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On the product of the conjugates outside the unit circle of an algebraic integer

by

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The aim of this paper is to extend some results of A. Schinzel [4] and to make them more precise.

Let K be a number field of degree $|K|$, let

$$P(z) = p_0 z^n + p_1 z^{n-1} + \ldots + p_n$$

be a polynomial over K with the content $C(P) = (p_0, \ldots, p_n)$, let G be the set of all isomorphic injections of K into the complex field C and, for $\sigma \in G$, let

$$\sigma P(z) = \sigma p_0 z^n + \ldots + \sigma p_n = \sigma p_0 \prod_{i=1}^{n} (z - \alpha_{\sigma i}).$$

Generalizing an argument of Smyth [5] concerning the fundamental case $K = Q$, Schinzel proved that if K is totally real, $P(z)$ is non-reciprocal, p_i are integers, $p_0 = 1$ and $p_n \neq 0$, then

$$\max_{\sigma \in G} \prod_{|\alpha_{\sigma j}| > 1} |\alpha_{\sigma j}| \geqslant \theta_0, \tag{1}$$

where θ_0 is the real root of the equation $\theta^3 - \theta - 1 = 0$. We extend this in the following manner.

Theorem 1. *Let K be a totally complex quadratic extension of a totally real field and $\sqrt{-3} \notin K$.*

If $P(z) \in K[z]$ is a monic polynomial with integer coefficients, $P(0) \neq 0$, $z^n \overline{P}(z^{-1}) \neq \mathrm{const}\, P(z)$, then (1) holds.

If $\sqrt{-3} \in K$ (1) needs not be satisfied, but $\Lambda \geqslant |\theta_1|$ where θ_1 is that root of the equation

$$\theta^2 + \frac{-1+\sqrt{-3}}{2}\theta - 1 = 0$$

which is greater in absolute value.

For K being a totally complex quadratic extension of a totally real field Schinzel considered the product

$$\Pi = \prod_{\sigma \in G} \prod_{|\alpha_{\sigma j}|>1} |\alpha_{\sigma j}|$$

and proved that if $p_n \neq 0$ and $|p_n| \neq |p_0|$, then

$$\Pi \geqslant \left(\frac{1+\sqrt{5}}{2}\right)^{|K|/2} \left(N_{K/Q}\frac{C(P)}{(p_0)}\right)^{\frac{1}{2}+\frac{1}{\sqrt{5}}} \left(N_{K/Q}\frac{(p_n)}{C(P)}\right)^{\frac{1}{2}-\frac{1}{\sqrt{5}}} \tag{2}$$

with the equality possible only if $\sqrt{5} \in K$, $C(P) = (p_0)$,

$$\left|\frac{p_n}{p_0}\right| = \frac{\pm 1+\sqrt{5}}{2}.$$

He made a conjecture about the possible form of the polynomial P for which the equality in (2) is attained.

We prove this conjecture as:

THEOREM 2. *The equality in* (2) *is attained if and only if*

$$P(z) = p_0\left(z^k + \varepsilon_1 \frac{1 \pm \sqrt{5}}{2}\right) \prod_{i=2}^{n-k+1} (z-\varepsilon_i) \tag{3}$$

where ε_i are roots of unity.

This theorem is an easy consequence of the following:

THEOREM 3. *Let K be a totally complex quadratic extension of a totally real field.*

If $P(z) \in K[z]$ is a monic polynomial with integer coefficients, $P(0) = \varepsilon \dfrac{1 \pm \sqrt{5}}{2}$ and

$$|\alpha_{\sigma j}| \geqslant 1 \quad \text{if} \quad \sigma(\sqrt{5}) = \pm\sqrt{5} \quad (j = 1, 2, \ldots, n),$$
$$|\alpha_{\sigma j}| \leqslant 1 \quad \text{if} \quad \sigma(\sqrt{5}) = \mp\sqrt{5} \quad (j = 1, 2, \ldots, n),$$

then (3) *holds.*

The proofs are based on several lemmata.

LEMMA 1. *Let $f(z) = \sum_i e_i z^i$ be a function holomorphic in an open disc containing $|z| \leqslant 1$ and satisfying $|f(z)| \leqslant 1$ for $|z| = 1$. Then*

$$|e_i| \leqslant 1 - |e_0|^2 \quad (i = 1, 2, \ldots), \tag{4}$$

$$\left|e_{2i} + \frac{e_i^2 \bar{e}_0}{1-|e_0|^2}\right| \leqslant 1 - |e_0|^2 - \frac{|e_i|^2}{1-|e_0|^2} \quad (i = 1, 2, \ldots). \tag{5}$$

Proof (due to A. Schinzel). Inequality (4) is proved in [4]. In order to prove (5) let us observe, following Smyth [5], that for all β_0, β_1

$$\int_{|z|=1} |f(z)|^2 |\beta_0 + \beta_1 z^i + z^{2i}|^2 dz \leqslant \int_{|z|=1} |\beta_0 + \beta_1 z^i + z^{2i}|\, dz;$$

thus

$$|e_0\beta_0|^2 + |e_0\beta_1 + e_1\beta_0|^2 + |e_0 + e_i\beta_1 + e_{2i}\beta_0|^2 \leqslant 1 + |\beta_0|^2 + |\beta_1|^2,$$

and setting

$$\beta_0 = pe_0^{-1}; \qquad \beta_1 = \frac{pe_i}{e_0(\delta - e_0)}$$

where $|p| = |\delta| = 1$, we obtain

$$\left|e_0 + \frac{e_i^2 p}{(\delta - e_0)e_0} + \frac{pe_{2i}}{e_0}\right| \leqslant |e_0|^{-1}.$$

Hence

$$\left|e_0^2 p^{-1} + \frac{e_i^2}{\delta - e_0} + e_{2i}\right| \leqslant 1. \tag{6}$$

For an arbitrary ε with $|\varepsilon| = 1$ the number

$$\delta = \frac{e_0\varepsilon + 1}{\varepsilon + \bar{e}_0}$$

has absolute value 1.

We set

$$\xi = e_{2i} + \frac{\bar{e}_0 e_i^2}{1-|e_0|^2}, \qquad p = \frac{|\xi|}{\xi}\,\frac{e_0^2}{|e_0|^2}, \qquad \varepsilon = \frac{\xi}{|\xi|}\,\frac{|e_i|^2}{e_i^2}.$$

Then

$$e_0^2 p^{-1} + \frac{e_i^2}{\delta - e_0} + e_{2i} = \frac{\xi}{|\xi|}|e_0|^2 + e_i^2 \frac{\bar{e}_0 + \varepsilon}{1-|e_0|^2} + e_{2i}$$
$$= \frac{\xi}{|\xi|}\left(|e_0|^2 + \frac{|e_i|^2}{1-|e_0|^2} + |\xi|\right).$$

Hence by (6)

$$\left|\frac{\xi}{|\xi|}\left(|\xi| + |e_0|^2 + \frac{|e_i|^2}{1-|e_0|^2}\right)\right| \leqslant 1 \quad \text{and} \quad |\xi| \leqslant 1 - |e_0|^2 - \frac{|e_i|^2}{1-|e_0|^2},$$

which proves the lemma.

LEMMA 2. *Let $P(z) = z^n + p_1 z^{n-1} + \ldots + p_n$, $|p_n| = 1$, $Q(z) = z^n \bar{P}(z^{-1}) \neq \text{const}\, P(z)$. Then*

$$\frac{\overline{P(0)}\,P(z)}{Q(z)} = \frac{f(z)}{g(z)},$$

where f and g are holomorphic in an open disc containing $|z| \leqslant 1$,

$$f(0) = g(0) = \prod_{|a_j|>1} |a_j|^{-1}, \quad |f(z)| = |g(z)| = 1 \quad \text{for } |z| = 1,$$

and if the coefficients of P are real, the coefficients of f and g are also real.

Proof (cf. [4], Lemma 2). We set

$$f(z) = \prod_{|a_j|<1} \frac{|a_j|}{(-a_j)} \frac{z-a_j}{1-\bar{a}_j z}, \quad g(z) = \prod_{|a_j|>1} \frac{(-a_j)}{|a_j|} \frac{1-\bar{a}_j z}{z-a_j},$$

and using the equalities

$$\prod_{i=1}^{n} (-a_j) = P(0), \quad |P(0)| = 1,$$

we easily verify all the assertions of the lemma. Note that if $|a_j| = 1$, the factor $z-a_j$ occurs both in $P(z)$ and in $Q(z)$. Also if $P(z)$ has real coefficients,

$$\overline{f(z)} = f(\bar{z}), \quad \overline{g(z)} = g(\bar{z}).$$

Lemma 3 (Kronecker [2]). *If $\alpha \neq 0$ is an algebraic integer with $\overline{|\alpha|} \leqslant 1$, then α is a root of unity.*

If α is a totally real algebraic integer with $\overline{|\alpha|} \leqslant 2$, then $\alpha = 2\cos w\pi$, where w is rational.

Lemma 4. *If $\alpha \neq 0$ is an algebraic integer of a field K satisfying the assumptions of Theorem 1, then either α is a root of unity or $\overline{|\alpha|} \geqslant \sqrt{2}$.*

Proof. By the first part of Lemma 3 we can assume that $\overline{|\alpha|} \geqslant 1$. For all $\sigma \in G$ we have

$$\sigma(|\alpha|^2) = \sigma\alpha \cdot \sigma\bar{\alpha} = \sigma\alpha\overline{\sigma\alpha} = |\sigma\alpha|^2;$$

thus $|\alpha|^2$ is totally real and totally positive, $|\alpha|$ is totally real and $\overline{|\alpha|} = \overline{||\alpha||}$.

On the other hand, by the second part of Lemma 3,

$$\overline{||\alpha||} \geqslant 2\cos\frac{2\pi}{4} = \sqrt{2}.$$

Proof of Theorem 1. Let

$$\Lambda = \max_{\sigma \in G} \prod_{|a_{\sigma j}|>1} |a_{\sigma j}|.$$

Since $\sqrt{2} > \theta_0$, we can assume that $\Lambda \leqslant \sqrt{2}$. It follows that $\overline{|P(0)|} = 1$ since otherwise by Lemma 4

$$\Lambda \geqslant \overline{|P(0)|} \geqslant \sqrt{2}.$$

The assumption $P(z)/Q(z) \neq \text{const}$ implies that

$$R(z) = \frac{\overline{P(0)}P(z)}{Q(z)} = 1 + a_k z^k + a_l z^l + \ldots, \tag{7}$$

where on the right-hand side we have infinitely many non-zero coefficients, a_k, a_l being the first two of them. Since a_i are integers of K, $\overline{|a_i|} \geqslant 1$ for $i = k, l$.

Using Lemma 4, we distinguish two cases.

The case $|a_k| \geqslant \sqrt{2}$. Since $\Lambda(P) = \Lambda(\sigma P)$, we can assume $|a_k| \geqslant \sqrt{2}$ replacing if necessary P by a suitable σP. Applying Lemma 2 to P, we get

$$\frac{\overline{P(0)}P(z)}{Q(z)} = \frac{f(z)}{g(z)} = \frac{c + c_1 z + c_2 z^2 + \ldots}{d + d_1 z + d_2 z^2 + \ldots}; \tag{8}$$

$$|f(z)| = |g(z)| = 1 \quad \text{for} \quad |z| = 1, \quad f, g \text{ holomorphic for } |z| \leqslant 1;$$

$$f(0) = g(0) = \prod_{|a_j|>1} |a_j|^{-1} = c = d.$$

Comparing (7) with (8), we get

$$\begin{cases} c_i = d_i \ (i = 1, 2, \ldots, k-1), \ c_k = d_k + a_k c, \\ c_{k+i} = d_{k+i} + a_k d_i \ (i = 1, 2, \ldots, l-k-1), \ c_l = d_l + a_l c + a_k d_{l-k}. \end{cases} \tag{9}$$

It follows from $c_k = d_k + a_k c$, by Lemma 1, that

$$\sqrt{2}c \leqslant |a_k||c| \leqslant |c_k| + |d_k| \leqslant 2 - 2|c|^2, \quad \sqrt{2} \leqslant c^{-1} \leqslant \Lambda.$$

The case $\overline{|a_k|} = 1$, a_k is a root of unity. Let η be a root of unity, $P_\eta(z) = \eta^n P(\eta^{-1}z)$. We have $\Lambda(P_\eta) = \Lambda(P)$, P_η and $K(\eta)$ satisfying the assumptions of the theorem.

Setting

$$R_\eta(z) = \frac{\overline{P_\eta(0)}P_\eta(z)}{z^n \bar{P}_\eta(z^{-1})} = 1 + \sum_{i=1}^{\infty} a_i' z^i,$$

we get

$$R_\eta(z) = \frac{\eta^{-n}\overline{P(0)}\eta^n P(\eta^{-1}z)}{z^n \eta^{-n} \bar{P}(\eta z^{-1})} = \frac{\overline{P(0)}P(\eta^{-1}z)}{Q(\eta^{-1}z)} = R(\eta^{-1}z);$$

hence for all i

$$a_i' = \bar{\eta}^i a_i, \quad |a_i| = |a_i'|. \tag{9a}$$

Taking $\eta = \sqrt[k]{a_k}$, we get

$$a_k' = 1, \quad a_i' = 0 \ (0 < i < k), \quad a_{2k}' \in K.$$

Therefore, without loss of generality we assume that $a_k = 1$ if $l < 2k$ and $a_k = \pm 1$, $a_{2k} \in K$ if $l \geqslant 2k$ (we admit both signs here for the sake of symmetry).

The case $l < 2k$, $a_k = 1$. Applying to P a suitable $\sigma \in G$, we can obtain $|a_l| \geqslant 1$. We shall exploit the following inequality, due to Smyth ([5], pp. 172, 173):

$$E = \frac{5}{4}|c|^2 + |c_{l-k} + \gamma c|^2 + \left|\frac{a_l c + c_{l-k}}{2} + \frac{\gamma c}{2} - c_{l-k} + \beta c\right|^2 \leqslant 2 + |\beta|^2 + |\gamma|^2, \tag{10}$$

where β and γ are arbitrary complex numbers.

Put $F(\beta, \gamma, c_{l-k}) = E - |\beta|^2 - |\gamma|^2$. $|t|^2 F\left(\frac{\beta}{t}, \frac{\gamma}{t}, \frac{c_{l-k}}{t}\right)$ is a hermitian form with the matrix

$$M = \begin{bmatrix} |c|^2 - 1 & \frac{|c|^2}{2} & -\frac{c}{2} & \bar{a}_l \frac{|c|^2}{2} \\ \frac{|c|^2}{2} & \frac{5}{4}|c|^2 - 1 & \frac{3}{4}c & \frac{1}{4}\bar{a}_l |c|^2 \\ -\frac{\bar{c}}{2} & \frac{3}{4}\bar{c} & \frac{5}{4} & -\frac{1}{4}\bar{a}_l \bar{c} \\ a_l \frac{|c|^2}{2} & \frac{1}{4}a_l |c|^2 & \frac{1}{4}a_l c & \frac{5}{4}|c|^2 + \frac{|a_l|^2 |c|^2}{4} \end{bmatrix}$$

with diagonal minors

$$\begin{aligned} &M_1 = |c|^2 - 1 < 0, \qquad M_2 = |c|^4 - \tfrac{9}{4}|c|^2 - 1 > 0, \\ &M_3 = \tfrac{5}{4} - 2|c|^2 > 0, \qquad M_4 = \tfrac{25}{16}|c|^2 - \tfrac{5}{2}|c|^4 + \frac{|c|^2 |a_l|^2}{4} \geqslant \tfrac{29}{16}|c|^2 - \tfrac{5}{2}|c|^4. \end{aligned} \tag{11}$$

In order to justify the second inequality we notice that the equation $c_k = a_k + c$ implies by Lemma 1

$$2 - 2c^2 \geqslant c, \qquad c \leqslant \frac{\sqrt{17} - 1}{4},$$

and since c is an algebraic integer,

$$c < \frac{\sqrt{17} - 1}{4}, \qquad |c|^4 - \frac{9}{4}|c|^2 + 1 > 0.$$

Now

$$F(\beta, \gamma, c_{l-k}) = M_1 |\beta + \ldots|^2 + \frac{M_2}{M_1}|\gamma + \ldots|^2 + \frac{M_3}{M_2}|c_{l-k} + \ldots|^2 + \frac{M_4}{M_3}$$

(see, e.g. [3], p. 461), from (10) we get

$$\frac{M_4}{M_3} = \min_{c_{l-k}} \max_{\beta, \gamma} F(\beta, \gamma, c_{l-k}) \leqslant 2$$

and from (11) we get

$$40|c|^4 - 93|c|^2 + 40 \geqslant 16(2M_3 - M_4) \geqslant 0.$$

As proved by Smyth, the latter inequality implies $\Lambda = c^{-1} > \theta_0$.

The case $l \geqslant 2k$, $a_k = \pm 1$, $a_{2k} \in K$. By (9)

$$c_{2k} = d_{2k} + a_k d_k + a_{2k} c.$$

On applying (5) to c_{2k} and d_{2k} and adding the resulting inequalities, we get

$$\left|c_{2k} - d_{2k} + \frac{c_k^2 - d_k^2}{1 - |c|^2} c\right| = \left|a_{2k} c + a_k d_k + \frac{c_k^2 - d_k^2}{1 - c^2} c\right| \leqslant 2 - 2c^2 - \frac{|c_k|^2}{1 - c^2} - \frac{|d_k|^2}{1 - c^2} \tag{12}$$

(c is real). We now set

$$c_k = c_k^{(1)} + ic_k^{(2)}, \qquad d_k = d_k^{(1)} + id_k^{(2)}, \qquad a_{2k} = a_{2k}^{(1)} + ia_{2k}^{(2)},$$

where $c_k^{(i)}$, $d_k^{(i)}$, $a_{2k}^{(i)}$ are real for $i = 1, 2$; and we get from $c_k = d_k + a_k c$ the equations

$$c_k^{(2)} = d_k^{(2)}, \qquad c_k^2 - d_k^2 = c_k^{(1)2} - d_k^{(1)2} + 2ia_k c d_k^{(2)}.$$

The inequality $|x| \geqslant |\mathrm{Re}\, x|$ applied to (12) gives

$$\left|a_{2k}^{(1)} c + a_k d_k^{(1)} + \frac{c_k^{(1)2} - d_k^{(1)2}}{1 - c^2} c\right| \leqslant 2 - 2c^2 - \frac{c_k^{(1)2}}{1 - c^2} - \frac{d_k^{(1)2}}{1 - c^2}. \tag{13}$$

The left-hand side of (13) is greater than or equal to

$$|a_{2k}^{(1)} c + a_k d_k^{(1)}| - \left|\frac{c_k^{(1)2} - d_k^{(1)2}}{1 - c^2}\right| c = |a_{2k}^{(1)} c + a_k d_k^{(1)}| \pm \left(\frac{c_k^{(1)2} - d_k^{(1)2}}{1 - c^2} c\right).$$

Hence

$$|a_{2k}^{(1)} c + a_k d_k^{(1)}| \leqslant 2 - 2c^2 - \min\left(\frac{c_k^{(1)2}}{1 + c} + \frac{d_k^{(1)2}}{1 - c}, \frac{c_k^{(1)2}}{1 - c} + \frac{d_k^{(1)2}}{1 + c}\right). \tag{14}$$

Since $c_k^{(1)} = \pm c + d_k^{(1)}$, we have

$$|c_k^{(1)}| + |d_k^{(1)}| = c, \tag{15}$$

for otherwise by Lemma 1

$$1 - c^4 \geqslant c, \qquad \Lambda = c^{-1} \geqslant \sqrt{2}.$$

Again by Lemma 1

$$(16)\qquad |c_k^{(1)}| \leqslant |c_k| \leqslant 1-c^2, \qquad |d_k^{(1)}| \leqslant |d_k| \leqslant 1-c^2.$$

By (15) and (16)

$$(17)\qquad c^2+c-1 \leqslant |c_k^{(1)}| \leqslant 1-c^2, \qquad c^2+c-1 \leqslant |d_k^{(1)}| \leqslant 1-c^2.$$

The further argument depends on $a_{2k}^{(1)}$.

We distinquish three cases: X. $|\overline{a_{2k}^{(1)}}| \geqslant 1$, Y. $a_{2k}^{(1)} = 0$, Z. $0 < |\overline{a_{2k}^{(1)}}| < 1$.

X. Applying to P a suitable $\sigma \in G$, we can obtain $|a_{2k}^{(1)}| \geqslant 1$. By (17)

$$(18)\qquad |a_{2k}^{(1)}c \pm d_k^{(1)}| \geqslant |a_{2k}^{(1)}|c - |d_k^{(1)}| \geqslant c^2+c-1.$$

By (14) and (18)

$$c^2+c-1 \leqslant M = \max_{c^2+c-1 \leqslant x \leqslant 1-c^2} \left(2-2c^2-\frac{x^2}{1+c}-\frac{(c-x)^2}{1-c}\right).$$

As proved by Smyth ([5], p. 175), the latter inequality implies $\Lambda = c^{-1} \geqslant \theta_0$.

Y. By (17)

$$|a_{2k}^{(1)}c \pm d_k^{(1)}| = |d_k^{(1)}| \geqslant c^2+c-1.$$

By (14)

$$c^2+c-1 \leqslant M \quad \text{and} \quad \Lambda \geqslant \theta_0$$

as in X.

Z. Since $|\overline{2a_{2k}^{(1)}}| < 2$ and $2a_{2k}^{(1)} = a_{2k}+\bar{a}_{2k}$ is a totally real algebraic integer, we have by Lemma 3

$$a_{2k}^{(1)} = \cos 2w\pi, \quad w \text{ rational}.$$

Since $c_k^{(1)} = \pm c + d_k^{(1)}$, we have by (14)

$$|c_k^{(1)} \pm (a_{2k}^{(1)}-1)c| \leqslant 2-2c^2-\min\left(\frac{c_k^{(1)2}}{1+c}+\frac{d_k^{(1)2}}{1-c}, \frac{c_k^{(1)2}}{1-c}+\frac{d_k^{(1)2}}{1+c}\right).$$

If $|\overline{a_{2k}^{(1)}-1}| \geqslant 1$ or $a_{2k}^{(1)}-1 = 0$, then the situation differs from that occurring in the case X or Y only by the permutation of $c_k^{(1)}$ and $d_k^{(1)}$. Let $|\overline{a_{2k}^{(1)}-1}| < 1$.

We have

$$a_{2k}^{(1)}-1 = \cos 2w\pi - 1 = -2\sin^2 w\pi;$$

hence

$$|\overline{\sin^2 w\pi}| < \frac{1}{2}, \quad |\overline{\sin w\pi}| < \frac{1}{\sqrt{2}}, \quad |\overline{2\sin w\pi}| < \sqrt{2}.$$

$2\sin w\pi$ is a totally real algebraic integer, and hence, by Lemma 4, $2\sin w\pi = \pm 1$, $a_{2k}^{(1)} = 1-2\sin^2 w\pi = \frac{1}{2}$, $a_{2k} = \frac{1}{2}+ia_{2k}^{(2)}$. If $\sqrt{-3} \notin K$, it is impossible to have $a_{2k}^{(2)} = \pm\sqrt{3}/2$, and thus a_{2k} is not a root of unity. Hence by Lemma 4 $|\overline{a_{2k}}| \geqslant \sqrt{2}$ and applying to P a suitable $\sigma \in G$, we can obtain

$$|a_{2k}| \geqslant \sqrt{2}, \qquad |a_{2k}^{(2)}| \geqslant \tfrac{1}{2}\sqrt{7}.$$

We now replace P by $P_\eta(z)$, where $\eta^k = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$. By (9a) we get

$$(19)\qquad a_k' = \left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)a_k, \quad a_{2k}' = ia_{2k}, \qquad a_i' = 0 \quad \text{for} \quad i \leqslant 2k,\ i \neq k, 2k.$$

$\tilde{P}(z) = P(z)\bar{P}(z)$ is a polynomial with totally real coefficients.

$$(20)\qquad \frac{\tilde{P}(z)}{\tilde{Q}(z)} = \frac{\overline{P(0)}P(z)}{Q(z)}\,\frac{P(0)\bar{P}(z)}{\bar{Q}(z)}$$
$$= (1+a_k'z^k+a_{2k}'z^{2k}+\ldots)(1+\bar{a}_k'z^k+\bar{a}_{2k}'z^{2k}+\ldots)$$
$$= 1+\sqrt{2}a_kz^k+(1+a_{2k}'+\bar{a}_{2k}')z^{2k}+\ldots = 1+b_kz^k+b_{2k}z^{2k}+\ldots$$

By Lemma 2

$$(21)\qquad \frac{P(z)}{Q(z)} = \frac{e_0+e_1z+\ldots}{f_0+f_1z+\ldots},$$

where $f_0 = e_0 = c^2$ and e_i, f_i are real.

The series occurring on the right side of (21) are convergent in an open disc containing $|z| \leqslant 1$ and have absolute value 1 on the circle $|z| = 1$.

By the inequality (14)

$$(22)\qquad |b_{2k}e_0+a_k\sqrt{2}f_k| \leqslant 2-2e_0^2-\min\left(\frac{e_k^2}{1+e_0}+\frac{f_k^2}{1-e_0}, \frac{e_k^2}{1-e_0}+\frac{f_k^2}{1+e_0}\right).$$

By (20) and (21)

$$(23)\qquad e_i = f_i,\ i<k; \qquad e_k = a_k\sqrt{2}e_0+f_k, \qquad e_{2k} = b_{2k}e_0+f_{2k}+a_k\sqrt{2}f_k.$$

The equality $e_k = f_k+a_k\sqrt{2}e_0$ implies

$$(24)\qquad |e_k|+|f_k| = \sqrt{2}e_0,$$

for otherwise, by Lemma 1, $\sqrt{2}e_0 \leqslant 1-e_0^2$, and

$$e_0^{-1} \geqslant \frac{\sqrt{2}+\sqrt{6}}{2} > 1.9 > \theta_0^2, \qquad \Lambda = c^{-1} > \theta_0.$$

By (24) and (22)

(25) $$|b_{2k}e_0 + a_k\sqrt{2}f_k| \leqslant \tilde{M} = \max\varphi(x),$$

where

$$\varphi(x) = 2-2e_0^2 - \frac{x^2}{1+e_0} - \frac{(\sqrt{2}e_0 - x)^2}{1-e_0}.$$

We have

$$\frac{1}{2}\varphi'(x) = \frac{-x}{1+e_0} - \frac{x-\sqrt{2}e_0}{1-e_0} = \frac{-2x+\sqrt{2}e_0(1+e_0)}{1-e_0^2};$$

thus the maximum of $\varphi(x)$ taken for $x = \frac{\sqrt{2}}{2}e_0(1+e_0)$ equals

$$M = \varphi\left(\frac{1}{\sqrt{2}}e_0(1+e_0)\right) = 2-2e_0^2 - \frac{1}{2}e_0^2(1+e_0) - \frac{1}{2}e_0^2(1-e_0) = 2-3e_0^2.$$

From the equality $f_k + a_k\sqrt{2}e_0 = e_k$ we get by (25)

$$|b'_{2k}e_0 + a_k\sqrt{2}e_k| \leqslant 2-3e_0^2, \quad \text{where} \quad b'_{2k} = b_{2k}-2.$$

Since $a'_{2k} = -a^{(2)}_{2k} + i\frac{1}{2}$, $b_{2k} = 1-2a^{(2)}_{2k}$, we have $b'_{2k} = -1-2a^{(2)}_{2k}$. Replacing, if necessary, f_k by e_k and b_{2k} by b'_{2k}, we can thus assume that $|b_{2k}| \geqslant 1+\sqrt{7}$.

Hence

$$|b_{2k}e_0 + a_k\sqrt{2}e_k| \geqslant |b_{2k}e_0| - \sqrt{2}|e_k| \geqslant \sqrt{2}e_0^2 + (1+\sqrt{7})e_0 - \sqrt{2}$$

and by (25)

$$2-3e_0^2 \geqslant \sqrt{2}e_0^2 + (1+\sqrt{7})e_0-\sqrt{2},$$

$$f(e_0^{-1}) \geqslant 0,$$

where $f(x) = (2+\sqrt{2})x^2 - (1+\sqrt{7})x - (3+\sqrt{2})$.

Since $f(\frac{16}{9}) < 0$ and $e_0 = c^2$, we have $\Lambda = c^{-1} > \frac{4}{3} > \theta_0$.

Consider now the case

$$a_k = \pm 1, \quad a_{2k} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \quad \sqrt{-3}\in K.$$

It follows from (9a) that

$$c_k^2 - d_k^2 = a_k c(2d_k^{(1)} + a_k c) + 2a_k c d_k^{(2)}$$

and from (15) that

$$|c_k|^2 + |d_k|^2 = |d_k^{(1)}|^2 + 2|d_k^{(2)}|^2 + (c-|d_k^{(1)}|)^2.$$

The inequality (12) implies in virtue of the above identities

$$M_1 = 2-2c^2 - \frac{|d_k^{(1)}|^2 + (c-|d_k^{(1)}|)^2 + 2|d_k^{(2)}|^2}{1-c^2} - \left|\left(\frac{\frac{1}{2}c+\frac{1}{2}c^3}{1-c^2} + a_k\frac{1+c^2}{1-c^2}d_k^{(1)}\right) + i\left(\pm\frac{\sqrt{3}}{2}c + a_k\frac{1+c^2}{1-c^2}d_k^{(2)}\right)\right| \geqslant 0.$$

Hence it follows that

$$M_2 = \max_{|d_k^{(2)}|}\max_{|d_k^{(1)}|}\left(2-2c^2 - \frac{|d_k^{(1)}|^2 + (c-|d_k^{(1)}|)^2 + 2|d_k^{(2)}|^2}{1-c^2} - \left|\frac{1+c^2}{1-c^2}\left(\frac{1}{2}c - |d_k^{(1)}|\right) + i\left(\frac{\sqrt{3}}{2}c - \frac{1+c^2}{1-c^2}|d_k^{(2)}|\right)\right|\right) \geqslant 0.$$

The inner maximum is attained for $|d_k^{(1)}| = c/2$ since then both

$$|d_k^{(1)}|^2 + (c-|d_k^{(1)}|)^2 \quad \text{and} \quad \left|\frac{1+c^2}{1-c^2}\left(\frac{1}{2}c - |d_k^{(1)}|\right)\right|$$

attain the minimal value. Thus

$$M_2 = \max_{|d_k^{(2)}|}\left(2-2c^2-\frac{c^2}{2(1-c^2)} - \frac{2|d_k^{(2)}|^2}{1-c^2} - \left|\frac{\sqrt{3}}{2}c - \frac{1+c^2}{1-c^2}|d_k^{(2)}|\right|\right)\geqslant 0.$$

We set

$$g(x) = 2-2c^2 - \frac{c^2}{2(1-c^2)} - \frac{2x^2}{1-c^2} - \left|\frac{\sqrt{3}}{2}c - \frac{1+c^2}{1-c^2}x\right|.$$

In the interval $\left(0, \frac{\sqrt{3}}{2}c\frac{1-c^2}{1+c^2}\right)$ the function $g(x)$ is increasing. Indeed we have in this interval

$$\frac{\sqrt{3}}{2}c - \frac{1+c^2}{1-c^2}x > 0, \quad g'(x) = -\frac{4x}{1-c^2} + \frac{1+c^2}{1-c^2} \geqslant \frac{-2\sqrt{3}c}{1+c^2} + \frac{1+c^2}{1-c^2}.$$

On the other hand, by the assumption $\Lambda < \sqrt{2}$ we have

$$c^2 > \tfrac{1}{2}, \quad (1+c^2)^2 > \tfrac{9}{4},$$

$$2\sqrt{3}c(1-c^2) < \sqrt{3}c < \sqrt{3} < \tfrac{9}{4} < (1+c^2)^2.$$

For $x > \frac{\sqrt{3}}{2}c\frac{1-c^2}{1+c^2}$ we have

$$\frac{\sqrt{3}}{2}c - \frac{1+c^2}{1-c^2}x < 0, \quad g'(x) = -\frac{4x}{1-c^2} - \frac{1+c^2}{1-c^2} < 0;$$

thus the function $g(x)$ is decreasing. Since it is continuous, we have

$$M_2 = g\left(\frac{\sqrt{3}}{2}c\frac{1-c^2}{1+c^2}\right),$$

$$g\left(\frac{\sqrt{3}}{2}c\frac{1-c^2}{1+c^2}\right) = 2-c^2-\frac{c^2}{2(1-c^2)}-\frac{3}{2}c^2\frac{(1-c^2)^2}{(1+c^2)^2(1-c^2)} \geqslant 0,$$

and on simplification

$$1-c^2-c^4-c^6+c^8 \geqslant 0,$$

whence $\Lambda^8-\Lambda^6-\Lambda^4-\Lambda^2+1 \geqslant 0$. The equation $x^4-x^3-x^2-x+1=0$ has only one real root greater than 1, namely

$$\tfrac{1}{4}\left(1+\sqrt{13}+\sqrt{2\sqrt{13}-2}\right).$$

It follows that

$$\Lambda \geqslant \tfrac{1}{2}\sqrt{1+\sqrt{13}+\sqrt{2\sqrt{13}-2}}.$$

On the other hand, the polynomial $z^2+\varrho z-1$, $\varrho = \dfrac{-1+\sqrt{-3}}{2}$ has two zeros given by the formula

$$\frac{-\varrho\pm\sqrt{\varrho^2+4}}{2} = \frac{1}{4}-i\frac{\sqrt{3}}{2}\pm\frac{1}{2}\left(\sqrt{\frac{\sqrt{13}}{2}+\frac{7}{4}}-i\sqrt{\frac{\sqrt{13}}{2}-\frac{7}{4}}\right)$$

$$=\left(\frac{1}{4}\pm\frac{1}{2}\sqrt{\frac{\sqrt{13}}{2}+\frac{7}{4}}\right)-i\left(\frac{\sqrt{3}}{4}\pm\frac{1}{2}\sqrt{\frac{\sqrt{13}}{2}-\frac{7}{4}}\right).$$

Hence

$$|\theta_1|^2 = \left(\frac{1}{4}+\frac{1}{2}\sqrt{\frac{\sqrt{13}}{2}+\frac{7}{4}}\right)^2+\left(\frac{\sqrt{3}}{4}+\frac{1}{2}\sqrt{\frac{\sqrt{13}}{2}-\frac{7}{4}}\right)^2$$

$$=\frac{1}{4}\left(1+\sqrt{13}+\sqrt{\frac{\sqrt{13}}{2}+\frac{7}{4}}+\sqrt{3\left(\frac{\sqrt{13}}{2}-\frac{7}{4}\right)}\right)$$

$$=\frac{1}{4}\left(1+\sqrt{13}+\sqrt{2\sqrt{13}-2}\right)$$

and we get

$$\Lambda \geqslant |\theta_1|.$$

It remains to note that the zeros of the polynomial $z^2+\varrho^2 z-1$ are complex conjugates of the zeros of $z^2+\varrho z-1$ and that

$$|\theta_1|^2 < 1.73 < \theta_0^2.$$

LEMMA 5. *$f(z) = a_nz^n+\ldots+a_0$ has all zeros inside the unit circle if and only if $\delta_k(f)>0$, $k=0,1,\ldots,n$ where*

$$(26)\qquad \delta_k(f) = \begin{vmatrix} a_n & & & & a_0 & a_1 & \cdots & a_{k-1} \\ a_{n-1} & a_n & & & & a_0 & & a_{k-2} \\ \vdots & & \ddots & & & & \ddots & \vdots \\ & & & & & & & a_1 \\ a_{n-k+1} & \cdots & a_{n-1} & a_n & & & & a_0 \\ \bar{a}_0 & & & & \bar{a}_n & \bar{a}_{n-1} & \cdots & \bar{a}_{n-k+1} \\ \bar{a}_1 & \bar{a}_0 & & & & \bar{a}_n & & \\ \vdots & & \ddots & & & & \ddots & \vdots \\ & & & & & & & \bar{a}_{n-1} \\ \bar{a}_{k-1} & \cdots & \bar{a}_1 & \bar{a}_0 & & & & \bar{a}_n \end{vmatrix}.$$

Proof see A. Cohn [1].

LEMMA 6. *If $A=(a_{ij})$, $i,j\leqslant n$, $n\geqslant l>k$, then*

$$(27)\qquad \det A = \sum_{i<j}(-1)^{i+j+k+l}A_{k,l;i,j}\begin{vmatrix} a_{ki} & a_{kj} \\ a_{li} & a_{lj}\end{vmatrix},$$

where $A_{k,l;i,j}$ is the determinant of the matrix obtained from A by crossing out the k-th and l-th rows and the i-th and j-th columns.

The proof follows from Laplace's theorem.

LEMMA 7. *Let K satisfy the assumptions of Theorem 3,*

$$f(z) = a_nz^n+\ldots+a_0,\qquad f^*(z) = z^n\bar{f}(z^{-1}) = \bar{a}_0z^n+\bar{a}_1z^{n-1}+\ldots+\bar{a}_n,$$

where a_i are integers of K. If $a_n=1$, $a_0=\varepsilon\dfrac{1+\sqrt{5}}{2}$, where ε is a root of unity, then for each $i\leqslant n$ the condition

$$(\mathrm{w}_i)\qquad \begin{array}{llll} \delta_k(\sigma f)>0 & \text{if} & \sigma\sqrt{5}=-\sqrt{5} & (k=1,2,\ldots,i), \\ \delta_k(\sigma f^*)>0 & \text{if} & \sigma\sqrt{5}=\sqrt{5} & (k=1,2,\ldots,i) \end{array}$$

implies

$$(28)\qquad \delta_k(\sigma f) = \delta_1(\sigma f)^k \quad\text{and}\quad a_{n-k+1}\bar{a}_0-\bar{a}_{k-1}=0$$

for all $k\leqslant i$ and all $\sigma\in G$.

Proof. We shall proceed by induction with respect to i. For $i=1$

$$\sigma(a_n\bar{a}_0-\bar{a}_0) = \sigma 0 = 0,$$

$$\delta_1(\sigma f) = \begin{vmatrix} 1 & \sigma a_0 \\ \sigma\bar{a}_0 & 1\end{vmatrix} = 1-|\sigma a_0|^2 = \begin{cases} -\dfrac{1+\sqrt{5}}{2} & \text{if}\quad \sigma\sqrt{5}=\sqrt{5}, \\[2ex] \dfrac{\sqrt{5}-1}{2} & \text{if}\quad \sigma\sqrt{5}=-\sqrt{5}.\end{cases}$$

Let us assume that the lemma is true for $i < i_0$. Then, by applying the formula from Lemma 6 to the rows i_0, $2i_0$ and omitting the terms which involve the coefficient

$$\begin{vmatrix} a_n & 0 \\ \overline{a}_0 & 0 \end{vmatrix} = 0,$$

we get

$$(29) \qquad \delta_{i_0}(f) = \delta_1(f)\,\delta_{i_0-1}(f) + \sum_{j=1}^{i_0-1} (-1)^j \begin{vmatrix} a_{n-i_0+j} & a_n \\ \overline{a}_{i_0-j} & \overline{a}_0 \end{vmatrix} (\delta_{i_0})_{i_0,2i_0;i_0,j}(f).$$

Now we apply (29) to σf and use the inductive assumption. We obtain

$$(30) \qquad \delta_{i_0}(\sigma f) = \delta_1(\sigma f)^{i_0} - \begin{vmatrix} \sigma a_{n-i_0+1} & \sigma a_n \\ \overline{\sigma a}_{n-i_0+1} & \overline{\sigma a}_0 \end{vmatrix} (\delta_{i_0})_{i_0,2i_0;1,i_0}(\sigma f),$$

$$\big(\delta_{i_0}(f)\big)_{i_0,2i_0;1,i_0}$$

$$= \begin{vmatrix}
0 & 0 & \cdots & 0 & 0 & a_0 & a_1 & & \cdots & a_{i-2} & a_{i-1} \\
a_n & 0 & \cdots & & & 0 & a_0 & a_1 & \cdots & a_{i-3} & a_{i-2} \\
a_{n-1} & a_n & \cdots & & & & & & & & \\
 & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & & \cdots & a_1 & a_2 \\
a_{n-i+3} & a_{n-i+2} & \cdots & a_{n-1} & a_n & 0 & 0 & & \cdots & a_0 & a_1 \\
0 & 0 & \cdots & & 0 & \overline{a}_n & \overline{a}_{n-1} & & \cdots & \overline{a}_{n-i+2} & \overline{a}_{n-i+1} \\
\overline{a}_0 & 0 & \cdots & & & 0 & \overline{a}_n & & \cdots & \overline{a}_{n-i+3} & \overline{a}_{n-i+2} \\
\overline{a}_1 & \overline{a}_0 & \cdots & \cdots & & & 0 & & \cdots & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\overline{a}_{i-3} & \overline{a}_{i-4} & \cdots & \overline{a}_1 & \overline{a}_0 & 0 & 0 & & \cdots & \overline{a}_n & \overline{a}_{n-1}
\end{vmatrix}$$

We develop the determinant according to the first row and the i_0-th row. When these rows are left out, the (i_0-1)-th column consists of zeros only, and hence it suffices to compute the minors $M_{1,i_0;i_0-1,i}$. For the elements a of K, we have $\sigma(\overline{a}) = \overline{(\sigma a)}$; thus

$$\begin{vmatrix} \sigma a_0 & \sigma a_i \\ \overline{\sigma a}_n & \overline{\sigma a}_{n-i} \end{vmatrix} = \sigma(a_0)\overline{\sigma(a_{n-i})} - \sigma(a_i)\overline{\sigma(a_n)}$$

$$= \sigma(a_0\overline{a}_{n-i} - a_i\overline{a}_n) = \overline{\sigma(\overline{a}_0 a_{n-i} - \overline{a}_i a_n)}.$$

By the inductive assumption

$$\delta_{i_0}(\sigma f)_{i_0,2i_0;1,i_0} = \delta_{i_0-2}(\sigma f) \begin{vmatrix} \sigma(a_0) & \sigma(a_{i_0-1}) \\ \overline{\sigma(a_n)} & \overline{\sigma(a_{n-i_0+1})} \end{vmatrix}$$

$$= \delta_1(\sigma f)^{i_0-2}\,\overline{\sigma(\overline{a}_0 a_{n-i_0+1} - \overline{a}_{i_0-1} a_n)}.$$

Substituting the computed value of $(\delta_{i_0})_{i_0,2i_0;1,i_0}$ into (30), we get

$$(31) \qquad \delta_{i_0}(\sigma f) = \delta_1(\sigma f)^{i_0} - \delta_1(\sigma f)^{i_0-2}\,|\sigma(\overline{a}_0 a_{n-i_0+1} - \overline{a}_{i_0-1} a_n)|^2.$$

Since $\delta_k(f^*)$ is obtained from $\delta_k(f)$ by a transposition of the ith row and the $(k+i)$-th row of the latter, we have

$$\delta_k(f^*) = (-1)^k \delta_k(f) \quad \text{and} \quad \delta_k(\sigma f^*) = (-1)^k \delta_k(\sigma f).$$

Hence the condition (w_{i_0}) is equivalent to the condition

$$\delta_{i_0}(\sigma f) > 0 \quad \text{if} \quad \sigma\sqrt{5} = -\sqrt{5},$$

$$(-1)^{i_0}\delta_{i_0}(\sigma f) > 0 \quad \text{if} \quad \sigma\sqrt{5} = \sqrt{5}.$$

We set

$$x = \overline{a}_0 a_{n-i_0+1} - \overline{a}_{i_0-1} a_n$$

and distinguish two cases:

A. i_0 is even. We have, for all $\sigma \in G$,

$$\delta_{i_0}(\sigma f) = \delta_1^{i_0}(\sigma f) - \delta_1^{i_0-2}(\sigma f)\,|\sigma x|^2 > 0,$$

and since

$$\delta_1(\sigma f)^{i_0} \quad \text{and} \quad \delta_1(\sigma f)^{i_0-2} > 0,$$

we get

$$|\sigma x|^2 < \delta_1^2(\sigma f) = \begin{cases} \dfrac{3+\sqrt{5}}{2} & (\sigma\sqrt{5} = \sqrt{5}), \\[2mm] \dfrac{3-\sqrt{5}}{2} & (\sigma\sqrt{5} = -\sqrt{5}), \end{cases}$$

$$\prod_{\sigma\in G} |\sigma x|^2 < \prod_{\sigma\in G} \delta_1^2(\sigma f).$$

The number $N = \prod_{\sigma\in G} |\sigma x|^2$ is a non-negative rational integer, as the square of the norm of x,

$$\prod_{\sigma\in G} \delta_1^2(\sigma f) = 1.$$

Since $\dfrac{3+\sqrt{5}}{2}$ and $\dfrac{3-\sqrt{5}}{2}$ occur equally often as factors in the product, we have $M = 0$ and $x = 0$.

By (31)

$$\delta_{i_0}(\sigma f) = \delta_1(\sigma f)^{i_0},$$

$$\overline{a}_0 a_{n-i_0+1} - \overline{a}_{i_0-1} a_n = 0, \qquad \sigma(\overline{a}_0 a_{n-i_0+1} - a_n \overline{a}_{i_0-1}) = 0$$

and the inductive assertion is proved.

B. i_0 in odd. We have

$$\delta_{i_0}(\sigma f) > 0 \quad \text{if} \quad \sigma\sqrt{5} = -\sqrt{5},$$
$$\delta_{i_0}(\sigma f) < 0 \quad \text{if} \quad \sigma\sqrt{5} = \sqrt{5}.$$

If

$$\sigma\sqrt{5} = -\sqrt{5}, \qquad \delta_1(\sigma f) = \frac{\sqrt{5}-1}{2} > 0,$$

we have

$$\delta_1(\sigma f)^{i_0} > 0, \qquad \delta_1(\sigma f)^{i_0-2} > 0$$

and the inequality

$$\delta_1(\sigma f)^{i_0} - \delta_1(\sigma f)^{i_0-2}|\sigma x|^2 > 0$$

implies

$$\frac{3-\sqrt{5}}{2} > |\sigma x|^2. \tag{32}$$

If

$$\sigma\sqrt{5} = \sqrt{5}, \quad \delta_1(\sigma f) = -\frac{1+\sqrt{5}}{2} < 0,$$

we have

$$\delta_1^{i_0}(\sigma f) < 0, \qquad \delta_1(\sigma f)^{i_0-2} < 0$$

and the inequality

$$\delta_1(\sigma f)^{i_0} - \delta_1(\sigma f)^{i_0-2}|\sigma x|^2 < 0$$

implies

$$\frac{3+\sqrt{5}}{2} > |\sigma x|^2. \tag{33}$$

The inductive assertion follows from (32) and (33) as in the case A.

LEMMA 8. *Let K satisfy the assumptions of Theorem 3,*

$$f(z) = z^n + a_{n-1}z^{n-1} + \ldots + \varepsilon\frac{1+\sqrt{5}}{2},$$

where a_i are integers of K, ε is a root of unity, and let

$$\sigma f(z) = \prod_{i=1}^{n}(z-\alpha_{\sigma j}), \qquad \sigma \in G.$$

If

$$(34) \qquad \begin{array}{lll} |\alpha_{\sigma j}| > 1 & \text{if} & \sigma\sqrt{5} = \sqrt{5}, \\ |\alpha_{\sigma j}| < 1 & \text{if} & \sigma\sqrt{5} = -\sqrt{5}, \end{array}$$

then

$$f(z) = z^n + \varepsilon\frac{1+\sqrt{5}}{2}.$$

Proof. $f(z)$ has all zeros outside the circle $|z| \geqslant 1$ if and only if $f^*(z) = z^n\bar{f}(z^{-1})$ has all zeros inside the circle $|z| < 1$. By Lemma 5 the condition (34) is equivalent to the condition

$$(34') \qquad \begin{array}{llll} \delta_k(\sigma f) > 0 & \text{if} & \sigma\sqrt{5} = -\sqrt{5} & (k = 1, 2, \ldots, n), \\ \delta_k(\sigma f) > 0 & \text{if} & \sigma\sqrt{5} = \sqrt{5} & (k = 1, 2, \ldots, n). \end{array}$$

The latter is the same as condition (w_n) considered in Lemma 7 and in virtue of that lemma

$$a_{n-i}\bar{a}_0 - \bar{a}_i = 0 \quad (i = 1, 2, \ldots, n-1), \tag{35}$$

where $a_0 = \varepsilon\frac{1+\sqrt{5}}{2}$.

(35) gives $a_i\bar{a}_0 - \bar{a}_{n-i} = 0$ and on passing to complex conjugates we get

$$-a_{n-i} + a_0\bar{a}_i = 0 \quad (i = 1, 2, \ldots, n-1). \tag{35'}$$

Since

$$\begin{vmatrix} \bar{a}_0 & -1 \\ -1 & a_0 \end{vmatrix} = |a_0|^2 - 1 = \frac{1+\sqrt{5}}{2} \neq 0,$$

(35) and (35′) imply $a_i = a_{n-i} = 0$ for $i = 1, 2, \ldots, n-1$. Hence $f(z) = z^n + a_0$.

Proof of Theorem 3. Assume first that

$$P(0) = \varepsilon\frac{1+\sqrt{5}}{2}.$$

Let

$$P(z) = f(z)\prod_{i=2}^{n-k+1}(z-\varepsilon_i),$$

where ε_i are roots of unity, but no zero of $f(z)$ is a root of unity. The product $\prod(z-\varepsilon_i)$ divides $(z^m-1)^m$ for a suitable m; hence

$$\prod_{i=2}^{n-k+1}(z-\varepsilon_i) = \big(P(z), (z^m-1)^m\big) \in K[z]$$

and $f(z) \in K[z]$, f is monic with integer coefficients and $f(0) = \varepsilon_1\frac{1+\sqrt{5}}{2}$.

For $\sigma \in G$ let

$$\sigma f(z) = \prod_{j=1}^{k}(z-\alpha_{\sigma j}). \tag{36}$$

By the assumption about $P(z)$, we have

$$(37)\qquad \begin{aligned} |\alpha_{\sigma j}| &\geqslant 1 \quad \text{if} \quad \sigma(\sqrt{5}) = \sqrt{5},\\ |\alpha_{\sigma j}| &\leqslant 1 \quad \text{if} \quad \sigma(\sqrt{5}) = -\sqrt{5}. \end{aligned}$$

Suppose that for a $\sigma_0 \in G$ and a $j_0 \leqslant k$ we have

$$(38)\qquad |\alpha_{\sigma_0 j_0}| = 1.$$

Consider the field $L = K^{\sigma_0}(\alpha_{\sigma_0 j_0})$ and any isomorphic injection τ of L into C. We have $\sigma_0\sqrt{5} = \overline{\sigma_0\sqrt{5}}$; thus $\tau\sigma_0\sqrt{5} = \tau\overline{\sigma_0\sqrt{5}}$.

If $\tau\sigma_0\sqrt{5} = \sqrt{5}$, then, since $\tau\sigma_0 f(\alpha_{\sigma_0 j_0}) = \tau\sigma_0 f(\tau\alpha_{\sigma_0 j_0}) = 0$, we have by (36) and (37)

$$|\tau\alpha_{\sigma_0 j_0}| \geqslant 1 \quad \text{and} \quad |\tau\overline{\alpha_{\sigma_0 j_0}}| \geqslant 1.$$

If $\tau\sigma_0\sqrt{5} = \tau\overline{\sigma_0\sqrt{5}} = -\sqrt{5}$ we have similarly

$$|\tau\alpha_{\sigma_0 j_0}| \leqslant 1 \quad \text{and} \quad |\tau\overline{\alpha_{\sigma_0 j_0}}| \leqslant 1.$$

On the other hand, by (38)

$$\tau\alpha_{\sigma_0 j_0}\cdot\tau\overline{\alpha_{\sigma_0 j_0}} = \tau(\alpha_{\sigma_0 j_0}\cdot\overline{\alpha_{\sigma_0 j_0}}) = \tau|\alpha_{\sigma_0 j_0}|^2 = 1;$$

thus $|\tau\alpha_{\sigma_0 j_0}| = 1$ for all τ. By Lemma 3, $\alpha_{\sigma_0 j_0}$ is a root of unity and by (36) a certain conjugate of it is a zero of $f(z)$, contrary to the definition of f. The contradiction obtained above shows that f satisfies all the assumptions of Lemma 8 and in virtue of that lemma

$$f(z) = z^k + \varepsilon_1\frac{1+\sqrt{5}}{2}.$$

Assume now that $P(0) = \varepsilon\dfrac{1-\sqrt{5}}{2}$. Then, for a σ with $\sigma\sqrt{5} = -\sqrt{5}$, σP satisfies the assumptions of the theorem and $\sigma P(0) = \dfrac{\sqrt{5}+1}{2}\sigma(\varepsilon)$. Thus, by the already proved case of the theorem, formula (3) holds.

Proof of Theorem 2. If (3) holds, we clearly have equality in (2). Suppose that for a polynomial $P \in K[z]$ with the leading coefficient p_0 the equality in (2) is obtained. By the equality $C(P) = (p_0)$ quoted in the introduction, $P_0(z) = p_0^{-1}P(z)$ has integral coefficients. Moreover $\sqrt{5} \in K$ and $|P_0(0)| = \dfrac{\pm 1+\sqrt{5}}{2}$. Since for $a \in K$, $\sigma \in G$

$$\sigma(|a|^2) = |\sigma a|^2,$$

we have for all $\sigma \in G$

$$\left|\sigma\left(\frac{P_0(0)^2}{\frac{3\pm\sqrt{5}}{2}}\right)\right| = \left|\sigma\left(\frac{P_0(0)}{\frac{1\pm\sqrt{5}}{2}}\right)\right|^2 = \sigma\left(\left|\frac{P_0(0)}{\frac{1\pm\sqrt{5}}{2}}\right|^2\right) = 1.$$

$\dfrac{P_0(0)^2}{\frac{3\pm\sqrt{5}}{2}} = P_0(0)^2\,\dfrac{3\mp\sqrt{5}}{2}$ is an integer, and hence by Lemma 4 $\dfrac{P_0(0)^2}{\frac{3\pm\sqrt{5}}{2}}$ is a root of unity and $\dfrac{P_0(0)}{\frac{1\pm\sqrt{5}}{2}}$ is also one.

Thus

$$P_0(z) = z^n + p_1 z^{n-1} + \ldots + \varepsilon\frac{1\pm\sqrt{5}}{2},$$

where p_i are integers of K. On the other hand,

$$(39)\qquad \prod_{\sigma\in G}\prod_{|\alpha_{\sigma j}|>1}|\alpha_{\sigma j}| = \prod_{\sigma(\sqrt{5})=\pm\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\prod_{|\alpha_{\sigma j}|<1}|\alpha_{\sigma j}|^{-1}\right)\prod_{\sigma\sqrt{5}=\pm\sqrt{5}}\prod_{|\alpha_{\sigma j}|>1}|\alpha_{\sigma j}|$$
$$= \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{|K|}{2}}\prod_{\sigma(\sqrt{5})=\pm\sqrt{5}}\prod_{|\alpha_{\sigma j}|<1}|\alpha_{\sigma j}|^{-1}\prod_{\sigma(\sqrt{5})=\pm\sqrt{5}}\prod_{|\alpha_{\sigma j}|>1}|\alpha_{\sigma j}|,$$

and the equality in (2) implies that both double products on the right-hand side of (39) are empty. Thus $P_0(z)$ satisfies the assumptions of Theorem 3 and in virtue of that Theorem 3 holds.

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