ACTA ARITHMETICA
XXX (1976)

On linear forms in the logarithms of algebraic numbers

by

T. N. SHOREY (Bombay)

1. Let $n > 1$ be an integer. Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers of heights less than or equal to $A_1, \ldots, A_n$ respectively, where each $A_i \geqslant 27$. Let $\beta_1, \ldots, \beta_{n-1}$ denote algebraic numbers of heights less than or equal to B ($\geqslant 27$). Suppose that $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_{n-1}$ all lie in a field of degree D over the rationals. Set

$$\Lambda = \log A_1 \ldots \log A_n \quad \text{and} \quad E = (\log \Lambda + \log\log B).$$

We prove:

THEOREM 1. *Given $\varepsilon > 0$, there exists an effectively computable number $C > 0$ depending only on ε such that*

$$|\beta_1 \log \alpha_1 + \ldots + \beta_{n-1} \log \alpha_{n-1} - \log \alpha_n|$$

exceeds

$$\exp\left(-(nD)^{Cn} \Lambda (\log \Lambda)^2 \big(\log(\Lambda B)\big)^2 E^{2n+2+\varepsilon}\right),$$

provided that the above linear form does not vanish.

It has been assumed that the logarithms have their principal values but the result would hold for any choice of logarithms if C were allowed to depend on their determinations. We shall follow the method of Stark [8] for the proof of Theorem 1. When n is large and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n-1}$ all lie in a fixed field, the theorem is sharper than all the known results in this direction. For example, see Baker [1] and Ramachandra [4]. The theorem is also of interest when both n and D are large. In [7], the author imposed certain restrictions on the linear form and obtained a result whose dependence on n was similar to that of Theorem 1.

Theorem 1 strengthens a result of Stark [8]. When n and D are fixed and $\beta_1, \ldots, \beta_{n-1}$ are rational numbers, the theorem is weaker than a recent result of Baker [2].

We mention an application of Theorem 1.

THEOREM 2. *Let $\varepsilon > 0$ be given. Suppose that u and $k \geqslant 2$ are positive integers satisfying $u > e^{k^\varepsilon}$. Then the number J of numbers among $(u+1)\ldots$ $\ldots(u+k)$, all of whose prime factors are less than or equal to k, satisfies*

$$J = O\left(\frac{k\log\log k}{(\log k)^2}\right).$$

Here the constant implied by O depends only on ε.

Ramachandra [4] proved Theorem 2 with $J = O\left(\frac{k}{\log k}\left(\frac{\log\log k}{\log k}\right)^{1/2}\right)$. For the proof of this, Ramachandra used a weaker version of Theorem 1. If that is replaced by Theorem 1 in [4], then Theorem 2 follows immediately from the arguments of [4]. An immediate corollary from Theorem 2 is the following:

COROLLARY. *Let ε, u and k be as in Theorem 2. Then the number of distinct prime factors of $(u+1)\ldots(u+k)$ exceeds*

$$k+\pi(k)\left(1+O\left(\frac{\log\log k}{\log k}\right)\right).$$

Here the constant implied by O depends only on ε.

2. In this section, we collect those results that we shall use from other sources.

LEMMA 1. *Suppose that the coefficients of the p linear forms*

$$y_k = a_{k,1}x_1+\ldots+a_{k,q}x_q \quad (k = 1, \ldots, p;\ p < q)$$

are integers in an algebraic number field of degree h and let $\overline{|a_{k,l}|} \leqslant A$. ($\overline{|a_{k,l}|}$ denotes the maximum of the absolute values of the conjugates of $a_{k,l}$.) Then there exist rational integers $x_1, \ldots, x_q$, not all zero, satisfying $y_1 = 0, \ldots$ $\ldots, y_p = 0$ and such that

$$|x_k| < 1+(2qA)^{\frac{ph(h+1)}{2q-ph(h+1)}}, \quad k = 1, \ldots, q,$$

provided that $2q > ph(h+1)$ and $A \geqslant 1$.

See, Ramachandra [5], p. 16.

LEMMA 2. *Let $a_1, \ldots, a_n$ be non-zero elements of an algebraic number field K and let $a_1^{1/p}, \ldots, a_n^{1/p}$ denote fixed p-th roots for some prime p. Further let $K' = K(a_1^{1/p}, \ldots, a_{n-1}^{1/p})$. Then either $K'(a_n^{1/p})$ is an extension of K' of degree p or we have*

$$a_1^{j_1}\ldots a_n^{j_n} = \eta^p$$

for some η in K and some integers $j_1, \ldots, j_n$ with $0 \leqslant j_i < p$ $(1 \leqslant i < n)$ and $j_n = 1$.

This is due to Baker and Stark [3].

LEMMA 3. *Let S be a set of primes with minimal element p_0 and maximal element P_0. Let $a_1, \ldots, a_n$ be $n \geqslant 1$ non-zero algebraic numbers with heights less than $A_1, \ldots, A_n$ respectively where $A_i \geqslant e$ $(1 \leqslant i \leqslant n)$ and suppose that $a_1, \ldots, a_n$ all belong to a field K of degree D. Suppose further that if p is in S, then there are rational integers $j_1, \ldots, j_n$, $0 \leqslant j_1 < p$, $j_n \neq 0$, with*

$$a_1^{j_1}\ldots a_n^{j_n} = \eta^p$$

for some η in K. Then there are two effectively computable numbers $c_0 = c_0(D, n)$ and $c = c(D, n)$ depending only on D and n such that if S has more than

$$D+\frac{c+(D^2+1)\log(\log A_1\ldots\log A_n)}{\log p_0}$$

elements, then there are rational integers $h_1, \ldots, h_n$ with $h_n \neq 0$,

$$|h_l| \leqslant c_0P_0^D\Big(\prod_{i=1}^{n}\log A_i\Big)^{2D-1}, \quad 1 \leqslant l \leqslant n,$$

such that

$$a_1^{h_1}\ldots a_n^{h_n} = 1.$$

Here

$$c_0 = 4D^2c_1^{D-1}(2Dn)^{2Dn},$$

$$c = \tfrac{1}{2}[D^2(n+1)+2n]\log(2Dn)+\tfrac{1}{2}D^2\log c_2,$$

where

$$c_2 = 2^Dc_1, \quad c_1 = D^2/\log(1+c_D), \quad c_D = \big(30D^2\log(6D)\big)^{-1}.$$

This is due to Stark [8].

3. In this section, we shall prove Theorem 1. The notation of this section is independent of the notation of § 1 and § 2. The size of an algebraic number a, denoted by $S(a)$, is defined as $\overline{|a|}+d(a)$, where $d(a)$ is the least positive integer such that $ad(a)$ is an algebraic integer. If a is an algebraic number of degree $\leqslant D$, then

$$S(a) \leqslant (D+1)H(a), \quad H(a) \leqslant 2^D\big(S(a)\big)^{2D}. \tag{1}$$

(See [6], p. 76.) So it does not matter whether we state our theorem in terms of size or height. Let $n > 1$ be an integer. We shall assume that $a_1, \ldots, a_n$ are non-zero algebraic numbers of sizes not exceeding $S_1, \ldots, S_n$, respectively, where each $S_i \geqslant 27$. Further we assume that $\beta_1, \ldots, \beta_{n-1}$ are algebraic numbers of sizes not exceeding $S(\geqslant 27)$. Assume that the numbers $a_1, \ldots, a_n$ and $\beta_1, \ldots, \beta_{n-1}$ all lie in a field K of degree D over the rationals. Set

$$\beta = |\beta_1\log a_1+\ldots+\beta_{n-1}\log a_{n-1}-\log a_n|$$

and assume that $\beta \leqslant \frac{1}{2}$. Let ε with $0 < \varepsilon < 1$ be given. Let A be a sufficiently large positive constant which can be explicitly determined in terms of ε. We shall assume, without loss of generality, that $S_1, \ldots, S_n$ and S exceeds some large positive constant S_0 which can be explicitly determined in terms of ε and A. We shall take S_0 large as compared with A. Set

$$\Lambda = \log S_1 \ldots \log S_n, \qquad E = (\log \Lambda + \log\log S),$$

$$T = [(nD)^5 E], \qquad T_1 = \left[\frac{(n^2+n)}{\varepsilon}\right]+2, \qquad R = [(nD)^9 E],$$

$$h = [(nD)^A E \log \Lambda \log S], \qquad h_1 = h,$$

$$h_{r+1} = [(nD)^{10\varepsilon/n} E^{\varepsilon/n} h_r]+1, \qquad r = 1, \ldots, T_1-1,$$

$$L_i = [(5D)^2 (nD)^{An+\frac{10\varepsilon(n-1)}{n}} \Lambda E^{n+\frac{\varepsilon(n-1)}{n}} (\log S_i)^{-1} \log S \log \Lambda], \qquad i = 1, \ldots, n,$$

$$L = \max_{1 \leqslant i \leqslant n} L_i,$$

$$k = [(5D)^2 (nD)^{A(n+1)+10\varepsilon} \Lambda E^{n+1+\varepsilon} \log S \log \Lambda],$$

$$k_1 = k, \qquad k_{r+1} = \left[\left(1-\frac{1}{T}\right)k_r\right], \qquad r = 1, \ldots, T-1.$$

$$\Delta = S_1^{L_1} \ldots S_n^{L_n}.$$

Observe that

(2) $$k_r \geqslant \frac{1}{18} k, \qquad r = 1, \ldots, T.$$

We consider the following auxiliary function

$$\varphi(z_1, \ldots, z_{n-1}) = \sum_{\lambda_1=0}^{L_1} \cdots \sum_{\lambda_n=0}^{L_n} p(\lambda_1, \ldots, \lambda_n)\, \alpha_1^{\gamma_1 z_1} \ldots \alpha_{n-1}^{\gamma_{n-1} z_{n-1}},$$

$$\gamma_i = \lambda_i + \lambda_n \beta_i \qquad (1 \leqslant i < n),$$

where $p(\lambda_1, \ldots, \lambda_n)$ are rational integers, not all zero, to be determined such that

(3) $$q(l, m_1, \ldots, m_{n-1}) = 0$$

for all integers l with $1 \leqslant l \leqslant h$ and for all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1 + \ldots + m_{n-1} \leqslant k$. Here

$$q(l, m_1, \ldots, m_{n-1}) = \sum_{\lambda_1=0}^{L_1} \cdots \sum_{\lambda_n=0}^{L_n} p(\lambda_1, \ldots, \lambda_n)\, \alpha_1^{\lambda_1 l} \ldots \alpha_n^{\lambda_n l} \gamma_1^{m_1} \ldots \gamma_{n-1}^{m_{n-1}}.$$

(3) is a set of $h(k+1)^{n-1}$ linear equations in $(L_1+1) \ldots (L_n+1)$ variables $p(\lambda_1, \ldots, \lambda_n)$. Observe that

$$(L_1+1) \ldots (L_n+1) > D(D+1)h(k+1)^{n-1}.$$

The absolute values of the conjugates of the coefficients of $p(\lambda_1, \ldots, \lambda_n)$ in (3) do not exceed $\Delta^h (2SL)^k$. Multiply each of the equations (3) by a natural number not exceeding $\Delta^h (2SL)^k$ so that the coefficients of $p(\lambda_1, \ldots, \lambda_n)$ are algebraic integers in K. Now it follows from Lemma 1 that there exist rational integers $p(\lambda_1, \ldots, \lambda_n)$, not all zero, satisfying (3) and

$$|p(\lambda_1, \ldots, \lambda_n)| \leqslant \Delta^{3h} (2SL)^{3k}.$$

LEMMA 4. *Assume that*

(4) $$\beta \leqslant \Lambda^{-k} n^{-k} (32)^{-2h_{T_1} k/T}.$$

Then

$$q(l, m_1, \ldots, m_{n-1}) = 0,$$

for all integers l with $1 \leqslant l \leqslant h_{T_1}$ and for all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1 + \ldots + m_{n-1} \leqslant k_{T_1}$.

Proof. First we prove that $q(l, m_1, \ldots, m_{n-1}) = 0$ for all integers l with $1 \leqslant l \leqslant h_2$ and for all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1 + \ldots + m_{n-1} \leqslant k_2$. If not, there exists an integer l with $h_1 < l \leqslant h_2$ and non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1 + \ldots + m_{n-1} \leqslant k_2$ such that $q(l, m_1, \ldots, m_{n-1}) \neq 0$. Now $q(l, m_1, \ldots, m_{n-1})$ is a non-zero algebraic number of degree $\leqslant D$. Each of its conjugates has absolute value not exceeding

$$\Delta^{5h_2} (2SL)^{5k}.$$

The denominator of $q(l, m_1, \ldots, m_{n-1})$ does not exceed

$$\Delta^{h_2} (2SL)^k.$$

Thus

$$|q(l, m_1, \ldots, m_{n-1})| \geqslant \left(\Delta^{6h_2} (2SL)^{6k}\right)^{-D}.$$

We write $\varphi_{m_1, \ldots, m_{n-1}}(z, \ldots, z)$ for the value of

$$\left(\frac{\partial}{\partial z_1}\right)^{m_1} \ldots \left(\frac{\partial}{\partial z_{n-1}}\right)^{m_{n-1}} \varphi(z_1, \ldots, z_{n-1})$$

at the point $z_r = z$, $1 \leqslant r < n$.

Observe that

$$|(\log \alpha_1)^{-m_1} \ldots (\log \alpha_{n-1})^{-m_{n-1}} \varphi_{m_1, \ldots, m_{n-1}}(l, \ldots, l) - q(l, m_1, \ldots, m_{n-1})| \leqslant \beta \Delta^{5h_2} (2SL)^{5k}.$$

Thus

(5) $$|\varphi_{m_1, \ldots, m_{n-1}}(l, \ldots, l)| \geqslant w\left(\Delta^{-6Dh_2} (2SL)^{-6Dk} - \beta \Delta^{5h_2} (2SL)^{5k}\right),$$

with

$$w = |(\log \alpha_1)^{m_1} \ldots (\log \alpha_{n-1})^{m_{n-1}}|.$$

We have

$$(6)\qquad \frac{1}{2\pi i}\int\limits_{\Gamma:|z|=5h_2}\frac{f(z)F(l)}{(z-l)F(z)}\,dz = f(l)+\sum_{r=1}^{h_1}\sum_{m=0}^{k_1-k_2}\frac{f^{(m)}(r)}{m!\,2\pi i}\int\limits_{\Gamma_r:|z-r|=\frac{1}{2}}\frac{(z-r)^m F(l)}{(z-l)F(z)}\,dz,$$

where

$$F(z) = \big((z-1)\ldots(z-h_1)\big)^{k_1-k_2+1} \quad\text{and}\quad f(z) = \varphi_{m_1,\ldots,m_{n-1}}(z,\ldots,z).$$

Now

$$\max_{|z|=5h_2}|f(z)| \leqslant w\Delta^{10h_2}(2SL)^{10k}.$$

Thus the absolute value of the integral does not exceed

$$w\Delta^{10h_2}(2SL)^{10k}2^{-h_1k_1/T}.$$

For an integer r with $1\leqslant r\leqslant h_1$ and non-negative integers $\mu_1,\ldots,\mu_{n-1}$ with $\mu_1+\ldots+\mu_{n-1}=m$ and $0\leqslant m\leqslant k_1-k_2$, we have

$$q(r, m_1+\mu_1,\ldots,m_{n-1}+\mu_{n-1}) = 0,$$

since

$$m_1+\mu_1+\ldots+m_{n-1}+\mu_{n-1}\leqslant k_2+m\leqslant k_1.$$

Thus

$$|(\log\alpha_1)^{-m_1-\mu_1}\ldots(\log\alpha_{n-1})^{-m_{n-1}-\mu_{n-1}}\varphi_{m_1+\mu_1,\ldots,m_{n-1}+\mu_{n-1}}(r,\ldots,r)| \leqslant \beta\Delta^{5h_1}(2SL)^{5k}.$$

Thus for every r with $1\leqslant r\leqslant h_1$ and for every m with $0\leqslant m\leqslant k_1-k_2$, we have

$$|f^{(m)}(r)|\leqslant \beta w n^k A^k\Delta^{5h_1}(2SL)^{5k}.$$

Observe that

$$(l-1)\ldots(l-h_1) = \binom{l-1}{h_1}h_1!\leqslant 2^{h_2}h_1!,$$

and when z lies on Γ_r,

$$|(z-1)\ldots(z-h_1)|\geqslant \tfrac{1}{8}(r-2)!(h_1-r-1)!\geqslant 4^{-h_1}h_1!.$$

Hence the absolute value of the double sum in (6) does not exceed

$$\beta w n^k A^k\Delta^{5h_1}(2SL)^{5k}(16)^{2h_2k_1/T},$$

since $k_1-k_2+1<k_1/T+2<2k_1/T$. (Observe that $k_r/T>2$, $r=1,\ldots,T$.) Hence

$$(7)\qquad |f(l)|\leqslant w\big(\Delta^{10h_2}(2SL)^{10k}2^{-h_1k_1/T}+\beta n^k A^k\Delta^{5h_1}(2SL)^{5k}(16)^{2h_2k_1/T}\big).$$

Now we show that (5) and (7) are inconsistent. For this, it is sufficient to show that

$$2^{h_1k_1/T}>\Delta^{16Dh_2}(2SL)^{16kD}\{1+\beta n^kA^k(32)^{2h_2k_1/T}\},$$

i.e. we should show

$$2^{h_1k_1/T}>2\Delta^{16Dh_2}(2SL)^{16kD},$$

provided that

$$\beta\leqslant n^{-k}A^{-k}(32)^{-2h_2k_1/T}.$$

We proceed by induction. It follows from (4) that

$$\beta\leqslant n^{-k}A^{-k}(32)^{-2h_{r+1}k_r/T},\qquad r=1,\ldots,T_1-1.$$

Therefore the lemma is proved if we show that for every integer r with $1\leqslant r<T_1$, the following inequality is satisfied

$$2^{h_rk_r/T}>2\Delta^{16Dh_{r+1}}(2SL)^{16kD}.$$

It is easy to see that this inequality is satisfied if A is large enough depending only on ε.

LEMMA 5. *Assume that β satisfies* (4). *Then*

$$q\left(\frac{a}{p}, m_1,\ldots,m_{n-1}\right) = 0,$$

for all integers a, p with $0<p\leqslant R$, $0<a/p\leqslant h$ and for all non-negative integers $m_1,\ldots,m_{n-1}$ with $m_1+\ldots+m_{n-1}\leqslant k_{T_1+1}$.

Proof. Suppose that the lemma is not true. Then there exist integers a, p with $0<p\leqslant R$, $0<a/p\leqslant h$ and non-negative integers $m_1,\ldots,m_{n-1}$ with $m_1+\ldots+m_{n-1}\leqslant k_{T_1+1}$ such that $q(a/p, m_1,\ldots,m_{n-1})\neq 0$. Observe that $q(a/p, m_1,\ldots,m_{n-1})$ is a non-zero algebraic number of degree $\leqslant DR^n$. The denominator of $q(a/p, m_1,\ldots,m_{n-1})$ does not exceed

$$\Delta^{DhR}(2SL)^k.$$

The absolute value of the conjugates of $q(a/p, m_1,\ldots,m_{n-1})$ does not exceed

$$\Delta^{5h}(2SL)^{5k}.$$

Further

$$\left|(\log\alpha_1)^{-m_1}\ldots(\log\alpha_{n-1})^{-m_{n-1}}\varphi_{m_1,\ldots,m_{n-1}}\left(\frac{a}{p},\ldots,\frac{a}{p}\right)-q\left(\frac{a}{p}, m_1,\ldots,m_{n-1}\right)\right| \leqslant \beta\Delta^{5h}(2SL)^{5k}.$$

Hence

$$(8)\qquad \left|\varphi_{m_1,\ldots,m_{n-1}}\left(\frac{a}{p},\ldots,\frac{a}{p}\right)\right| \geqslant w\left(\Delta^{-6D^2hR^{n+1}}(2SL)^{-6kDR^n}-\beta\Delta^{5h}(2SL)^{5k}\right),$$

where

$$w = |(\log a_1)^{m_1}\ldots(\log a_{n-1})^{m_{n-1}}|.$$

Set

$$f(\zeta) = \varphi_{m_1,\ldots,m_{n-1}}(\zeta,\ldots,\zeta) \quad \text{and} \quad F(\zeta) = \left((\zeta-1)\ldots(\zeta-h_{T_1})\right)^{k_{T_1}-k_{T_1+1}+1}.$$

For every z with $|z| = 2h_{T_1}$, we have

$$\frac{1}{2\pi i}\int_{\Gamma:|\zeta|=7h_{T_1}}\frac{f(\zeta)F(z)}{(\zeta-z)F(\zeta)}\,d\zeta = f(z)+\sum_{r=1}^{h_{T_1}}\sum_{m=0}^{k_{T_1}-k_{T_1+1}}\frac{f^{(m)}(r)}{m!\,2\pi i}\int_{\Gamma_r:|\zeta-r|=\frac{1}{2}}\frac{(\zeta-r)^m F(z)}{(\zeta-z)F(\zeta)}\,d\zeta.$$

Note that the absolute value of the integral does not exceed

$$\Delta^{11h_{T_1}}(2SL)^{10k}2^{-h_{T_1}k_{T_1}/T}.$$

Using Lemma 4, absolute value of the double sum in the above formula does not exceed

$$\beta w n^k A^k \Delta^{5h_{T_1}}(2SL)^{5k}(16)^{2h_{T_1}k_{T_1}/T}.$$

Thus by maximum modulus principle,

$$(9)\qquad \left|f\left(\frac{a}{p}\right)\right| \leqslant w\left(\Delta^{11h_{T_1}}(2SL)^{10k}2^{-h_{T_1}k_{T_1}/T}+\right.$$
$$\left.+\beta n^k A^k \Delta^{5h_{T_1}}(2SL)^{5k}(16)^{2h_{T_1}k_{T_1}/T}\right).$$

Observe that

$$h_{T_1} > \left((nD)^{10\varepsilon/n}E^{\varepsilon/n}\right)^{T_1-1}h > (nD)^{10(n+1)}E^{n+1}h > R^{n+1}h.$$

We show that (8) and (9) are inconsistent. Using the above inequality and (4), it is sufficient to show that

$$2^{h_{T_1}k_{T_1}/T} > \Delta^{17D^2h_{T_1}}(2SL)^{17kDR^n}.$$

This inequality is satisfied, if Δ, depending only on ε, is large enough. This completes the proof of Lemma 5.

Remark. Let $p'(\lambda_1,\ldots,\lambda_n)$, $0\leqslant\lambda_i\leqslant L_i$ $(i=1,\ldots,n)$ be integers satisfying

$$|p'(\lambda_1,\ldots,\lambda_n)| \leqslant \Delta^{3h}(2SL)^{3k}.$$

Consider

$$q' = q'(z, m_1,\ldots,m_{n-1}) = \sum_{\lambda_1=0}^{L_1}\ldots\sum_{\lambda_n=0}^{L_n}p'(\lambda_1,\ldots,\lambda_n)\alpha_1^{\lambda_1 z}\ldots\alpha_n^{\lambda_n z}\gamma_1^{m_1}\ldots\gamma_{n-1}^{m_{n-1}}.$$

Suppose that $q'(l, m_1,\ldots,m_{n-1}) = 0$ for all integers l with $1\leqslant l\leqslant h$ and for all non-negative integers $m_1,\ldots,m_{n-1}$ with $m_1+\ldots+m_{n-1}\leqslant k$. If β satisfies (4), then our argument shows that

$$q'\left(\frac{a}{p}, m_1,\ldots,m_{n-1}\right) = 0,$$

for all integers a, p with $0<p\leqslant R$, $0<a/p\leqslant h$ and for all non-negative integers $m_1,\ldots,m_{n-1}$ with $m_1+\ldots+m_{n-1}\leqslant k_{T_1+1}$. We shall call q' is associated to $p'(\lambda_1,\ldots,\lambda_n)$.

LEMMA 6. *Assume that β satisfies* (4). *Let $p'(\lambda_1,\ldots,\lambda_n)$ be integers with $0\leqslant\lambda_i\leqslant L_i$ $(i=1,\ldots,n)$ satisfying*

$$|p'(\lambda_1,\ldots,\lambda_n)| \leqslant \Delta^{3h}(2SL)^{3k}.$$

Let $q' = q'(z, m_1,\ldots,m_{n-1})$ be associated to $p'(\lambda_1,\ldots,\lambda_n)$. Assume that for x_0 with $0<x_0\leqslant 1$, we have

$$q'(x_0+l, m_1,\ldots,m_{n-1}) = 0,$$

for all integers l with $0\leqslant l\leqslant h-1$ and for all non-negative integers $m_1,\ldots$ $\ldots,m_{n-1}$ with $m_1+\ldots+m_{n-1}\leqslant k_1$. Then

$$q'\left(\frac{a}{p}, m_1,\ldots,m_{n-1}\right) = 0,$$

for all integers a, p with $0<p\leqslant R$, $0<a/p\leqslant h$ and for all non-negative integers $m_1,\ldots,m_{n-1}$ with $m_1+\ldots+m_{n-1}\leqslant k_{T_1+2}$.

Proof. It is sufficient to prove that $q'(l, m_1,\ldots,m_{n-1}) = 0$ for all integers l with $1\leqslant l\leqslant h$ and non-negative integers $m_1,\ldots,m_{n-1}$ with $m_1+\ldots+m_{n-1}\leqslant k_2$. (Then the lemma would follow by the above remark.) Define

$$f(z) = \varphi'_{m_1,\ldots,m_{n-1}}(z,\ldots,z) \quad \text{with} \quad m_1+\ldots+m_{n-1}\leqslant k_2,$$

where

$$\varphi'_{m_1,\ldots,m_{n-1}}(z_1,\ldots,z_{n-1}) = \sum_{\lambda_1=0}^{L_1}\ldots\sum_{\lambda_n=0}^{L_n}p'(\lambda_1,\ldots,\lambda_n)\alpha_1^{\gamma_1 z_1}\ldots\alpha_{n-1}^{\gamma_{n-1}z_{n-1}}.$$

For every z with $|z| = 2h$, we have

$$\frac{1}{2\pi i}\int_{\Gamma:|\zeta|=\ldots}\frac{f(\zeta)F(z)}{(\zeta-z)F(\zeta)}\,d\zeta = f(z)+\sum_{r=0}^{h-1}\sum_{m=0}^{k_1-k_2}\frac{f^{(m)}(r+x_0)}{m!\,2\pi i}\int_{\Gamma_r}\frac{(\zeta-r-x_0)^m F(z)}{(\zeta-z)F(\zeta)}\,d\zeta,$$

where Γ_r denotes the circle with centre $(r+x_0)$ and radius $\frac{1}{2}$,

$$F(\zeta) = \prod_{u=0}^{h-1} (\zeta - u - x_0)^{k_1-k_2+1}.$$

From this formula and by maximum modulus principle, we have for an integer l with $1 \leqslant l \leqslant h$,

$$(10) \qquad |f(l)| \leqslant w\big(\Delta^{11h}(2SL)^{10k}2^{-hk/T} + \beta n^k A^k \Delta^{5h}(2SL)^{5k}(16)^{2hk/T}\big).$$

Further for an integer l with $1 \leqslant l \leqslant h$, we have

$$(11) \qquad |f(l)| \geqslant w\big(\Delta^{-6Dh}(2SL)^{-6Dk} - \beta \Delta^{5h}(2SL)^{5k}\big),$$

provided that $f(l) \neq 0$. (For details of these inequalities, one can refer to Lemma 4.) Observe that β satisfies (4). Proceed exactly as in Lemma 4 and show that (10) and (11) are inconsistent. This completes the proof of the lemma.

LEMMA 7. *Set* $f = \left[\frac{2\log L_n}{\log R}\right]+1$. *Assume that* β *satisfies* (4). *Then the number of primes* p *satisfying* (i) $R^{1/2} < p \leqslant R$, (ii) $K'(\alpha_n^{1/p})$ *is an extension of* $K' = K(\alpha_1^{1/p}, \ldots, \alpha_{n-1}^{1/p})$ *of degree* p, *is less than* f.

Proof. We shall assume that the number of primes satisfying the assumptions of the lemma is greater than or equal to f and shall arrive at a contradiction. Out of these, choose f primes and denote them by $p_1, \ldots, p_f$. For every integer a with $(a, p_1) = 1$ and $0 < a/p_1 \leqslant h$, we have from Lemma 5,

$$\sum_{\lambda_n=0}^{L_n}\Big(\sum_{\lambda_1=0}^{L_1}\cdots\sum_{\lambda_{n-1}=0}^{L_{n-1}} p(\lambda_1, \ldots, \lambda_n)\alpha_1^{\lambda_1 a/p_1}\ldots\alpha_{n-1}^{\lambda_{n-1}a/p_1}\gamma_1^{m_1}\ldots\gamma_{n-1}^{m_{n-1}}\Big)\alpha_n^{\lambda_n a/p_1} = 0,$$

for all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1 + \ldots + m_{n-1} \leqslant k_{T_1+1}$. Since $(a, p_1) = 1$, the above sum is still zero when λ_n is summed over any single residue class $\bmod p_1$. Let $\Lambda_1, \ldots, \Lambda_n$, $0 \leqslant \Lambda_i \leqslant L_i$, be arbitrary. Then

$$\sum_{\substack{\lambda_n=0\\ \lambda_n\equiv\Lambda_n(p_1)}}^{L_n}\Big(\sum_{\lambda_1=0}^{L_1}\cdots\sum_{\lambda_{n-1}=0}^{L_{n-1}} p(\lambda_1, \ldots, \lambda_n)\alpha_1^{\lambda_1 a/p_1}\ldots\alpha_{n-1}^{\lambda_{n-1}a/p_1}\gamma_1^{m_1}\ldots\gamma_{n-1}^{m_{n-1}}\Big)\alpha_n^{\lambda_n a/p_1} = 0,$$

for all integers a with $0 < a/p_1 \leqslant h$, $(a, p_1) = 1$ and non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1 + \ldots + m_{n-1} \leqslant k_{T_1+1}$. Here $\lambda_n \equiv \Lambda_n(p_1)$ stands for $\lambda_n \equiv \Lambda_n (\bmod p_1)$. Define

$$p_1(\lambda_1, \ldots, \lambda_n) = \begin{cases} p(\lambda_1, \ldots, \lambda_n), & \text{if} \quad \lambda_n \equiv \Lambda_n(p_1),\\ 0, & \text{otherwise}\end{cases}$$

and call q_1 the function associated to $p_1(\lambda_1, \ldots, \lambda_n)$. Then

$$q_1\left(\frac{1+lp_1}{p_1}, m_1, \ldots, m_{n-1}\right) = 0,$$

for all integers l with $0 \leqslant l \leqslant h-1$ and for all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1 + \ldots + m_{n-1} \leqslant k_{T_1+1}$. By Lemma 6, we have

$$q_1\left(\frac{a}{p'}, m_1, \ldots, m_{n-1}\right) = 0,$$

for all integers a, p' with $0 < p' \leqslant R$, $0 < a/p' \leqslant h$ and for non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1 + \ldots + m_{n-1} \leqslant k_{2(T_1+2)}$. Define

$$p_2(\lambda_1, \ldots, \lambda_n) = \begin{cases} p_1(\lambda_1, \ldots, \lambda_n), & \text{if} \quad \lambda_n \equiv \Lambda_n(p_2),\\ 0, & \text{otherwise.}\end{cases}$$

Proceed as above and conclude that

$$q_2\left(\frac{1+lp_2}{p_2}, m_1, \ldots, m_{n-1}\right) = 0,$$

for all integers l with $0 \leqslant l < h$ and $0 \leqslant m_1 + \ldots + m_{n-1} \leqslant k_{2(T_1+2)}$. Proceeding inductively, define

$$p_f(\lambda_1, \ldots, \lambda_n) = \begin{cases} p_{f-1}(\lambda_1, \ldots, \lambda_n), & \text{if} \quad \lambda_n \equiv \Lambda_n(p_f),\\ 0, & \text{otherwise.}\end{cases}$$

Then we have

$$(12) \qquad q_f\left(\frac{1+lp_f}{p_f}, m_1, \ldots, m_{n-1}\right) = 0,$$

for integers l with $0 \leqslant l < h$ and $m_1 + \ldots + m_{n-1} \leqslant k_{(T_1+2)f}$, $m_i \geqslant 0$. Observe that

$$p_1 \ldots p_f > R^{f/2} > L_n.$$

Hence

$$p_f(\lambda_1, \ldots, \lambda_n) = \begin{cases} p(\lambda_1, \ldots, \lambda_n), & \text{if} \quad \lambda_n = \Lambda_n,\\ 0, & \text{otherwise.}\end{cases}$$

Recall that A is a sufficiently large constant depending only on ε and $S_1, \ldots, S_n, S$ exceeds some absolute constant depending only on ε and A. Observe that

$$\log L_n \leqslant 3An\log(5nD) + 2\log A + \log\log S + 2n\log E$$
$$\leqslant 3An\log(5nD) + 3nE.$$

Thus

$$(T_1+2)f \leqslant \frac{16n^2}{\varepsilon}\frac{\log L_n}{\log R} \leqslant n^4 DE < T.$$

Therefore it follows from (2) that

$$k_{(T_1+2)f} \geqslant \frac{1}{18}k \geqslant L_1+\ldots+L_n.$$

Setting $l = 0$ in (12) and writing p for p_f, we have

$$\sum_{\lambda_1=0}^{L_1}\cdots\sum_{\lambda_{n-1}=0}^{L_{n-1}} p(\lambda_1,\ldots,\lambda_n)\alpha_1^{\lambda_1/p}\ldots\alpha_{n-1}^{\lambda_{n-1}/p}\gamma_1^{m_1}\ldots\gamma_{n-1}^{m_{n-1}} = 0,$$

for integers $m_1,\ldots,m_{n-1}$ with $0 \leqslant m_i \leqslant L_i$. Now proceed as in [8], p. 288, and conclude that $p(\lambda_1,\ldots,\lambda_n) = 0$ for all $(\lambda_1,\ldots,\lambda_n)$. This is a contradiction. This completes the proof of the lemma.

Denote by $\mathscr{A}$ the set of primes p satisfying (i) $R^{1/2} < p \leqslant R$, (ii) if p is in $\mathscr{A}$, then there are integers $j_1,\ldots,j_n$ with $0 \leqslant j_i < p$ $(1 \leqslant i < n)$, $j_n = 1$ such that $\alpha_1^{j_1}\ldots\alpha_n^{j_n} = \eta^p$ for some η in K.

LEMMA 8. *Suppose that β satisfies* (4). *Then the number of elements in $\mathscr{A}$ exceeds* $\frac{1}{4}\frac{R}{\log R}$.

Proof. By Lemma 2 and Lemma 7, the number of primes between $R^{1/2}$ and R which are not in $\mathscr{A}$ does not exceed $(2\log L_n)/\log R$. But the number of primes between $R^{1/2}$ and R exceeds $R/(2\log R)$. The lemma follows, since

$$8\log L_n \leqslant 120An^2D+24nE \leqslant 25n^2DE < R.$$

Now we shall apply Lemma 3 to the set $\mathscr{A}$ and $\alpha_1,\ldots,\alpha_n$ and obtain the following.

LEMMA 9. *Suppose that β satisfies* (4). *Then there exist rational integers* $b_1,\ldots,b_n$, $b_n \neq 0$,

$$|b_l| \leqslant (2Dn)^{8Dn}R^D A^{2D-1}, \quad 1 \leqslant l \leqslant n,$$

such that

$$\alpha_1^{b_1}\ldots\alpha_n^{b_n} = 1.$$

Proof. In the notation of Lemma 3, set $S = \mathscr{A}$, $p_0 \geqslant R^{1/2}$ and $P_0 \leqslant R$. Further $A_i = 2^D S_i^{2D}$, $i = 1,\ldots,n$ (see (1)). We show that the assumptions of Lemma 3 are satisfied. Observe that

$$\log A_1\ldots\log A_n \leqslant (3D)^n A$$

and so

$$\log(\log A_1\ldots\log A_n) \leqslant 4nD\log A.$$

Further $c_1 \leqslant 180D^5$ and so $c_2 \leqslant 180D^5 2^D$. Therefore

$$c \leqslant 4D^3n^2+\frac{D^2}{2}(\log 180+5\log D+D\log 2)$$
$$\leqslant 4D^3n^2+\frac{D^2}{2}(10+6D) \leqslant 4D^3n^2+8D^3 \leqslant 6D^3n^2.$$

Thus

$$D+\frac{c+(D^2+1)\log(\log A_1\ldots\log A_n)}{\log p_0} \leqslant D+\frac{2}{\log R}(6D^3n^2+8nD^3\log A)$$
$$\leqslant \frac{D\log R+20n^2D^3\log A}{\log R}.$$

By Lemma 8, the assumptions of Lemma 3, are satisfied if we show that

$$\tfrac{1}{4}R > D\log R+20n^2D^3\log A.$$

This follows from the definition of R. Now

$$c_0 = 4D^2c_1^{D-1}(2Dn)^{2Dn} \leqslant (3D)^{5D}(2Dn)^{3Dn} \leqslant (2Dn)^{6Dn}.$$

Hence it follows from Lemma 3 that there exist integers $b_1,\ldots,b_n$; $b_n \neq 0$, such that $\alpha_1^{b_1}\ldots\alpha_n^{b_n} = 1$. Further for $l = 1,\ldots,n$, we have

$$|b_l| \leqslant (2Dn)^{6Dn}(3D)^{2nD}R^D A^{2D-1} \leqslant (2Dn)^{8Dn}R^D A^{2D-1}.$$

LEMMA 10. *Assume that*

$$\beta \leqslant \exp\left(-(nD)^{u_1 n}A(\log A)^2(\log S)^2 E^{2(n+1+\varepsilon)}\right), \tag{13}$$

where u_1 is a large positive constant depending only on ε. Then there exist integers $b_1,\ldots,b_n$, $b_n \neq 0$,

$$|b_l| \leqslant (2Dn)^{8Dn}R^D A^{2D-1}, \quad 1 \leqslant l \leqslant n,$$

such that

$$\alpha_1^{b_1}\ldots\alpha_n^{b_n} = 1.$$

Proof. Set $M = (nD)^{10\varepsilon/n}E^{\varepsilon/n}$. Observe that

$$h_{T_1} \leqslant M^{T_1-1}h+M^{T_1-2}+\ldots+1 \leqslant 2T_1M^{T_1-1}h.$$

It is easy to check that the inequality (4) follows from (13). Hence the lemma follows from Lemma 9.

Remark. If $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent, then

$$\beta > \exp\left(-(nD)^{u_1 n} A (\log A)^2 (\log S)^2 E^{2(n+1+\varepsilon)}\right).$$

Denote by $u_2, u_3, \ldots$ effectively computable constants > 1 depending only on ε.

Proof of Theorem 1. Suppose that β satisfies (13). Then by Lemma 10, there exist rational integers $b_1, \ldots, b_n$, $b_n \neq 0$,

$$|b_l| \leqslant 2(Dn)^{u_2 Dn} A^{3D} (\log\log S)^D, \quad 1 \leqslant l \leqslant n,$$

such that

$$\alpha_n^{b_n} = \alpha_1^{b_1} \ldots \alpha_{n-1}^{b_{n-1}}.$$

Taking logarithm, we have

$$b_n \log \alpha_n = b_0 \log \alpha_0 + b_1 \log \alpha_1 + \ldots + b_{n-1} \log \alpha_{n-1},$$

where $\alpha_0 = -1$ and b_0 is an integer that compensates for using principal values of logarithm. We have

$$|b_0| \leqslant \sum_{l=1}^{n} |b_l| \leqslant (Dn)^{u_3 Dn} (\log\log S)^D A^{3D}.$$

Now our linear form becomes

$$\beta = \left| -\frac{b_0}{b_n} \log \alpha_0 + \left(\beta_1 - \frac{b_1}{b_n}\right) \log \alpha_1 + \ldots + \left(\beta_{n-1} - \frac{b_{n-1}}{b_n}\right) \log \alpha_{n-1} \right|.$$

Suppose that the coefficients of $\log \alpha_1, \ldots, \log \alpha_{n-1}$ are not all zero. Assume (if necessary, after renumbering $\alpha_1, \ldots, \alpha_{n-1}$) that the coefficient of $\log \alpha_{n-1}$ is not zero. We divide the linear form by the coefficient of $\log \alpha_{n-1}$ and obtain a new linear form

(14) $$|\beta_0^{(1)} \log \alpha_0 + \ldots + \beta_{n-2}^{(1)} \log \alpha_{n-2} - \log \alpha_{n-1}|.$$

It follows from (1) that the sizes of $\beta_0^{(1)}, \ldots, \beta_{n-2}^{(1)}$ do not exceed

$$S^{(1)} = (Dn)^{u_4 D^2 n} A^{9D^2} S^{4D^2},$$

and they lie in K. Set $E_1 = (\log A + \log\log S^{(1)})$. If we assume that

$$\beta \leqslant (S^{(1)})^{-2D} \exp\left(-(Dn)^{u_1 n} A (\log A)^2 (\log S^{(1)})^2 E_1^{2(n+1+\varepsilon)}\right),$$

then (14) does not exceed

$$\exp\left(-(Dn)^{u_1 n} A (\log A)^2 (\log S^{(1)})^2 E_1^{2(n+1+\varepsilon)}\right).$$

Then again appealing to Lemma 10, there exist integers $b_1^{(1)}, \ldots, b_{n-1}^{(1)}$, $b_{n-1}^{(1)} \neq 0$,

$$|b_l^{(1)}| \leqslant 2(Dn)^{u_2 Dn} A^{3D} (\log\log S^{(1)})^D, \quad 1 \leqslant l < n,$$

such that

$$\alpha_{n-1}^{b_{n-1}^{(1)}} = \alpha_1^{b_1^{(1)}} \ldots \alpha_{n-2}^{b_{n-2}^{(1)}}.$$

(Actually there exists such a multiplicative relation between $\alpha_0, \ldots, \alpha_{n-1}$, but this can be reduced to a multiplicative relation between $\alpha_1, \ldots, \alpha_{n-1}$ by squaring, if necessary.) Define inductively

$$S^{(i+1)} = (Dn)^{u_4 D^2 n} A^{9D^2} (S^{(i)})^{4D^2}, \quad 1 \leqslant i < n.$$

Put

$$E_n = (\log A + \log\log S^{(n)}).$$

Assume that

$$\beta \leqslant (S^{(n)})^{-2nD} \exp\left(-(nD)^{u_1 n} A (\log A)^2 (\log S^{(n)})^2 E_n^{2(n+1+\varepsilon)}\right).$$

Earlier restrictions on β follow from here. Now it is possible to continue the reduction process, eliminating at least one logarithm, until we arrive at

(15) $$|a \log \alpha_0| \leqslant \exp\left(-(nD)^{u_1 n} A (\log A)^2 (\log S^{(n)})^2 E_n^{2(n+1+\varepsilon)}\right),$$

where a is an algebraic number of K of size $\leqslant S^{(n)}$. The left-hand side of this inequality does not vanish, as our original linear form does not vanish. So

(16) $$|a \log \alpha_0| \geqslant u_5 (S^{(n)})^{-2D}.$$

Observe that (15) and (16) contradict each other. Thus

$$\beta \geqslant (S^{(n)})^{-2Dn} \exp\left(-(nD)^{u_1 n} A (\log A)^2 (\log S^{(n)})^2 E_n^{2(n+1+\varepsilon)}\right).$$

Observe that

$$\log S^{(n)} \leqslant (Dn)^{u_6 n} \log(AS), \quad \log\log S^{(n)} \leqslant u_6 (Dn)^2 E \quad \text{and} \quad E_n \leqslant u_7 (Dn)^2 E.$$

Hence

$$\beta \geqslant \exp\left(-(nD)^{u_8 n} A (\log A)^2 \left(\log(AS)\right)^2 E^{2(n+1+\varepsilon)}\right),$$

where u_8 is a large effectively computable constant depending only on ε. This completes the proof of Theorem 1.

References

[1] A. Baker, *Linear forms in the logarithms of algebraic numbers IV*, Mathematika 15 (1968), pp. 204–216.

[2] — *A sharpening of the bounds for linear forms in logarithms III*, Acta Arith. 27 (1975), pp. 247–252.

[3] A. Baker and H. M. Stark, *On a fundamental inequality in number theory*, Ann. Math. 94 (1971), pp. 190–199.

[4] K. Ramachandra, *Application of Baker's theory to two problems considered by Erdös and Selfridge*, J. Indian Math. Soc. 37 (1973), pp. 25–34.
[5] — *Lectures on Transcendental Numbers*, Ramanujan Institute, Madras 1969.
[6] — *Contributions to the theory of transcendental numbers (II)*, Acta Arith. 14 (1968), pp. 73–88.
[7] T. N. Shorey, *On gaps between numbers with a large prime factor II*, ibid. 25 (1974), pp. 365–373.
[8] H. M. Stark, *Further advances in the theory of linear forms in logarithms*, Diophantine Approximations and its Applications, New York 1973, pp. 255–294.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH
Bombay, India

Received on 18. 9. 1974 (619)

ACTA ARITHMETICA
XXX (1976)

On the product of the conjugates outside the unit circle of an algebraic integer

by

A. Bazylewicz (Warszawa)

The aim of this paper is to extend some results of A. Schinzel [4] and to make them more precise.

Let K be a number field of degree $|K|$, let

$$P(z) = p_0 z^n + p_1 z^{n-1} + \ldots + p_n$$

be a polynomial over K with the content $C(P) = (p_0, \ldots, p_n)$, let G be the set of all isomorphic injections of K into the complex field C and, for $\sigma \in G$, let

$$\sigma P(z) = \sigma p_0 z^n + \ldots + \sigma p_n = \sigma p_0 \prod_{i=1}^{n} (z - a_{\sigma i}).$$

Generalizing an argument of Smyth [5] concerning the fundamental case $K = Q$, Schinzel proved that if K is totally real, $P(z)$ is non-reciprocal, p_i are integers, $p_0 = 1$ and $p_n \neq 0$, then

$$\max_{\sigma \in G} \prod_{|a_{\sigma j}|>1} |a_{\sigma j}| \geqslant \theta_0, \tag{1}$$

where θ_0 is the real root of the equation $\theta^3 - \theta - 1 = 0$. We extend this in the following manner.

THEOREM 1. *Let K be a totally complex quadratic extension of a totally real field and $\sqrt{-3} \notin K$.*

If $P(z) \in K[z]$ is a monic polynomial with integer coefficients, $P(0) \neq 0$, $z^n \overline{P}(z^{-1}) \neq \mathrm{const}\, P(z)$, then (1) holds.

If $\sqrt{-3} \in K$ (1) needs not be satisfied, but $\Lambda \geqslant |\theta_1|$ where θ_1 is that root of the equation

$$\theta^2 + \frac{-1+\sqrt{-3}}{2}\theta - 1 = 0$$

which is greater in absolute value.