{1}{n}) |P_0|^{1/2} Z_n(P_0 \gamma, s - \frac{1}{2}) |P_0|^{-1/2} +$$

$$+ d_K^m \sum_{\gamma \in \mathcal{O}_K^*} \exp(2\pi i \gamma P_0 Q^{(0)}) \int_{u \in \mathcal{O}_K^*} \prod_{j=1}^{m} \left( T^{(0)}[Q^{(0)}]^{(0)} \right)^{(n-2)\alpha_j} \times$$

$$\times \exp \left[ 2\pi i \gamma^{(0)} Q^{(0)} a_0^{(0)} \right] K \mathfrak{n}_{m-2}(2\pi i T^{(0)}[Q^{(0)}]^{(0)} - \{b_0^{(0)}\}) \right].$$

where the sum is over non-equivalent non-zero vectors $a$ in $O_K^{-1}$ with equivalence defined by componentwise multiplication by units in $\mathcal{O}_K$, and over non-zero vectors $b$ with component $\gamma$ in the inverse different. Finally $H_n(P, s)$ is an entire function of $s$.

**Proof.** The proof is an exact analogue of that of Theorem 1 of [12], page 480. We split the summation over the variable $g$ in (1.5) into two parts $g = \left( \begin{array}{c} a \\ b \end{array} \right)$, with $a$ in $O_K^{-1}$ and $b$ in $O_K$. Then from (2.2) it follows that

$$\prod_{j=1}^{m} \left( T^{(0)}[Q^{(0)}]^{(0)} \right)^{(n-2)\alpha_j} = \prod_{j=1}^{m} \left( T^{(0)}[Q^{(0)}]^{(0)} + P^{(0)}[Q^{(0)} a_0^{(0)} + b_0^{(0)}] \right).$$

Thus

$$Z_n(P, s) = \sum_{a \in \mathcal{O}_K^* \mathcal{O}_K^{-1}} + \sum_{a \in \mathcal{O}_K^* \mathcal{O}_K^{-1}} \cdot$$

So

$$Z_n(P, s) = Z_n(P_0, s) +$$

$$+ \sum_{a \in \mathcal{O}_K^* \mathcal{O}_K^{-1}} \prod_{j=1}^{m} \left( T^{(0)}[Q^{(0)}]^{(0)} + P^{(0)}[Q^{(0)} a_0^{(0)} + b_0^{(0)}] \right).$$

Now let $w_1, \ldots, w_m$ be a $Z$-basis of $O_K$. And let $w_1^*, \ldots, w_m^*$ be a dual basis with respect to the trace. Then

$$Z_n(P, s) = Z_n(P_0, s) +$$

$$+ \sum_{a \in \mathcal{O}_K^* \mathcal{O}_K^{-1}} \prod_{j=1}^{m} \left( T^{(0)}[Q^{(0)}]^{(0)} + P^{(0)}[Q^{(0)} a_0^{(0)} + b_0^{(0)}] \right).$$
Next apply Poisson's summation formula to the sum over $b$ in $O_{K}$

\[(\sum_{y=1}^{\infty} \sum_{x=1}^{\infty} \prod_{j=1}^{m} (T[x_{j}]_{[d]} + P_{x_{j}}[d] + \sum_{y=1}^{\infty} a_{y}[x_{j}])\] \[\times \exp \left\{ 2\pi i \sum_{j=1}^{m} \frac{t_{j}}{d} \right\} \frac{dx_{1} \ldots dx_{m}}{d}.

The natural course of action is to separate variables in the integral. So we make the change of variables

\[v_{j} = \sum_{k=1}^{m} x_{k} w_{k}^{j}, \quad j = 1, \ldots, m,
\]

where $v_{j}$ and $x_{j}$ are in $\mathbb{R}^{r}$ for $j = 1, \ldots, m$. If we think of $v = (v_{1}, \ldots, v_{m}) = (v\ldots v')$ and $a = (a_{1}, \ldots, a_{m}) = (a\ldots a')$ as $m \times r$ matrices, then $v = \left( M v \ldots M v' \right)$, where $M = (w_{k}^{j})_{j,k=1,\ldots,m}$. We already have the inverse for this transformation thanks to (2.5), namely

\[M^{-1} = (w_{k}^{j})_{j,k=1,\ldots,m}.
\]

It follows that, as in [12], pages 480–481:

\[f(a, c) = 2^{-2m} |P|^{-1/2} \prod_{j=1}^{m} (T[x_{j}]_{[d]} + P_{x_{j}}[d] + \sum_{y=1}^{\infty} a_{y}[x_{j}]) \times \exp \left\{ 2\pi i \frac{t_{j}}{d} \right\} \frac{dx_{1} \ldots dx_{m}}{d},
\]

and if $c$ is a non-zero $m \times r$ matrix of integers, then:

\[f(a, c) = 2^{-2m} |P|^{-1/2} \prod_{j=1}^{m} (T[x_{j}]_{[d]} + P_{x_{j}}[d] + \sum_{y=1}^{\infty} a_{y}[x_{j}]) \times \exp \left\{ 2\pi i \frac{t_{j}}{d} \right\} \frac{dx_{1} \ldots dx_{m}}{d},
\]

with

\[t_{j}([b^{(j)} \ldots b^{(m)}]) = (M^{*} b \ldots M^{*} b')\]

if $M = (w_{k}^{j})_{j,k=1,\ldots,m}$ and $c = (c\ldots c') \neq 0$ in $\mathbb{Z}^{mr}$. Thus $b^{(j)}$ is the $j$th conjugate of an arbitrary element $b$ of $(\mathbb{Z}^{r})^{*}$. This completes the proof of the theorem except for the statement that $H_{s}(P, s)$ is entire in $s$. For this one needs the exponential decay of the modified Bessel function of the second kind.

We leave to the reader the task of generalizing the other results of [12] to totally real algebraic number fields. It will be convenient however to state the results of Theorem 1 in the special case $m = 2$ separately. This is Asai's result ([1], formula 13, pages 203–204) essentially. One has to translate from binary quadratic forms $P > 0$ to complex numbers $x$ in the upper half plane in order to go from our result to Asai's. This is achieved by looking at solutions $z$ in the upper half plane of $P \left( \frac{z}{1} \right) = 0$.

We make the simplifying assumption that $K$ is a totally real algebraic number field with class number one. Then the formula (1.1) is a correct expression for the Dedekind zeta function $\zeta_{K}$ of $K$. Thus, setting $n = 2$ and $r = 1$ in Theorem 1 yields:

\[\zeta_{K}(s) = \left[ P \left( \frac{z}{1} \right) \right]^{-1} \prod_{j=1}^{m} \frac{\prod_{j=1}^{m} \left( T[x_{j}]_{[d]} + P_{x_{j}}[d] + \sum_{y=1}^{\infty} a_{y}[x_{j}] \right) \times \exp \left\{ 2\pi i \frac{t_{j}}{d} \right\} \frac{dx_{1} \ldots dx_{m}}{d}}{d},
\]

where

\[H_{s}(P, s) = \left[ P \left( \frac{z}{1} \right) \right]^{-1} \prod_{j=1}^{m} \frac{\prod_{j=1}^{m} \left( T[x_{j}]_{[d]} + P_{x_{j}}[d] + \sum_{y=1}^{\infty} a_{y}[x_{j}] \right) \times \exp \left\{ 2\pi i \frac{t_{j}}{d} \right\} \frac{dx_{1} \ldots dx_{m}}{d}}{d},
\]

To study the situation when the class number is bigger than one, Epstein zeta functions associated with ideals as in Tamagawa [11] will no doubt be necessary. We leave this question open.

3. Some formulas relating $\zeta_{K}(2s)$ and $\zeta_{K}(2s-1)$. We first prove generalizations of Proposition 2 of [33] to totally real algebraic number fields of class number one. The result appears more complicated than (1.2) because we do not assume $m = [K : Q]$ is odd.

**Theorem 2.** Let $\zeta_{K}$ be the Dedekind zeta function of a totally real algebraic number field $K$ with class number one. Then

\[\sum_{n \in D} \frac{1}{n^{s}} \zeta_{K}(2s) + \left( 1 - (2s - 1)^{m} \right) \frac{1}{d} \prod_{j=1}^{m} \frac{\prod_{j=1}^{m} \left( T[x_{j}]_{[d]} + P_{x_{j}}[d] + \sum_{y=1}^{\infty} a_{y}[x_{j}] \right) \times \exp \left\{ 2\pi i \frac{t_{j}}{d} \right\} \frac{dx_{1} \ldots dx_{m}}{d}}{d},
\]

**Here** $\zeta_{K}(s) = \frac{d}{ds} K_{s}(\mathfrak{a})$. 

functions of totally real fields with class number one. The result appears more complicated than (1.3) because we are not assuming $m = \lceil K : \mathcal{O} \rceil$ is odd.

**Corollary 2.** If $\zeta_K$ is the Dedekind zeta function for the totally real algebraic number field $K$ of class number one, then

$$\zeta_K(2) = \frac{m}{4} (2\pi)^m \frac{R_K}{d_K} + d_K^{1/2} \sum_{\omega \in \mathcal{O}_K^* \cap \mathcal{O}_K^{\text{rational}}} \frac{N\omega}{|\omega|^{2/3}} \mid \prod_{j=1}^m \left[ (2\pi |\omega(j)|^{2})^{1/2} + \prod_{j=1}^m \left[ (2\pi |\omega(j)|^{2})^{1/2} - 1 \right] \right],$$

where $R_K$ is the regulator of $K$.

**Proof.** Substitute $s = 1$ into the preceding Corollary, using the following formula from Lang [7], Theorem 2, page 259:

$$\lim_{s \to 1} \left( s \Gamma \left( \frac{m}{2} \right) \right) \zeta_K(s) = -2^{m-2} R_K,$$

since $K$ is totally real and thus has only two roots of unity.

One obtains

$$\zeta_K(2) = \zeta_K(2) - \frac{m}{4} (2\pi)^m \frac{R_K}{d_K}.$$

And the preceding Corollary implies that

$$\zeta_K(2) - \frac{m}{4} (2\pi)^m \frac{R_K}{d_K} = d_K^{1/2} \sum_{\omega \in \mathcal{O}_K^* \cap \mathcal{O}_K^{\text{rational}}} \frac{N\omega}{|\omega|^{2/3}} \mid \prod_{j=1}^m \left[ (2\pi |\omega(j)|^{2})^{1/2} + \prod_{j=1}^m \left[ (2\pi |\omega(j)|^{2})^{1/2} - 1 \right] \right].$$

From $K_{1/2}(s) = \sqrt{\frac{\pi}{2s}} e^{-s}$, it follows that

$$K_{1/2}(s) \sim - \sqrt{\frac{\pi}{2s}} e^{-s} \left( \frac{1}{1 + \frac{1}{2s}} \right).$$

Thus

$$K_{1/2}(s) - 2s K_{1/2}(s) = \sqrt{2\pi e^{-s}} (s^{-1} + 1),$$

$$3K_{1/2}(s) + 2s K_{1/2}(s) = \sqrt{2\pi e^{-s}} (s^{-1} - 1).$$

Substitution of these results into the term in braces completes the proof of the corollary.
Next we prove another generalization of the Schöflich-Ramanujan formula. This result is equivalent to (1.3).

**Corollary 3.** If $K$ is a totally real algebraic number field of class number one and odd degree $m = [K : Q] > 1$, then

$$
\zeta_K(2) = d_K^{-1/2}(2\pi)^m \sum_{a \in \mathcal{O}_K^*} \left| N_A \right| \exp \left\{ -\pi \sum_{j=1}^m |a_j|d_K \right\}.
$$

**Proof.** Multiply both sides of the formula of Corollary 1 to Theorem 2 by $\Gamma(s)^m$ and let $s$ approach zero to obtain:

$$
2\pi K(2\pi)^{-m} \zeta_K(2) = -2d_K^{1/2}(2\pi)^m \sum_{a \in \mathcal{O}_K^*} \left| N_A \right| \Gamma(s)^{-m} \iota_K\Gamma(s)^{1-m} \times
$$

$$
\prod_{j=1}^m \left[ \frac{1}{2} K_{1j}(2\pi |a_j|d_K) + \pi |a_j|d_K \right]_2 K_{1j}(2\pi |a_j|d_K)).
$$

Since $K_{1j}(s) + 2\pi K_{1j}(s) = -V\pi \pi e^{-i\theta}$, the proof of Corollary 3 is complete.

Next we prove formula (1.2).

**Theorem 3.** If $\zeta_K$ is the Dedekind zeta function of a totally real algebraic number field $K$ of odd degree $m = [K : Q]$ and having class number one, then:

$$
\zeta_K(2s) = d_K^{1/2} \Gamma(s) \Gamma(s-1/2)^{1-m} \zeta_K(2s-1) (1-s)^m = -d_K^{1/2} \Gamma(s)^{-1} \sum_{a \in \mathcal{O}_K^*} \left| N_A \right| \Gamma(s-1/2)^{-m} \times
$$

$$
\prod_{j=1}^m \left[ K_{1j}(2\pi |a_j|d_K) + \pi |a_j|d_K \right]_2 K_{1j-1}(2\pi |a_j|d_K)).
$$

**Proof.** Proceed as in the proof of Theorem 2, except multiply (3.1) by $|a_j|^{1/2}$ before applying $\frac{\partial}{\partial x_1} \ldots \frac{\partial}{\partial x_m} |x_1^{m-1} = \ldots = x_m^{1}$. Thus (3.1) is replaced by

$$
\zeta_K(2s) = (2\pi |a_j|^{1/2} + |a_j|^{1/2}) +
$$

$$
+ d_K^{1/2} \Gamma(s - 1/2)^{-m} \zeta_K(2s-1)(|a_j|^{1/2} - |a_j|^{1/2}) = d_K^{1/2} \Gamma(s)^{-1} \sum_{a \in \mathcal{O}_K^*} \left| N_A \right| \Gamma(s-1/2)^{-m} \times
$$

$$
\prod_{j=1}^m \left[ K_{1j}(2\pi |a_j|d_K) + \pi |a_j|d_K \right]_2 K_{1j-1}(2\pi |a_j|d_K)).
$$

with $|a_j| = \prod_{j=1}^m |a_j|$.  

**Corollary 1.** If $\zeta_K$ is the Dedekind zeta function of the totally real algebraic number field $K$ of odd degree $m = [K : Q]$ and class number one, then

$$
\zeta_K(2s) = d_K^{1/2} \Gamma(s) \Gamma(s-1/2)^{1-m} \zeta_K(2s-1) (1-s)^m = -d_K^{1/2} \Gamma(s)^{-1} \sum_{a \in \mathcal{O}_K^*} \left| N_A \right| \Gamma(s-1/2)^{-m} \times
$$

$$
\prod_{j=1}^m \left[ K_{1j}(2\pi |a_j|d_K) + \pi |a_j|d_K \right]_2 K_{1j-1}(2\pi |a_j|d_K)).
$$

**References**


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