

Distribution mod 1 of additive functions on the set of divisors

by

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For any real number x , let $\{x\} = x - [x]$ denote the fractional part of x , and $\|x\| = \min(\{x\}, 1 - \{x\})$ the distance of x to the nearest integer. Let d denote a general divisor of the integer n , and the symbol \sum_d denote a summation extended over the all positive divisors of n . Let $\tau(n)$ be the number of divisors of n .

Let $f(n)$ be an additive function. Our aim is to investigate the distribution of $\{f(d)\}$.

Let

$$(1) \quad h_n(x) = \frac{1}{\tau(n)} \sum_{\{f(d)\} < x} 1.$$

We shall give a necessary and sufficient condition to that on a sequence of asymptotic density 1,

$$(2) \quad h_n(x) \rightarrow x$$

uniformly for $x \in [0, 1]$. We define the discrepancy

$$(3) \quad \Delta(n) = \sup_{0 \leq \alpha < \beta < 1} |h_n(\beta) - h_n(\alpha) - (\beta - \alpha)|.$$

It is well-known that the relation (2) is equivalent to the relation

$$(4) \quad \Delta(n) \rightarrow 0.$$

THEOREM. *The relation $\Delta(n) \rightarrow 0$ holds on a sequence of asymptotic density 1 if and only if the series*

$$(5) \quad \sum_p \frac{\|2mf(p)\|^2}{p} = \infty$$

for $m = 1, 2, \dots$

Proof. We need the following result of Erdős and Turán [1], which we state now as

LEMMA 1. Let x_1, \dots, x_N be any real numbers,

$$l(x) = \frac{1}{N} \sum_{\{x_j\} < x} 1,$$

$$S_m = \frac{1}{N} \sum_{j=1}^N e^{2\pi i m x_j} \quad (m = 1, 2, \dots),$$

and

$$\Delta = \sup_{0 \leq \alpha < \beta \leq 1} |l(\beta) - l(\alpha) - (\beta - \alpha)|.$$

Then

$$(6) \quad \Delta \leq c \left(\frac{1}{T} + \sum_{m=1}^T \frac{|S_m|}{m} \right),$$

c being an absolute constant.

LEMMA 2. Let $g(n)$ be a multiplicative function and satisfy

$$0 \leq g(n) \leq 1.$$

Then

$$\sum_{n \leq x} g(n) \leq c_1 x \exp \left(\sum_{p < x} \frac{g(p) - 1}{p} \right),$$

where c_1 denotes an absolute positive constant.

For the proof see for example [2].

Let

$$s_m(n) = \prod_{p^a || n} (1 + e^{2\pi i m f(p)} + \dots + e^{2\pi i m f(p^a)}),$$

and

$$(7) \quad g_m(n) = \frac{|s_m(n)|}{\tau(n)}.$$

We obtain immediately that $0 \leq g_m(n) \leq 1$, and

$$g_m(p) = |\cos 2\pi m f(p)|.$$

First we consider the sum

$$A_m(x) = \sum_{p < x} \frac{1 - g_m(p)}{p}.$$

Since $|\cos \pi a|$ is an even function and periodical mod 1, we can put

$$g_m(p) = |\cos(\pi \|2mf(p)\|)|.$$

By elementary calculation we have

$$(8) \quad 1 - |\cos \pi a| \geq c_2 a^2, \quad 0 \leq a \leq 1/2,$$

c_2 being a positive constant.

Hence

$$A_m(x) \geq c_2 B_m(x) \quad (c_2 > 0),$$

where

$$(9) \quad B_m(x) = \sum_{p \leq x} \frac{\|2mf(p)\|^2}{p}.$$

From Lemma 1 we get

$$K(x) \stackrel{\text{def}}{=} \sum_{n \leq x} \Delta(n) \leq c \frac{x}{T} + c \sum_{m \leq T} \frac{1}{m} \sum_{n \leq x} g_m(n),$$

whence by Lemma 2 we get

$$(10) \quad K(x) \leq c \frac{x}{T} + c \cdot c_1 \sum_{m \leq T} \frac{1}{m} \exp(-A_m(x)).$$

Suppose now that the condition (5) holds. Then $A_m(x) \rightarrow \infty$ for every fixed m , and so uniformly for $m \leq T = T(x)$, when $T(x)$ tends to infinity sufficiently slowly. Thus we get

$$\frac{K(x)}{x} \rightarrow 0 \quad (x \rightarrow \infty),$$

whence the assertion of our theorem immediately follows. Now we prove the necessity of (5). Suppose that

$$(11) \quad \sum_p \frac{\|2mf(p)\|^2}{p} < \infty$$

for a fixed m . This is equivalent to the condition

$$\sum_p \frac{1 - |\cos 2\pi m f(p)|}{p} < \infty.$$

Thus for the multiplicative function $g_m(n)$ the following conditions hold:

$$0 \leq g_m(n) \leq 1, \quad \sum_p \frac{1 - g_m(p)}{p} < \infty.$$

Then, by using a theorem of H. Delange,

$$\frac{1}{x} \sum_{\substack{n \leq x \\ (n, 2) = 1}} g_m(n) \rightarrow M(g) \neq 0.$$

Thus by a suitable positive δ we get that

$$\frac{|s_m(n)|}{\tau(n)} > \delta$$

on a set of integers having positive density. As it is well-known, in this set

$$\Delta(n) > \delta.$$

By this the proof of the necessity of condition (5) is completed.

References

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 [2] R. R. Hall, *Halving an estimate obtained from Selberg's upper bound method*, *Acta Arith.* 25 (1974), pp. 347-351.

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