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Algebraic function fields with equal class number

by

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1. Introduction. Let F/K be a field of algebraic functions of one variable having a finite field K with q elements as its exact field of constants. The group C_{0F} of divisor classes of degree zero of such a field is finite. Its order h_F is called the class number of the field. Let E/L be a finite separable extension of F/K. In this paper, we discuss, and almost completely answer, the following question: When is $h_E = h_F$? A field of genus zero has class number one. The special case $g_F = 0$, $g_E \neq 0$ has been completely solved in [6], [7]. There are, up to isomorphism, seven possibilities for such E. We shall, therefore, assume $g_F \neq 0$. The extension E/L can be obtained from F/K in two steps, a purely constant extension followed by a purely geometric extension, i.e. no new constants are introduced. For fields of genus larger than one, we shall treat the two cases separately. Our main results are the following:

THEOREM 1. *Let F/K be a function field of genus one. Let E/L be a finite separable extension of F/K. If $L = K$ and the extension is unramified, then $h_E = h_F$. If $E = FL$, then $h_E > h_F$ if any of the conditions $q > 4$; $q = 3, 4$, $[L:K] > 2$; $q = 2$, $[L:K] > 3$ is satisfied.*

THEOREM 2. *Let E/K be a purely geometric extension of F/K with $g_F > 1$. Then $h_E > h_F$ in each of the following cases:* (a) $q > 5$, $g_F > 1$; (b) $q = 4$ *or* 5, $g_F \geqslant 3$, $g_E \geqslant 2g_F+1$; (c) $q = 3$, $g_F \geqslant 3$, $g_E \geqslant 3g_F$ *or* $g_F = 2$, *and* $g_E \geqslant 7$; (d) $q = 2$, $g_F \geqslant 3$, $g_E \geqslant 5g_F$ *or* $g_F = 2$ *and* $g_E \geqslant 11$.

THEOREM 3. *Let F/K be a function field with $g_F > 1$ and $E = FL$ be a constant extension. Then $h_E > h_F$ if any one of the following is satisfied:* (a) $q \geqslant 4$; (b) $q = 3$, $[L:K] > 2$; (c) $q = 3$, $[L:K] = 2$, $g_F > 20$; (d) $q = 2$, $[L:K] > 3$; (e) $q = 2$, $[L:K] = 3$, $g_F > 9$; (f) $q = 2$ *or* 3, $[L:K] = 2$, *F hyperelliptic and not isomorphic to* $K(x, y)$, $y^2+y = x^5+x^3$ *or* $(x^2+x)(x^3+x+1)^{-1}$.

Proofs of these theorems are given in § 2. In § 3, we make some remarks and give examples. Among these examples is one of a field of genus 3

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defined over GF(2) for which the class number remains unchanged in a constant extension of degree 2. Also, we present examples of fields of genus 2 defined over GF(2) and GF(3) which are contained in geometric extensions of genus 5 and 3, respectively, and there is no change in the class number.

2. Proofs of theorems. We begin with the

Proof of Theorem 1. Since E/K is a geometric unramified separable extension of the function field F/K of genus one, we have the Riemann–Hurwitz genus formula ([1]; p. 106)

$$2g_E-2 = [E:F][2g_F-2] = 0.$$

Thus, $g_E = g_F = 1$. By F. K. Schmidt's theorem [10], a function field over a finite field of constants contains a divisor of degree one. By the Riemann–Roch Theorem, each class of degree one of a field of genus one has dimension one and, therefore, contains precisely one prime of degree one. Thus, the class number is the number of primes of degree one.

To prove $h_E = h_F$, consider first the case when E/F is normal. Then, $\mathrm{gal}(E/F)$ is isomorphic ([3], p. 65) to a subgroup of the group of translations of $E\tilde{K}/\tilde{K}$, $\tilde{K}$ denoting the algebraic closure of K. Thus, E/F is abelian. To prove $h_E = h_F$, we can assume $[E:F] = l$ is a prime. In this case, as Moriya [8] has shown, the equality of class numbers is an immediate consequence of class field theory. (See [9], [11] for the standard results of class field theory.) Namely, the primes of degree one of E are obtained from the primes of degree one of F which decompose. The primes which decompose are precisely the primes which are norms. Also, the norm index is l. Thus,

$$h_E = l\cdot\frac{h_F}{l} = h_F.$$

Turning, now, to the case when E/F is non-normal, we can assume that there is no field strictly between E and F. Let $T/\bar{K}$ be the normal closure of E/F. The extension T/F is also unramified and $g_T = g_E = g_F = g_{\bar{E}} = g_{\bar{F}} = 1$, where $\bar{E} = E\bar{K}$, $\bar{F} = F\bar{K}$. The extension $T/\bar{F}$ is normal, unramified, geometric and, hence, abelian. Thus, $\bar{E}/\bar{F}$ is also abelian. Since there is no field between E and F, it follows $[E:F] = [\bar{E}:\bar{F}] = l$, a prime. Let $[\bar{\bar{K}}:\bar{K}] = l^2$, $\bar{\bar{E}} = \bar{\bar{K}}E$, $\bar{\bar{F}} = \bar{\bar{K}}F$, $G = \mathrm{gal}(\bar{\bar{K}}/K)$, $H = \mathrm{gal}(\bar{K}/K)$. Let N denote the kernel of the canonical conorm map $C_{0F}\to C_{0E}$. Also, let $\bar{N}$, $\bar{\bar{N}}$ have similar meaning. We know, $h_{E\bar{K}} = h_{F\bar{K}}$. If $\bar{N}$ is trivial, con: $C_{0\bar{F}}\to C_{0\bar{E}}$ is an isomorphism. It follows $C_{0F} = C_{0\bar{F}}^{H}$ (the invariant subgroup under H) $\cong C_{0\bar{E}}^{H} = C_{0E}$. Thus, $h_F = h_E$. If $\bar{N}$ is nontrivial. its order is l. Then, $[\bar{\bar{N}}:1] = l$. Let $[\bar{K}_1:\bar{K}] = l$. The class number of $\bar{F}\bar{K}_1$, is divisible by, at least, l^2([8]; Satz 1). This implies that the l-rank

of $C_{0,\bar{F}\bar{K}_1}$ is two and that the elements of $C_{0\bar{E}}$ are lth powers in $C_{0\bar{\bar{E}}}$ ([8]; Hilfssatz 6 and 7). Also $C_{0\bar{\bar{F}}}/\bar{\bar{N}}$ being of index l in $C_{0\bar{\bar{E}}}$, it follows that $C_{0\bar{E}}$, and hence C_{0E}, is contained in $C_{0\bar{\bar{F}}}/\bar{\bar{N}}$. Consider, now, the exact sequence of G-modules

$$1\to\bar{\bar{N}}\to C_{0\bar{\bar{F}}}\to\frac{C_{0\bar{\bar{F}}}}{\bar{\bar{N}}}\to 1. \tag{1}$$

By Herbrand's Lemma

$$[H^1(G, C_{0\bar{\bar{F}}}):1] = [H^0(G, C_{0\bar{\bar{F}}}):1] = 1,$$

because, the norm map from $C_{0\bar{\bar{F}}}$ to C_{0F} is surjective. Thus, in cohomology, (1) gives the exact sequence.

$$1\to\bar{\bar{N}}^G\to C_{0\bar{\bar{F}}}^G\to\left(\frac{C_{0\bar{\bar{F}}}}{\bar{\bar{N}}}\right)^G\to H^1(G,\bar{\bar{N}})\to H^1(G, C_{0\bar{\bar{F}}}) = 1. \tag{2}$$

We claim $[H^1(G,\bar{\bar{N}}):1] = [N:1]$. If $N = 1$, consider $[H^0(G,\bar{\bar{N}}):1]$ which, by Herbrand's Lemma, equals $[H^1(G,\bar{\bar{N}}):1]$. The group $H^0(G,\bar{\bar{N}}) = \dfrac{\text{invariant elements}}{\text{norms}} = \dfrac{\bar{\bar{N}}^G}{\text{norms}} = 1$ for $\bar{\bar{N}}^G = N = 1$. If $[N:1] = l$, then G operates trivially on $\bar{\bar{N}} = N$. Thus,

$$H^1(G,\bar{\bar{N}})\cong\mathrm{Hom}(G,\bar{\bar{N}}) \quad\text{and}\quad [H^1(G,\bar{\bar{N}}):1] = l.$$

Considering that $C_{0\bar{\bar{F}}}^G = C_{0F}$ and $\left(\dfrac{C_{0\bar{\bar{F}}}}{\bar{\bar{N}}}\right)^G = C_{0E}$, (2) gives, in each case, $h_E = h_F$. This completes proof of the first part of the theorem.

For the second part of the statement, we recall that by the Riemann Hypothesis [4], h_F = the number of primes of degree one of $F \leqslant (\sqrt{q}+1)^2$, $h_E \geqslant (q^{[L:K]/2}-1)^2$. To complete the proof, we observe that $q^{[L:K]/2} > \sqrt{q}+2$ for the values of q and $[L:K]$ in the statement of the theorem.

Proof of Theorem 2. Let $g_0 = g_F$, $g = g_E$. We consider a constant extension $\bar{E}/\bar{K}$ of E/K of degree $2g-1$. Since K is perfect, there is no change in the genus. Using the Riemann Hypothesis to obtain a lower estimate for the number of primes of degree one of $\bar{E}$ and considering the decomposition of primes of E in $\bar{E}$, we show, as in [7], that E has, at least,

$$\frac{q^{2g-1}+1-2g\cdot q^{(2g-1)/2}}{2g-1} \tag{3}$$

integral divisors of degree $2g-1$. Again, by the Riemann Hypothesis,

$$h_F \leqslant (\sqrt{q}+1)^{2g_0}. \tag{4}$$

It is an easy consequence ([2], p. 64) of the Riemann–Roch Theorem that a class of degree $2g-1$ has exactly $(q^g-1)(q-1)^{-1}$ integral divisors. Thus, we can conclude from (3) and (4), that $h_E > h_F$ whenever

$$T(g,q) = (q-1)[q^{2g-1}+1-2g\cdot q^{(2g-1)/2}]-(2g-1)(q^g-1)(q^{1/2}+1)^{2g_0} \tag{5}$$

is positive.

For $q \geqslant 5$, we can assume that $[E:F]>2$ and, hence, $g \geqslant 3g_0-2$. Otherwise $h_E \geqslant h_F(\sqrt{q}-1)^{g_0-1} > h_F$, since, in this case, the zeta function of F divides the zeta function of E. Then, $T(g,q)$ is easily seen to be positive for $q \geqslant 7$.

The following facts are easily verified:

(6) $T(5,5)$, $T(9,3)$, $T(7,3)$, $T(5,4)$, $T(15,2)$, $T(11,2)$ are positive;

$$\frac{\partial T}{\partial g} = 2(q^{1/2}+1)^{2g_0}+2q^{(2g-1)/2}\cdot S(q,g), \tag{7}$$

where

$$S(q,g) = (q-1)[q^{(2g-1)/2}\ln q-1-g\ln q]- \\ -(q^{1/2}+1)^{2g_0}q^{1/2}[1+\tfrac{1}{2}(2g-1)\ln q]; \tag{8}$$

$$\begin{aligned}\frac{\partial S}{\partial g} &= (q-1)[(\ln q)^2 q^{(2g-1)/2}-\ln q-(q^{1/2}+1)^{2g_0}q^{1/2}\ln q]\\ &\geqslant (q-1)\ln q\,[q^{(2g-1)/2}\ln q-1-q^{2g_0}] \quad \text{if} \quad q\geqslant 3,\\ \frac{\partial S}{\partial g} &= \ln 2\,[2^{(2g-1)/2}\ln 2-1-2^{1/2}(2^{1/2}+1)^{2g_0}] \quad \text{if} \quad q=2.\end{aligned} \tag{9}$$

From (7), (8), (9), it is seen that $T(g,q)$ is an increasing function for the values of g_0, g and q in the statement of the theorem, g_0, g varying under the restrictions imposed by the genus formula. It follows from (6) and (5) that the proof of Theorem 2 is complete.

Proof of Theorem 3. We give the proof in three steps.

Step 1. It follows from the Riemann Hypothesis [4] that the polynomial numerator of the zeta function $\zeta(s)$ of a function field of genus g can be written

$$L(u) = 1+a_1u+\ldots+q^gu^{2g} = \prod_{i=1}^{g}(1-2q^{1/2}u\cos\theta_i+qu^2), \tag{10}$$

where $u=q^{-s}$. The class number $h_F = L(1)$. Writing $n=[L:K]$, it follows from (10),

$$h_F \leqslant (q^{1/2}+1)^{2g_F}, \qquad h_E \geqslant (q^{n/2}-1)^{2g_E}. \tag{11}$$

Since finite fields are perfect, genus does not change in a constant extension. Thus, $g_E = g_F$ and (11) implies $h_E > h_F$ whenever $q^{n/2} > q^{1/2}+2$. (b), (d) and also (a), except for the case $n=2$, $q=4$, follow from this inequality.

Step 2. Considering the constant extension of E/L of degree $2g-1$, we can show, as in the proof of Theorem 2, that $h_E > h_F$ whenever

$$T^*(g,q) = (q^n-1)[(q^n)^{2g-1}+1-2g(q^n)^{(2g-1)/2}]-(2g-1)(q^{ng}-1)(q^{1/2}+1)^{2g}$$

is positive. Direct verification shows that (c) and (e) and also the exceptional case of Step 1 for $g>2$ follow from this.

Step 3. We observe that a field of genus 2 is necessarily, hyperelliptic since the dimension as well as the degree of the canonical class is two, the quotient of two integral divisors in it determines an x such that $[F:K(x)]=2$.

It remains to prove (f) and (a) for $q=4$, $g=2$, $[L:K]=2$. To that end, together with F, we consider also the function field F' of the same genus defined by

$$y^2+y = 1+f(x), \qquad y^2+y = \eta+f(x), \qquad y^2 = 2f(x),$$

for $q=2, 4, 3$, respectively, where $f(x)$ defines the function field F/K in the normal form [5] and η denotes a primitive third root of unity. Then, $FL = F'L$. The Euler product representation of the zeta function and the decomposition behavior of primes shows that $L_{F'}(u) = L_F(-u)$. Thus [2], the product of the L-polynomials of F, F' gives the L-polynomial of FL. In particular, $h_{FL} = h_F h_{F'}$. For $q=4$, $g=2$, and $q=3$, $g>1$ holds $h_{F'}>1$. For $q=2$, the two exceptions [6] correspond to those listed in the statement. This completes proof of Theorem 3.

3. Remarks and examples.

(A) Example 1. Let $q=3=|K|$, $G=K(x,\sqrt{x^3+2x+2})$,

$$H = K(x,\sqrt{x^2+1}), \quad \text{and} \quad F = K\big(x,\sqrt{(x^2+1)(x^3+2x+2)}\big).$$

Then, $g_G=1$, $g_{F_1}=2$. We shall show $h_F=h_{FG}$, $g_{FG}=3$. We have

$$FG = H(\sqrt{x^3+2x+2}) = K(\sqrt{x^2+1}+x, \sqrt{x^3+2x+2}).$$

Let $Y=\sqrt{x^3+2x+2}$, $Z=\sqrt{x^2+1}+x$. Then,

$$Y^2 = x^3+2x+2 = \left(\frac{Z-Z^{-1}}{2}\right)^3+(Z-Z^{-1})+2,$$

$$Z^4Y^2 = (Z^2Y)^2 = 2Z^7+Z^5+2Z^4-Z^3-2Z.$$

Which shows that $g_{FG} = 3$. It is also easily verified that the number of primes of degree 1 of F, $N_{1,F} = 5$, $N_{2,F} = 6$, $N_{1,G} = 1$. This gives [6]

$$L_F(u) = 1+u+4u^2+3u^3+9u^4, \quad L_G(u) = 1-3u+3u^2.$$

These two polynomials are relatively prime and each divides $L_{FG}(u)$. Since $\deg L_{FG}(u) = 2g_{FG} = 6$, we have $L_{FG}(u) = L_F(u)L_G(u)$ and, hence, $h_{FG} = h_F$.

Example 2. Let $|K| = q = 2$ and the fields F, G, H be defined over $K(x)$ in Hasse's normal form [5], respectively, by the functions x^5+x, x^5+x^3+1, x^3+x+1. We shall show $g_{FG} = 5$, $h_{FG} = h_F$. Any two of the fields F, G, H gives the same composite $E = FG$. The infinite prime is the only ramified prime. It ramifies fully in E. It follows from the genus formula and the arithmetic theory [5] that $g_F = 2 = g_G$, $g_H = 1$. To calculate g_E, let

$$H = K(X, Y), \quad Y^2+Y = X^3+X+1,$$
$$E = H(Z), \quad Z^2+Z = X^5+X^3+1.$$

The last equation is not in the normal form for P_∞, the infinite prime of H. However addition of $\left(\frac{Y}{X}\right)^5+\left(\frac{Y}{X}\right)^{10}$ to each side reduces it to the normal form for P_∞. One obtains that the degree of the different of E/H is 8. The different of $E/K(X)$ is the product of the different of $H/K(X)$ and E/H. This gives, by the genus formula,

$$2g_E-2 = 4(-2)+16, \quad \text{i.e. } g_E = 5.$$

Verification of the following facts is left to the reader.

$$N_{1,F} = 5, \quad N_{2,F} = 2, \quad L_F(u) = 1+2u+4u^2+4u^3+4u^4.$$

We know [6],

$$L_G(u) = 1-2u+2u^2-4u^3+4u^4, \quad L_H(u) = 1-2u+2u^2.$$

$L_F(u)$, $L_G(u)$, $L_H(u)$ are relatively prime and each of them divides $L_E(u)$, a polynomial of degree 10. Therefore,

$$L_E(u) = L_F(u)L_G(u)L_H(u), \quad \text{and hence,} \quad h_E = h_F h_G h_H = h_F.$$

Example 3. Let $|K| = 4$,

$$F = K(X, Z), \quad Z^2+Z = X^3+X+\eta,$$
$$G = K(X, Y), \quad Y^2+Y = X^3+\eta,$$

η primitive 3rd root of unity, $E = FG$. Then,

$$g_E = 2, \quad g_F = 1 = g_G, \quad h_E = h_F.$$

Example 4. Let $|K| = 3$,

$$F = K(X, Z), \quad Z^2 = X(X^3+2X+2),$$
$$G = K(X, Y), \quad Y^2 = X^3+2X+2,$$

$E = FG$. Then,

$$g_E = 2, \quad g_F = 1 = g_G, \quad h_E = h_F.$$

Example 5. Let $|K| = 2$,

$$F = K(X, Z), \quad Z^2+Z = X^3+1,$$
$$G = K(X, Y), \quad Y^2+Y = X^3+X+1,$$

$E = FG$. Then,

$$g_E = 2, \quad g_F = 1 = g_G, \quad h_E = h_F.$$

(B) In a constant extension $E = FL$ of degree n,

$$h_{FL}h_F^{-1} = \prod_w (1+a_1w+a_2w^2+\ldots+q^g w^{2g}),$$

where $w \neq 1$ varies over the nth roots of unity [2].

This gives:

(i) $n = 2 \Rightarrow h_{FL} = h_F$ iff $L(-1) = 1$.

For $g = 1$, this means $a_1 = a_2$, i.e. $N_1-(q+1) = q$, so $h_F = 2q+1$;

(ii) $n = 3$, $g = 1 \Rightarrow h_{FL} = h_F$ iff $a_1^2+q^2 = a_1+q(1+a_1)$.

Example 6. We give an example of a function field of genus 3 defined over GF(2) for which the class number is equal to that of the quadratic constant extension. Consider the projective plane curve of degree 4 defined over GF(2) by the equation

$$Y^3Z+Y^2(X^2+Z^2)+XYZ^2 = X^3Z+XZ^3.$$

It is easily checked to be non-singular. Therefore the number of primes of degree 1, 2, 3 of the corresponding function field $F = K(X, Y)$ of genus 3 defined by

$$Y^3+Y^2(X^2+1)+XY = X^3+X$$

can be directly calculated. One finds

$$N_1 = 7, \quad N_2 = 0, \quad N_3 = 1.$$

Using these values to calculate a_1, a_2, a_3, we obtain

$$L_F(u) = 1+4u+9u^2+15u^3+18u^4+16u^5+8u^6, \quad L_F(-1) = 1.$$

Thus by (i), $h_F = h_{FL}$, for $[L:K] = 2$.

EXAMPLE 7. Let $|K| = 4$, $F = K(X, Y)$, $Y^2+Y = X^3$, $[L:K] = 2$. Then,

$$g_F = 1, \quad h_{FL} = h_F = 9.$$

EXAMPLE 8. Let $|K| = 3$, $F = K(X, Y)$, $Y^2 = 2X^3+X+1$, $[L:K] = 2$. Then,

$$g_F = 1, \quad h_{FL} = h_F = 7.$$

EXAMPLE 9. Let $|K| = 2$, $F = K(X, Y)$, $Y^2+Y = X^3+X^2$, $[L:K] = 2$. Then,

$$g_F = 1, \quad h_F = h_{FL}.$$

EXAMPLE 10. Let $|K| = 2$, $F = K(X, Y)$, $Y^2+Y = X^5+X^3$, $[L:K] = 2$. Then,

$$g_F = 2, \quad h_{FL} = h_F.$$

EXAMPLE 11. Let $|K| = 2$, $Y^2+Y = (X^2+X)(X^3+X+1)^{-1}$, $F = K(X, Y)$, $[L:K] = 2$. Then,

$$g_F = 2, \quad h_F = h_{FL}.$$

EXAMPLE 12. Let $|K| = 2$, $F = K(X, Y)$, $Y^2+Y = X^3+X^2$, $[L:K] = 3$. Then,

$$h_F = 5 = h_{FL}, \quad g_F = 1.$$

Examples 7–12 are easily verified using (i), (ii) or by using, as in the proof of (f) of Theorem 3, the information [6] of the fields of class number one. Examples 10, 11 are the exceptions in Theorem 3 (f).

(C) Let F/K be a function field of genus one and E/L a finite separable extension. In Theorem 1, we have shown that $L = K$, E/F unramified implies $h_E = h_F$. The above examples show that for each value of $q \leqslant 4$, neither of these conditions is necessary. However, for $q > 4$, each of these conditions is necessary. The necessity of $L = K$ has been established in Theorem 1. Also, the function $T(g, q)$, introduced in the proof of Theorem 2, is positive for $g > 1$, $q > 7$ and $g > 2$, $q > 4$. To prove that $g_E > 1$ implies $h_E > h_F$, we have, therefore, to consider only the two cases $q = 5$, $q = 7$ for $g_E = 2$. It is known that if a field of genus two has a subfield of genus one, then, it has one more subfield of genus one and its zeta function is the product of the zeta functions of the two subfields. This implies $h_E > h_F$ because, for $q > 4$, there are no fields of genus one and class number one. For $q = 7$, we indicate an independent proof. The equality $h_E = h_F$ implies $T(2, 7)$ is not positive. Together with the Riemann Hypothesis applied to F, we obtain 12 and 13 as the two possibilities for $h_E = h_F$. The non-existence of function fields of genus 2 defined over GF(7) of class number 12 or 13 is established using the method of [7], p. 428. The method fails for $q = 5$.

(D) For $|K| = q = 4$, $E = FL$, $[L:K] = 2$, $g_E = g_F > 1$ implies $h_E > h_F$. This statement is a special case of Theorem 3. We give an alternate proof for it.

If $h_{FL} = h_E$, then (10) implies

$$L_F(-1) = 1 = \prod_{i=1}^{g} (5+4\cos\theta_i).$$

Thus, $\cos\theta_i = -1$, $i = 1, \ldots, g$. Substitution in the expressions for a_1, a_2 obtained by comparing coefficients in (10) gives

$$a_1 = 4g, \quad a_2 = 8g^2-4g. \tag{12}$$

Also,

$$\begin{aligned} a_1 &= N_1-(q+1) = N_1-5, \\ 2a_2 &= N_1^2-(2q+1)N_1+2N_2+2q = N_1^2-9N_1+2N_2+8. \end{aligned} \tag{13}$$

From (12) and (13), we obtain $N_2 = 6(1-g)$, a contradiction because N_2 is non-negative.

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