On the product of the conjugates outside the unit circle of an algebraic integer

by

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The aim of this paper is to extend some results of A. Schinzel [4] and to make them more precise.

Let $K$ be a number field of degree $|K|$, let

$$P(x) = p_0 x^n + p_1 x^{n-1} + \ldots + p_n$$

be a polynomial over $K$ with content $C(P) = \langle p_0, \ldots, p_n \rangle$, let $G$ be the set of all isomorphic injections of $K$ into the complex field $C$ and, for $\sigma \in G$, let

$$\sigma P(x) = \sigma_0 x^n + \ldots + \sigma_n = \sigma_0 \prod_{i=1}^{n} (x - a_i).$$

Generalizing an argument of Smyth [3] concerning the fundamental case $K = Q$, Schinzel proved that if $K$ is totally real, $P(x)$ is non-reciprocal, $p_i$ are integers, $p_0 = 1$ and $p_n \neq 0$, then

$$\max_{\sigma \in G} \prod_{|a_i| > 1} |\mu_{\sigma}| \geq \theta_{p},$$

where $\theta_{p}$ is the real root of the equation $\theta^3 - \theta - 1 = 0$. We extend this in the following manner.

**Theorem 1.** Let $K$ be a totally complex quadratic extension of a totally real field and $\sqrt{-3} \notin K$.

If $P(x) \in K[x]$ is a monic polynomial with integer coefficients, $P(0) \neq 0$, $x^3 P(x^{-1}) \neq \text{const} P(x)$, then (1) holds.

If $\sqrt{-3} \notin K$ (1) needs not be satisfied, but $A \geq |\theta_1|$ where $\theta_1$ is that root of the equation

$$\theta^2 + \frac{-1 + \sqrt{-3}}{2} \theta - 1 = 0$$

which is greater in absolute value.
For \( K \) being a totally complex quadratic extension of a totally real field Schinzel considered the product

\[
II = \prod_{\alpha} \prod_{|\alpha| > 1} |\alpha_{\alpha}|
\]

and proved that if \( p \neq 0 \) and \(|p_n| \neq |p_0|\), then

\[
II \geq \left( \frac{1 + \sqrt{5}}{2} \right)^{|K|/2} \left( N_{K/Q}(p_0) \right)^{1 - \frac{1}{\sqrt{5}}} \left( N_{K/Q}(p) \right)^{1 - \frac{1}{\sqrt{5}}}
\]

with the equality possible only if \( \sqrt{5} \in K \), \( C(P) = (p_0) \),

\[
\frac{|p_n|}{|p_0|} = \frac{1 + \sqrt{5}}{2}.
\]

He made a conjecture about the possible form of the polynomial \( P \) for which the equality in (2) is attained.

We prove this conjecture as:

**Theorem 2.** The equality in (2) is attained if and only if

\[
P(z) = p_0 \left( z^n + \varepsilon_1 \frac{1 + \sqrt{5}}{2} \right) \prod_{\varepsilon_2} (z - \varepsilon_2)
\]

where \( \varepsilon_i \) are roots of unity.

This theorem is an easy consequence of the following:

**Theorem 3.** Let \( K \) be a totally complex quadratic extension of a totally real field. If \( P(z) \in K[z] \) is a monic polynomial with integer coefficients, \( P(0) = 1 \) and

\[
|\alpha_{\alpha}| \geq 1 \quad \text{if} \quad \sigma(\sqrt{5}) = \pm \sqrt{5} \quad (j = 1, 2, \ldots, n),
\]

\[
|\alpha_{\alpha}| \leq 1 \quad \text{if} \quad \sigma(\sqrt{5}) = \mp \sqrt{5} \quad (j = 1, 2, \ldots, n),
\]

then (3) holds.

The proofs are based on several lemmata.

**Lemma 1.** Let \( f(z) = \sum \varepsilon_i z^j \) be a function holomorphic in an open disc containing \( |z| \leq 1 \) and satisfying \( |f(z)| \leq 1 \) for \( |z| = 1 \). Then

\[
|\varepsilon_i| \leq 1 - |\varepsilon_i|^2 \quad (i = 1, 2, \ldots),
\]

(4)

\[
|\varepsilon_i^2 + \frac{\varepsilon_i^2}{1 - |\varepsilon_i|^2}| \leq 1 - |\varepsilon_i|^2 - \frac{1 - |\varepsilon_i|^2}{1 - |\varepsilon_i|^2} \quad (i = 1, 2, \ldots).
\]

(5)

Proof (due to A. Schinzel). Inequality (4) is proved in [4]. In order to prove (5) let us observe, following Smyth [5], that for all \( \beta_0, \beta_1 \)

\[
\left| \sum_{\alpha \in K} |f(\alpha)|^2 |\beta_0^\alpha + \bar{\beta}_0^\alpha + \bar{\beta}_1^\alpha + \beta_1^\alpha| \frac{d\alpha}{|\alpha|}
\]

thus

\[
|\varepsilon_0^\beta_0 1^\beta_1 1^\beta_1 + \varepsilon_0^\beta_0 1^\beta_1 1^\beta_1 + \varepsilon_0^\beta_0 1^\beta_1 1^\beta_1 + \varepsilon_0^\beta_0 1^\beta_1 1^\beta_1| |\varepsilon_0| |\beta_0^2 + \beta_1^2| \leq 1 + |\beta_0^2| + |\beta_1^2|,
\]

and setting

\[
\beta_0 = p_{\varepsilon_0}^{-1}; \quad \beta_1 = \frac{p_{\varepsilon_0}}{\varepsilon_0 (\delta - \varepsilon_0)}
\]

where \( |p| = |\delta| = 1 \), we obtain

\[
|\varepsilon_0 + \frac{\varepsilon_0^2}{\delta - \varepsilon_0} \varepsilon_0 + \frac{p_{\varepsilon_0}}{\varepsilon_0} | \leq |\varepsilon_0|^{-1}.
\]

Hence

\[
|\varepsilon_0^{p_{\varepsilon_0}} \delta^{-1} + \frac{\varepsilon_0^2}{\delta - \varepsilon_0} + \varepsilon_0 | \leq 1.
\]

For an arbitrary \( \varepsilon \) with \( |\varepsilon| = 1 \) the number

\[
\delta = \frac{\varepsilon \delta + 1}{\varepsilon + \varepsilon_0}
\]

has absolute value 1.

We set

\[
\varepsilon = \varepsilon_0^\beta_0 1^\beta_1 1^\beta_1, \quad \delta = \frac{|\varepsilon|}{|\varepsilon|} \varepsilon_0^\beta_0 1^\beta_1 1^\beta_1, \quad \varepsilon = \frac{\varepsilon}{|\varepsilon|} \varepsilon_0^\beta_0 1^\beta_1 1^\beta_1.
\]

Then

\[
\varepsilon_0 \delta^{-1} + \frac{\varepsilon_0^2}{\delta - \varepsilon_0} + \varepsilon_0 = \frac{\varepsilon}{|\varepsilon|} \varepsilon_0^\beta_0 1^\beta_1 1^\beta_1 + \frac{\varepsilon_0^2}{1 - |\varepsilon_0|^2} + \varepsilon_0
\]

\[
= \frac{\varepsilon}{|\varepsilon|} \left( |\varepsilon_0|^2 + \frac{|\varepsilon_0|^2}{1 - |\varepsilon_0|^2} + \varepsilon_0 \right).
\]

Hence by (6)

\[
\left| \frac{\varepsilon}{|\varepsilon|} \left( |\varepsilon| + |\varepsilon_0|^2 + \frac{|\varepsilon_0|^2}{1 - |\varepsilon_0|^2} \right) \right| \leq 1 \quad \text{and} \quad |\varepsilon| \leq 1 - |\varepsilon_0|^2 - \frac{|\varepsilon_0|^2}{1 - |\varepsilon_0|^2},
\]

which proves the lemma.

**Lemma 2.** Let \( P(z) = x^n + p_1 x^{n-1} + \ldots + p_n \), \( |p_n| = 1 \), \( Q(z) = z^n P(x^{-1}) \neq \text{const} \ P(x) \). Then

\[
\frac{P(0) P(z)}{Q(z)} = \frac{f(z)}{g(z)}
\]
where \( f \) and \( g \) are holomorphic in an open disc containing \( |z| < 1 \),
\[
f(0) = g(0) = \prod_{|a_j| > 1} |a_j|^{-1}, \quad |f(z)| = |g(z)| = 1 \quad \text{for} \quad |z| = 1,
\]
and if the coefficients of \( P \) are real, the coefficients of \( f \) and \( g \) are also real.

Proof (cf. [4], Lemma 2). We set
\[
f(z) = \prod_{|a_j| < 1} \frac{z - a_j}{1 - a_j z}, \quad g(z) = \prod_{|a_j| > 1} \frac{1 - a_j z}{z - a_j},
\]
and using the equalities
\[
\prod_{i=1}^n (-a_j) = P(0), \quad |P(0)| = 1,
\]
we easily verify all the assertions of the lemma. Note that if \(|a_j| = 1\), the factor \(z - a_j\) occurs both in \(P(z)\) and in \(Q(z)\). Also if \(P(z)\) has real coefficients,
\[
f(z) = f(\bar{z}), \quad g(z) = g(\bar{z}).
\]

**Lemma 3 (Kronecker [2]).** If \( a \neq 0 \) is an algebraic integer with \(|a| \leq 1\), then \( a \) is a root of unity.

If \( a \) is a totally real algebraic integer with \(|a| \leq 2\), then \( a = 2 \cos \theta \), where \( \theta \) is rational.

**Lemma 4.** If \( a \neq 0 \) is an algebraic integer of a field \( K \) satisfying the assumptions of Theorem 1, then either \( a \) is a root of unity or \(|a| > \sqrt{2}\).

Proof. By the first part of Lemma 3 we can assume that \(|a| \geq 1\).
For all \( \sigma \in G \) we have
\[
\sigma(|a|^2) = |\sigma a| \overline{\sigma a} = |\sigma a|^2;
\]
thus \(|a|^3\) is totally real and totally positive, \(|a|\) is totally real and \(|a| = |\sigma a|\).

On the other hand, by the second part of Lemma 3,
\[
|a| \geq 2 \cos \frac{2 \pi}{d} = \sqrt{2}.
\]

**Proof of Theorem 1.** Let
\[
A = \max_{a \in G} \prod_{|a_j| > 1} |a_j|.
\]
Since \( \sqrt{2} > \theta_0 \), we can assume that \( A \leq \sqrt{2} \). It follows that \(|P(0)| = 1\)
since otherwise by Lemma 4
\[
A \geq |P(0)| \geq \sqrt{2}.
\]

The assumption \( P(z)/Q(z) \neq \text{const} \) implies that
\[
K(z) = \frac{P(0)P(z)}{Q(z)} = 1 + a_n z^n + a_2 z^2 + \ldots,
\]
where on the right-hand side we have infinitely many non-zero coefficients, \( a_n = a_2 \) being the first two of them. Since \( a_i \) are integers of \( K \), \( |a_i| \geq 1 \) for \( i = k, l \).

Using Lemma 4, we distinguish two cases.

The case \(|a_i| > \sqrt{2} \). Since \( A(P) = A(\sigma P) \), we can assume \(|a_i| > \sqrt{2} \) replacing if necessary \( P \) by a suitable \( \sigma P \). Applying Lemma 2 to \( P \), we get
\[
\frac{P(0)P(z)}{Q(z)} = \frac{\overline{f(z)}}{\overline{g(z)}} = \frac{c + c_2 z + c_3 z^2 + \ldots}{\overline{d} + \overline{d}_2 z + \overline{d}_3 z^2 + \ldots};
\]
\[
|f(z)| = |g(z)| = 1 \quad \text{for} \quad |z| = 1, \quad f, g \text{ holomorphic for } |z| < 1;
\]
\[
f(0) = g(0) = \prod_{|a_j| > 1} |a_j|^{-1} = c = \overline{d}.
\]

Comparing (7) with (8), we get
\[
\begin{align*}
\delta_i &= d_i \quad (i = 1, 2, \ldots, k - 1), \quad c_k = \overline{d}_k + \overline{a}_k, \\
\delta_{k+1} &= d_{k+1} + \overline{a}_k d_k \quad (i = 1, 2, \ldots, k - 1), \quad c_l = d_l + a_l c_k + \overline{a}_k d_{l-k}.
\end{align*}
\]
It follows from \( c_k = \overline{d}_k + a_k c \), by Lemma 1, that
\[
\sqrt{2} \leq |a_k| |c| < |c_k| + |d_k| < 2 - 2 |z|^2, \quad \sqrt{2} \leq c_k < A.
\]

The case \(|a_i| = 1\). \( a_k \) is a root of unity. Let \( \eta \) be a root of unity,
\( P(\eta^{-2}) = \eta^n P(\eta^{-1}) \). We have \( A(P_\eta) = A(P) \), \( P_\eta \) and \( K(\eta) \) satisfying the assumptions of the theorem.

Setting
\[
R_\eta(z) = \frac{P(0)P(z)}{P_\eta(z)}, \quad R_\eta(z) = 1 + \sum_{i=1}^\infty a_i \eta^i,
\]
we get
\[
R_\eta(z) = \frac{\eta^n P(0)\eta^n P(\eta^{-1})}{\eta^n P_\eta(\eta^{-1})} = \frac{P(0)P(\eta^{-1})}{Q(\eta^{-1})} = R(\eta^{-1}) z;
\]
hence for all \( i \)
\[
(9a) \quad a_i = \eta a_i, \quad |a_i| = |a_i|.
\]
Taking \( \eta = \sqrt{2} \), we get
\[
\delta_i = 1, \quad a'_i = 0 \quad (0 < i < k), \quad \delta_k, \delta_l, \in K.
\]
Therefore, without loss of generality we assume that \( a_k = 1 \) if \( l < 2k \) and \( a_k = \pm 1 \), \( a_k \neq K \) if \( l \geq 2k \) (we admit both signs here for the sake of symmetry).

The case \( l < 2k \), \( a_k = 1 \). Applying to \( P \) a suitable \( \sigma \in G \), we can obtain \( |a_k| \geq 1 \). We shall exploit the following inequality, due to Smyth ([5], pp. 172, 173):

\[
E = \frac{5}{4} |c|^2 + |c_{l-k} + \gamma c|^2 + \left| \frac{a_l c + c_{l-k}}{2} + \frac{c}{2} - c_{l-k} + \beta \right|^2 \\
\leq 2 + |\beta|^2 + |\gamma|^2,
\]

where \( \beta \) and \( \gamma \) are arbitrary complex numbers.

Put \( F(\beta, \gamma, c_{l-k}) = E - |\beta|^2 - |\gamma|^2, \) \( |\tau|^2 F(\frac{\beta}{\tau}, \frac{\gamma}{\tau}, \frac{c_{l-k}}{\tau}) \) is a hermitian form with the matrix

\[
M = \begin{pmatrix}
|c|^2 - 1 & \frac{|c|^2}{2} & -\frac{c}{2} & \frac{1}{2} a_l |c|^2 \\
\frac{5}{4} |c|^2 - 1 & \frac{\tilde{a}_l}{2} |c|^2 \\
\frac{3}{4} |c|^2 + \frac{1}{2} a_l c & \frac{5}{4} |c|^2 + \frac{1}{2} a_l c \\
\frac{a_l |c|^2}{2} & \frac{1}{4} a_l |c|^2 \\
\end{pmatrix}
\]

with diagonal minors

\[
M_1 = |c|^2 - 1 < 0, \quad M_2 = |c|^4 - \frac{9}{4} |c|^2 - 1 > 0, \\
M_3 = \frac{5}{2} |c|^2 - 2 |c|^4 > 0, \quad M_4 = \frac{25}{16} |c|^4 - \frac{5}{4} |c|^2 + \frac{25}{16} |c|^4 - \frac{9}{4} |c|^2.
\]

In order to justify the second inequality we note that the equation \( c_k = a_k + c \) implies by Lemma 1

\[
2 - 2e^2 \geq c, \quad c \leq \frac{\sqrt{17} - 1}{4},
\]

and since \( c \) is an algebraic integer,

\[
o < \frac{\sqrt{17} - 1}{4}, \quad |c|^4 - \frac{9}{4} |c|^2 + 1 > 0.
\]

Now

\[
F(\beta, \gamma, c_{l-k}) = M_1 |\beta|^2 + \ldots + \frac{M_2}{2} |\gamma|^2 + \ldots + \frac{M_4}{M_3} |c_{l-k} + \ldots|^2 + \frac{M_4}{M_3} |\beta|^2,
\]

(see, e.g. [3], p. 461), from (10) we get

\[
\frac{M_4}{M_3} = \min_{a_k \neq K} \max_{\gamma, c_{l-k}} F(\beta, \gamma, c_{l-k}) \leq 2
\]

and from (11) we get

\[
40 |c|^4 - 93 |c|^2 + 40 \geq 16 (2 M_3 - M_4) 0.
\]

As proved by Smyth, the latter inequality implies \( A = \sigma^2 > \theta_k \).

The case \( l \geq 2k \), \( a_k = \pm 1, a_{2k} \neq K \). By (9)

\[
o_{2k} = d_k + a_k d_k + d_{2k} \epsilon.
\]

On applying (5) to \( c_{2k} \) and \( d_{2k} \) and adding the resulting inequalities, we get

\[
|o_{2k} - d_{2k} + \frac{d_k - d_k^2}{1 - |c|^2} \epsilon | \leq |a_k \epsilon + a_k d_k + \frac{d_k - d_k^2}{1 - |c|^2} \epsilon |
\]

\[
\leq 2 - 2e^2 - \frac{|c|^2}{1 - e^2} - \frac{|d_{2k}|^2}{1 - e^2}.
\]

(\( \epsilon \) is real). We now set

\[
o_{2k} = d_k^0 + i d_k^0, \quad d_{2k} = d_k^0 + i d_k^0, \quad a_{2k} = d_k^0 + i d_k^0,
\]

where \( d_k^0, d_k^0, d_k^0 \) are real for \( i = 1, 2 \), and we get from \( c_k = d_k + o_k \epsilon \) the equations

\[
\frac{d_k^0}{d_k} = d_k^0, \quad \frac{d_{2k}^0}{d_{2k}} = d_{2k}^0 - d_{2k}^0 + 2i a_k d_{2k}^0.
\]

The inequality \( |z| \geq |\Re z| \) applied to (12) gives

\[
|d_k^0 \epsilon + a_k d_k^0 + \frac{d_k^0 - d_k^0}{1 - e^2} \epsilon | \leq 2 - 2e^2 - |d_k^0|^2 \frac{d_k^0}{1 - e^2} - \frac{d_k^0}{1 - e^2}.
\]

The left-hand side of (13) is greater than or equal to

\[
|a_k^0 d_k^0 + a_k d_k^0| - \frac{d_k^0}{1 - e^2} - \frac{d_k^0}{1 - e^2} \epsilon - \frac{d_k^0}{1 - e^2} \epsilon
\]

Hence

\[
|a_k^0 \epsilon + a_k d_k^0| \leq 2 - 2e^2 - \min \left( \frac{d_k^0}{1 + e}, \frac{d_k^0}{1 - e}, \frac{d_k^0}{1 - e}, \frac{d_k^0}{1 + e} \right).
\]

Since \( d_k^0 = \pm 1 + c d_k^0, \) we have

\[
|a_k^0| + |d_k^0| = c,
\]

for otherwise by Lemma 1

\[
1 - e^2 \geq c, \quad A = \sigma^2 > \frac{1}{2}.
\]
Again by Lemma 1
\begin{align}
|c_0^{(1)}| &\leq |a_b| \leq 1 - \epsilon^2, \\
|a_b^{(1)}| &\leq |d_k| \leq 1 - \epsilon^2.
\end{align}
By (16) and (10)
\begin{align}
\epsilon^2 + \epsilon - 1 &\leq |c_0^{(1)}| \leq 1 - \epsilon^2, \\
\epsilon^2 + \epsilon - 1 &\leq |d_k^{(1)}| \leq 1 - \epsilon^2.
\end{align}
The further argument depends on \(a_{2k}^{(1)}\).

We distinguish three cases: \(X\), \(a_{2k}^{(1)} \geq 1\), \(Y\), \(a_{2k}^{(1)} = 0\), \(Z\), \(0 < |a_{2k}^{(1)}| < 1\).

\(X\). Applying to \(P\) a suitable \(\sigma \in \mathcal{G}\), we can obtain \(|a_{2k}^{(1)}| \geq 1\).

\(Y\). By (17)
\begin{align}
|a_{2k}^{(1)}| &\geq |d_k^{(1)} - a_{2k}^{(1)}| \geq \epsilon^2 + \epsilon - 1.
\end{align}
By (14) and (18)
\begin{align}
\epsilon^2 + \epsilon - 1 &\leq M = \max_{\epsilon^2 + \epsilon - 1 \leq \epsilon^2 < 1} \left(2 - 2\epsilon^2 - \frac{\epsilon^2}{1 + \epsilon} - \frac{(\epsilon - \epsilon^2)^2}{1 - \epsilon}\right).
\end{align}
As proved by Smyth [15], p. 175, the latter inequality implies \(A = \epsilon^{-1} \geq \theta_0\).

\(Z\). Since \(2|a_{2k}^{(1)}| < 2\) and \(2a_{2k}^{(1)} = a_{2k} + \overline{a}_{2k}\) is a totally real algebraic integer, we have by Lemma 3
\begin{align}
a_{2k}^{(1)} &= \cos 2\omega_n, \quad \text{as rational.}
\end{align}
Since \(c_0^{(1)} = \pm \epsilon + d_k^{(1)}\), we have by (14)
\begin{align}
|d_k^{(1)} \pm (a_{2k}^{(1)} - 1)| \leq 2 - 2\epsilon^2 - \min \left\{\frac{d_k^{(1)} + a_{2k}^{(1)} - 1}{1 + \epsilon}, \frac{d_k^{(1)} - a_{2k}^{(1)} + 1}{1 - \epsilon}\right\}.
\end{align}
If \(|a_{2k}^{(1)} - 1| \geq 1\) or \(|a_{2k}^{(1)} - 1| = 0\), then the situation differs from that occurring in the case \(X\) or \(Y\) only by the permutation of \(c_0^{(1)}\) and \(d_k^{(1)}\). Let \(|a_{2k}^{(1)} - 1| < 1\).

We have
\begin{align}
a_{2k}^{(1)} - 1 &= \cos 2\omega_n - 1 = -2\sin^2\omega_n;
\end{align}
\begin{align}
\sin^2\omega_n &< \frac{1}{2}, \quad \sin^2\omega_n < \frac{1}{\sqrt{2}}, \quad 2\sin^2\omega_n < \sqrt{2}.
\end{align}

2\sin^2\omega_n is a totally real algebraic integer, and hence, by Lemma 4, 2\sin^2\omega_n = \pm 1, \(a_{2k}^{(1)} = 1 - 2\sin^2\omega_n = \frac{1}{2}, a_{2k} = \frac{1}{2} + i\alpha_{2k}\). If \(1 \neq a_k\), it is impossible to have \(a_{2k}^{(1)} = \pm \sqrt{2}/2\), and thus \(a_{2k}\) is not a root of unity. Hence by Lemma 4 \(|a_{2k}| \geq \sqrt{2}\) and applying to \(P\) a suitable \(\sigma \in \mathcal{G}\), we can obtain
\begin{align}
|a_{2k}| \geq \sqrt{2}, \quad |a_{2k}^{(1)}| \geq \frac{1}{2}\sqrt{7}.
\end{align}

We now replace \(P\) by \(P_\eta(z)\), where \(\eta = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\). By (9a) we get
\begin{align}
a_k &= \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) a_\eta, \quad a_{2k} = i a_{2k},
\end{align}
\begin{align}
a_i = 0 \quad \text{for} \quad i \leq 2k, \quad i \not= k, 2k.
\end{align}

\(P(z) = P_\eta(z)P(z)\) is a polynomial with totally real coefficients.

\begin{align}
\frac{P(z)}{Q(z)} &= \frac{P(0)P(z)}{Q(0)Q(z)} \\
&= \left(1 + a_0 + a_2z^2 + \cdots \right) \left(1 + a_0^* + a_2^*z^2 + \cdots \right) \\
&= 1 + a_0^* + a_2^*z^2 + \cdots = 1 + b_0 x + b_2 x^2 + \cdots
\end{align}

By Lemma 2
\begin{align}
P(z) &= \frac{e_0 + e_2 z^2 + \cdots}{f_0 + f_2 z^2 + \cdots},
\end{align}
where \(f_0 = e_0 = \epsilon^2\) and \(e_i, f_i\) are real.

The series occurring on the right side of (21) are convergent in an open disc containing \(|z| < 1\) and have absolute value 1 on the circle \(|z| = 1\).

By the inequality (14)
\begin{align}
|b_{2k} + a_{2k}^2 f_k| &\leq 2 - 2\epsilon^2 - \min \left\{\frac{e_k^2}{1 - \epsilon}, \frac{f_k^2}{1 - \epsilon}, \frac{e_k^2}{1 - \epsilon}, \frac{f_k^2}{1 + \epsilon}\right\}.
\end{align}
By (20) and (21)
\begin{align}
e_i = f_i, \quad i < k; \quad e_k = a_k \sqrt{2} e_\eta + f_k, \quad e_{2k} = b_{2k} e_0 + f_{2k} + a_k \sqrt{2} f_\eta.
\end{align}
The equality \(e_k = f_k + a_k \sqrt{2} e_\eta\) implies
\begin{align}
|e_k| + |f_k| = \sqrt{2} e_\eta,
\end{align}
for otherwise, by Lemma 1, \(\sqrt{2} e_\eta \leq 1 - \epsilon_0^2\), and
\begin{align}
\epsilon_0^{-1} \geq \frac{\sqrt{2} + \sqrt{6}}{2} > 1.9 > \theta_0, \quad A = \epsilon^{-1} > \theta_0.
\end{align}
By (24) and (22)
\begin{equation}
|b_{2k}e_0 + a_02f_0| \leq \tilde{M} = \max \varphi(x),
\end{equation}
where
\[ \varphi(x) = 2 - 2e_0 - \frac{x^2}{1 + e_0} - \frac{(\sqrt{2}e_0 - x)^2}{1 - e_0}. \]
We have
\[ \frac{1}{2} \varphi'(x) = -\frac{x}{1 + e_0} - \frac{x - \sqrt{2}e_0}{1 - e_0} = \frac{-2x + \sqrt{2}e_0(1 + e_0)}{1 - e_0}, \]
thus the maximum of \( \varphi(x) \) taken for \( x = \frac{\sqrt{2}}{2} e_0(1 + e_0) \) equals
\[ M = \varphi\left(\frac{1}{2} e_0(1 + e_0)\right) = 2 - 2e_0 - \frac{1}{2} e_0^2(1 + e_0) - \frac{1}{2} e_0^2(1 - e_0) = 2 - 3e_0^2. \]

From the equality \( f_k + a_0\sqrt{2}e_0 = e_0 \) we get by (25)
\[ |b_{2k}e_0 + a_0\sqrt{2}e_0| \geq |b_{2k}e_0| - |a_0\sqrt{2}e_0| \geq \sqrt{2}e_0(1 + \sqrt{2})e_0 - \sqrt{2} \]
and by (25)
\[ 2 - 3e_0^2 \geq \sqrt{2}e_0^2 + (1 + \sqrt{2})e_0 - \sqrt{2}, \]
\[ f(e_0^{-1}) \geq 0, \]
where \( f(x) = (2 + \sqrt{2})x^2 - (1 + \sqrt{2})x - (3 + \sqrt{2}) \).
Since \( f\left(\frac{\sqrt{3}}{2}\right) < 0 \) and \( e_0 = e^s \), we have \( A = e^{-1} > \frac{s}{2} > 0 \).

Consider now the case
\[ a_k = \pm 1, \quad a_{2k} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \quad \sqrt{-3}e_0K. \]
It follows from (9a) that
\[ c_k + d_k = a_k(e(2d_k^0 + a_0e) + 2a_02d_k^0) \]
and from (15) that
\[ |c_k|^2 + |d_k|^2 = |d_k^0|^2 + 2|d_k^0|^2 + (c - |d_k^0|)^2. \]

The inequality (12) implies in virtue of the above identities
\[ M = 2 - 2e_0 - \frac{|d_k^0|^2 + (c - |d_k^0|)^2 + 2|d_k^0|^2}{1 - e_0^2} \geq \frac{1}{2} \left( \frac{1 + \sqrt{3}}{2} - \frac{1 + e^s}{1 - e^s} \right) \left( \pm \frac{\sqrt{3}}{2} c + a_k \frac{1 + e^s}{1 - e^s} d_k^0 \right) \geq 0. \]

Hence it follows that
\[ M = \max_{\omega_k | d_k^0 |} \left( 2 - 2e_0 - \frac{|d_k^0|^2 + (c - |d_k^0|)^2 + 2|d_k^0|^2}{1 - e_0^2} \right) \]
\[ \geq \frac{1 + e^s}{2} \left( \frac{1}{2} - \frac{1 + e^s}{1 - e^s} |d_k^0| \right) \geq 0. \]
The inner maximum is attained for \( |d_k^0| = e^s/2 \); since then both
\[ |d_k^0|^2 + (c - |d_k^0|)^2 \] and
\[ \frac{1 + e^s}{1 - e^s} \left( \frac{1}{2} - \frac{1 + e^s}{1 - e^s} |d_k^0| \right) \]
attain the minimal value. Thus
\[ M = \max_{\omega_k | d_k^0 |} \left( 2 - 2e_0 - \frac{c^2}{2(1 - e_0^2)} - \frac{2|d_k^0|^2}{1 - e_0^2} - \frac{\sqrt{3}}{2} \frac{c - (1 + e^s)|d_k^0|}{1 - e_0^2} \right) \geq 0. \]
We set
\[ g(x) = 2 - 2e_0 - \frac{x^2}{2(1 - e_0^2)} - \frac{2e_0^2}{1 - e_0^2} - \frac{\sqrt{3}}{2} \frac{1 + e^s}{1 - e_0^2} x. \]
In the interval \( \left( 0, \frac{\sqrt{3}}{2} \frac{1 - e_0^2}{1 + e^s} \right) \) the function \( g(x) \) is increasing.

Indeed we have in this interval
\[ \frac{\sqrt{3}}{2} \frac{1 + e^s}{1 - e_0^2} x > 0, \quad g'(x) = - \frac{4x}{1 - e_0^2} + \frac{1 + e^s}{1 - e_0^2} \geq - 2\frac{\sqrt{3}e_0}{1 - e_0^2} \frac{1 + e^s}{1 - e_0^2} \]
On the other hand, by the assumption \( A < \sqrt{2} \) we have
\[ e^s > \frac{1}{2}, \quad \frac{1 + e^s}{1 - e_0^2} > \frac{3}{2} \]
\[ 2\frac{\sqrt{3}e_0}{1 - e_0^2} < \sqrt{3}e_0 < 2 < \frac{1}{2} < (1 + e^s)^2 \]
For \( x > \frac{\sqrt{3}}{2} \frac{1 - e_0^2}{1 + e^s} \) we have
\[ \frac{\sqrt{3}}{2} \frac{1 + e^s}{1 - e_0^2} x < 0, \quad g'(x) = - \frac{4x}{1 - e_0^2} + \frac{1 + e^s}{1 - e_0^2} \frac{1 + e^s}{1 - e_0^2} < 0; \]
thus the function $g(x)$ is decreasing. Since it is continuous, we have

$$M_3 = g\left(\frac{\sqrt{3}}{2} \frac{1-c^2}{1+c^2}\right),$$

$$g\left(\frac{\sqrt{3}}{2} \frac{1-c^2}{1+c^2}\right) = 2 - c^2 - \frac{c^2}{2(1-c^2)} - \frac{3}{2} c^2 (1-c^2) \frac{(1-c^2)}{(1+c^2)} \geq 0,$$

and on simplification

$$1 - c^2 - c^4 - c^6 \geq 0,$$

whence $A^8 - A^6 - A^4 - A^2 + 1 \geq 0$. The equation $x^8 - x^6 - x^4 - x^2 + 1 = 0$ has only one real root greater than 1, namely

$$\frac{1}{4}(1 + \sqrt{13} + \sqrt{21} + \sqrt{13} - 2).$$

It follows that

$$A \geq \frac{1}{4}(1 + \sqrt{13} + \sqrt{21} + \sqrt{13} - 2).$$

On the other hand, the polynomial $x^2 + c_2 x - 1$, $c_2 = -\frac{1 + \sqrt{3}}{2}$ has two zeros given by the formula

$$\frac{-c \pm \sqrt{c^2 + 4}}{2} = \frac{1}{4} \pm \frac{1}{2} \left(\sqrt{\frac{\sqrt{13} + 7}{2}} - \frac{3}{4} \pm \frac{1}{2} \sqrt{\frac{\sqrt{13} - 7}{2}} \right).$$

Hence

$$|\theta_2|^2 = \left(\frac{1}{4} + \frac{1}{2} \sqrt{\frac{\sqrt{13} + 7}{2}} \right)^2 + \left(\frac{3}{4} + \frac{1}{2} \sqrt{\frac{\sqrt{13} - 7}{2}} \right)^2 = \frac{1}{4} \left(1 + \sqrt{13} + \sqrt{\frac{\sqrt{13} + 7}{2}} + \sqrt{3 \left(\frac{\sqrt{13} - 7}{2}\right)}\right)$$

$$= \frac{1}{4} \left(1 + \sqrt{13} + \sqrt{2 \sqrt{13} - 2}\right)$$

and we get

$$A \geq |\theta_2|.$$ 

It remains to note that the zeros of the polynomial $x^2 + c_2 x - 1$ are complex conjugates of the zeros of $x^2 + c_2 - 1$ and that

$$|\theta_2|^2 < 1.73 < \theta_2^*.$$  

**Lemma 5.** $f(x) = a_0 x^n + \ldots + a_n$ has all zeros inside the unit circle if and only if $\delta_k(f) > 0$, $k = 0, 1, \ldots, n$ where

$$\begin{align*}
\delta_k(f) &= \frac{a_{n-k} a_n - a_k}{\prod_{j=0}^{n-k} \bar{a}_j} \\
&= \frac{\bar{a}_n \bar{a}_{n-k} - \bar{a}_k}{\prod_{j=0}^{n-k} a_j}.
\end{align*}$$

**Proof.** see A. Cohn [1].

**Lemma 6.** If $A = (a_{ij})$, $i, j \leq n$, $n \geq 1 > k$, then

$$\det A = \sum_{i=0}^{n-k} (-1)^{i+k} A_{i+k,i} \left| a_{0i} \ a_{ki} \right|$$

where $A_{i+k,i}$ is the determinant of the matrix obtained from $A$ by crossing out the $i$-th and $k$-th rows and the $i$-th and $j$-th columns.

The proof follows from Laplace's theorem.

**Lemma 7.** Let $K$ satisfy the assumptions of Theorem 3, $f(x) = a_n x^n + \ldots + a_0$, $f^n(x) = x^n (x^{-1} = a_{n-k} x^{n-k} + \ldots + a_n x_{n-k+1})$ where $a_0, \ldots, a_n$ are integers of $K$. If $a_n = 1$, $a_0 = 1 + \frac{1+\sqrt{5}}{2}$, where $e$ is a root of unity, then for each $i \leq n$ the condition

$$(w_i) \quad \delta_i(af) > 0 \quad \text{if} \quad a\sqrt{5} = -\sqrt{5} \quad (k = 1, 2, \ldots, i),$$

implies

$$\delta_k(af^n) > 0 \quad \text{if} \quad a\sqrt{5} = \sqrt{5} \quad (k = 1, 2, \ldots, i).$$

for all $k \leq i$ and all $a \in G$.

**Proof.** We shall proceed by induction with respect to $i$. For $i = 1$

$$\sigma(a\bar{a}_0 - a_0) = 0,$$

$$\delta_1(af) = \left| \frac{1}{aar{a}_0} \right| = 1 - |aar{a}_0|^2 = \begin{cases} \frac{1 + \sqrt{5}}{2} & \text{if} \quad a\sqrt{5} = -\sqrt{5} \\ \frac{\sqrt{5} - 1}{2} & \text{if} \quad a\sqrt{5} = \sqrt{5}. \end{cases}$$
Let us assume that the lemma is true for $i < i_q$. Then, by applying the formula from Lemma 6 to the rows $i_q$, $2i_q$ and omitting the terms which involve the coefficient $a_{a_0}$, we get

\[ \delta_{i_q}(f) = \delta_1(f) \delta_{i_q-1}(f) + \sum_{j=1}^{i_q-1} (-1)^j \frac{a_{a_n-n-q+j}}{a_{a_n-j}} \frac{a_n}{a_{a_n-n-q+j}} \delta_{i_q}(f_{i_q-i_q-j}(f)). \]

Now we apply (29) to $\sigma f$ and use the inductive assumption. We obtain

\[ \delta_{i_q}(\sigma f) = \delta_1(\sigma f)^{i_q} - \frac{a_{a_n-n-q+i_q}}{a_{a_n-n-q+i_q}} \delta_{i_q}(f_{i_q-i_q-i_q+i_q}(f)). \]

Hence the condition (w_q) is equivalent to the condition

\[ \delta_{i_q}(\sigma f) > 0 \text{ if } \sigma \sqrt{5} = -\sqrt{5}, \]

\[ (-1)^{i_q} \delta_{i_q}(\sigma f) > 0 \text{ if } \sigma \sqrt{5} = \sqrt{5}. \]

We set

\[ x = \bar{a}_0 a_{a_n-n-q+i_q} + \bar{a}_{a_{n-1}} a_n \]

and distinguish two cases:

1. $i_q$ is even. We have, for all $\sigma \in G$,

\[ \delta_{i_q}(\sigma f) = \delta_0^2(\sigma f) - \delta_0^{i_q-2}(\sigma f)|\sigma x|^2 > 0, \]

and since

\[ \delta_1(\sigma f)^{i_q} \text{ and } \delta_1(\sigma f)^{i_q-1} > 0, \]

we get

\[ |\sigma x|^2 < \delta_1^2(\sigma f) = \begin{cases} \frac{3+\sqrt{5}}{2} (\sigma \sqrt{5} = \sqrt{5}), \\ \frac{3-\sqrt{5}}{2} (\sigma \sqrt{5} = -\sqrt{5}) \end{cases} \]

\[ \prod_{\sigma \in G} |\sigma x|^2 < \prod_{\sigma \in G} \delta_1^2(\sigma f). \]

The number $N = \prod_{\sigma \in G} |\sigma x|^2$ is a non-negative rational integer, as the square of the norm of $x$,

\[ \prod_{\sigma \in G} \delta_1^2(\sigma f) = 1. \]

Since $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$ occur equally often as factors in the product, we have $M = 0$ and $x = 0$.

By (31)

\[ \delta_{i_q}(f) = \delta_1(\sigma f)^{i_q}, \]

\[ \bar{a}_0 a_{a_n-n-q+i_q} + \bar{a}_{a_{n-1}} a_n = 0, \]

and the inductive assertion is proved.
B. $i_2$ in odd. We have
\[ \delta_4(\sigma f) > 0 \quad \text{if} \quad \sigma \sqrt[5]{5} = -\sqrt[5]{5}, \]
\[ \delta_4(\sigma f) < 0 \quad \text{if} \quad \sigma \sqrt[5]{5} = \sqrt[5]{5}. \]
If
\[ \sigma \sqrt[5]{5} = -\sqrt[5]{5}, \quad \delta_4(\sigma f) = \frac{\sqrt[5]{5} - 1}{2} > 0, \]
we have
\[ \delta_4(\sigma f)^{s+2} > 0, \quad \delta_4(\sigma f)^{s-2} > 0 \]
and the inequality
\[ \delta_4(\sigma f)^{s-2} - \delta_4(\sigma f)^{s+2} > 0 \]
implies
\[ \frac{3 - \sqrt[5]{5}}{2} > |\sigma z|^2. \]
If
\[ \sigma \sqrt[5]{5} = \sqrt[5]{5}, \quad \delta_4(\sigma f) = -\frac{1+\sqrt[5]{5}}{2} < 0, \]
we have
\[ \delta_4(\sigma f) < 0, \quad \delta_4(\sigma f)^{s-2} < 0 \]
and the inequality
\[ \delta_4(\sigma f)^{s-2} - \delta_4(\sigma f)^{s+2} < 0 \]
implies
\[ \frac{3 + \sqrt[5]{5}}{2} > |\sigma z|^2. \]
The inductive assertion follows from (32) and (33) as in the case A.

**Lemma 8.** Let $K$ satisfy the assumptions of Theorem 3,
\[ f(z) = z^n + a_{n-1}z^{n-1} + \ldots + e^{\frac{1+\sqrt[5]{5}}{2}}, \]
where $a_i$ are integers of $K$, $e$ is a root of unity, and let
\[ \sigma f(z) = \prod_{i=1}^n (z - a_i), \quad \sigma \in G. \]
If
\[ |a_i| > 1 \quad \text{if} \quad \sigma \sqrt[5]{5} = \sqrt[5]{5}, \]
\[ |a_i| < 1 \quad \text{if} \quad \sigma \sqrt[5]{5} = -\sqrt[5]{5}, \]
then
\[ f(z) = z^n + e^{\frac{1+\sqrt[5]{5}}{2}}. \]

**Proof.** $f(z)$ has all zeros outside the circle $|z| \geq 1$ if and only if $f'(z) = z^n f(z^{-1})$ has all zeros inside the circle $|z| < 1$. By Lemma 5 the condition (34) is equivalent to the condition
\[ \delta_4(\sigma f) > 0 \quad \text{if} \quad \sigma \sqrt[5]{5} = -\sqrt[5]{5}, \quad (k = 1, 2, \ldots, n), \]
\[ \delta_4(\sigma f) > 0 \quad \text{if} \quad \sigma \sqrt[5]{5} = \sqrt[5]{5}, \quad (k = 1, 2, \ldots, n). \]
The latter is the same as condition (w_n) considered in Lemma 7 and in virtue of that lemma
\[ a_{n-i} \overline{a}_i - \overline{a}_{n-i} = 0 \quad (i = 1, 2, \ldots, n-1), \]
where $\overline{a}_i = e^{\frac{1+\sqrt[5]{5}}{2}}$.

(35) gives $a_0 \overline{a}_i - \overline{a}_{n-i} = 0$ and on passing to complex conjugates we get
\[ -a_{n-i} \overline{a}_i - a_0 \overline{a}_i = 0 \quad (i = 1, 2, \ldots, n-1). \]
Since
\[ |a_i|^{n-1} = |a_i|^{n-1} = \frac{1+\sqrt[5]{5}}{2} \neq 0, \]
(35) and (35') imply $a_i = a_0 = 0$ for $i = 1, 2, \ldots, n-1$. Hence $f(z) = z^n + a_0$.

**Proof of Theorem 3.** Assume first that
\[ P(0) = e^{\frac{1+\sqrt[5]{5}}{2}}. \]
Let
\[ P(z) = f(z) \prod_{i=1}^{n-k+1} (z - \varepsilon_i), \]
where $\varepsilon_i$ are roots of unity, but no zero of $f(z)$ is a root of unity. The product $\prod (z - \varepsilon_i)$ divides $(z^{n-1})^m$ for a suitable $m$; hence
\[ \prod_{i=1}^{n-k+1} (z - \varepsilon_i) = [P(z), (z^{n-1})^m] \in K[z] \]
and $f(z) \in K[z]$, $f$ is monic with integer coefficients and $f(0) = e^{\frac{1+\sqrt[5]{5}}{2}}$.

For $\sigma \in G$ let
\[ \sigma f(z) = \prod_{i=1}^n (z - a_i), \]
By the assumption about \( P(x) \), we have
\[
|a_{\alpha}| \geq 1 \text{ if } \sigma(\sqrt{5}) = \sqrt{5}, \\
|a_{\alpha}| \leq 1 \text{ if } \sigma(\sqrt{5}) = -\sqrt{5}.
\]
(37)

Suppose that for a \( \sigma \in G \) and a \( j \leq k \) we have
\[
|a_{\alpha_{j}}| = 1.
\]
(38)

Consider the field \( L = \mathbb{K}^{\tau}(a_{\alpha_{j}}) \) and any isomorphic injection \( \tau \) of \( L \)
into \( C \). We have \( \sigma_{\alpha_{j}} \tau \sqrt{5} = \alpha_{j} \sqrt{5} \); thus \( \tau \sigma_{\alpha_{j}} \sqrt{5} = \tau \alpha_{j} \sqrt{5} \).

If \( \tau \sigma_{j} \sqrt{5} = \sqrt{5} \), then, since \( \tau \sigma_{\alpha_{j}}(\alpha_{\alpha_{j}}) = \tau \sigma_{\alpha_{j}}(\tau \alpha_{j}) = 0 \), we have
by (36) and (37)
\[
|\tau \alpha_{\alpha_{j}}| \geq 1 \quad \text{and} \quad |\tau \alpha_{\alpha_{j}}| \geq 1.
\]

If \( \tau \alpha_{\alpha_{j}} = \tau \alpha_{\alpha_{j}} \sqrt{5} = -\sqrt{5} \) we have similarly
\[
|\tau \alpha_{\alpha_{j}}| \leq 1 \quad \text{and} \quad |\tau \alpha_{\alpha_{j}}| \leq 1.
\]

On the other hand, by (38)
\[
|\tau \alpha_{\alpha_{j}}| \tau \alpha_{\alpha_{j}} = \tau(a_{\alpha_{j}} \alpha_{\alpha_{j}}) = |\tau \alpha_{\alpha_{j}}|^{2} = 1;
\]

thus \( |\tau \alpha_{\alpha_{j}}| = 1 \) for all \( \tau \). By Lemma 3, \( a_{\alpha_{j}} \) is a root of unity and by
(36) a certain conjugate of it is a zero of \( f(x) \), contrary to the definition of \( f \). The contradiction obtained above shows that \( f \) satisfies all the assumptions of Lemma 8 and in virtue of that lemma
\[
f(x) = e^{x} + e^{-x} \frac{1 + \sqrt{5}}{2}.
\]

Assume now that \( P(0) = \frac{e^{\frac{1}{2}} - \sqrt{5}}{2} \). Then, for a \( \sigma \) with \( \sigma \sqrt{5} = -\sqrt{5} \),
\[
\sigma P \text{ satisfies the assumptions of the theorem and } \sigma P(0) = \frac{1 - \sqrt{5}}{2} = \sigma(\varepsilon).
\]

Thus, by the already proved case of the theorem, formula (3) holds.

Proof of Theorem 2. If (3) holds, we clearly have equality in (2). Suppose that for a polynomial \( P \in K[x] \) with the leading coefficient \( p_{n} \), the equality in (2) is obtained. By the equality \( C(P) = (p_{n}) \) quoted in the introduction, \( P_{0}(x) = p_{n}^{-1}P(x) \) has integral coefficients. Moreover \( \sqrt{5} \in K \) and \( |P_{0}(0)| = \frac{1 + \sqrt{5}}{2} \). Since for \( \alpha \in K \), \( \sigma \in G \nabla \)
\[
\sigma |a_{\alpha}|^{2} = |\sigma a_{\alpha}|^{2},
\]
we have for all \( \sigma \in G \)
\[
\frac{1 + p_{n} \varepsilon^{-1}}{2} \in C \text{ is an integer, and hence by Lemma 4, } \frac{1 + \sqrt{5}}{2} \text{ is also one.}
\]

Thus
\[
P_{0}(x) = e^{x} + p_{1} e^{-x} + \ldots + e^{x} \frac{1 + \sqrt{5}}{2},
\]
where \( p_{i} \) are integers of \( K \). On the other hand,
\[
\prod_{\alpha \in \sigma} \prod_{|a_{\alpha}| \geq 1} |a_{\alpha}|^{2} = \prod_{\alpha \in \sigma} \frac{1 + \sqrt{5}}{2} \prod_{|a_{\alpha}| = 1} \prod_{|a_{\alpha}| < 1} \prod_{|a_{\alpha}| = 1} |a_{\alpha}|^{2} = 1,
\]

and the equality in (2) implies that both double products on the right-hand side of (39) are empty. Thus \( P_{0}(x) \) satisfies the assumptions of Theorem 3 and in virtue of that Theorem 3 holds.

References


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