Conspectus materiae tomi XXX, fasciculi 1

La revue est consacrée à la Théorie des Nombres
The journal publishes papers on the Theory of Numbers
Die Zeitschrift veröffentlicht Arbeiten aus der Zahlentheorie
Журнал посвящен теории чисел

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ACTA ARITHMETICA
ul. Śniadeckich 8, 00-950 Warszawa

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PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

ACTA ARITHMETICA
XXX (1976)

A remark on certain Hecke L-series which are non-negative on the real axis

by

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1. We consider certain special series of the type

$$\sum \frac{\chi(N\mathfrak{a})}{(N\mathfrak{a})^s}$$

where $\mathfrak{a}$ runs over all integral ideals of an algebraic number field K, $N\mathfrak{a}$ denotes the norm of the ideal $\mathfrak{a}$; and further χ is an ordinary Dirichlet character.

THEOREM. *Let p denote a prime $\equiv 1 \pmod 4$ and let the class-number of $Q(\sqrt{p})$ be 1. Then the series*

$$L(s, \Psi) = \sum \frac{\Psi(\mathfrak{a})}{(N\mathfrak{a})^s}$$

where $\Psi(\mathfrak{a})$ is defined as $\chi(N\mathfrak{a})$ and χ is a character of exact order 4 is a Hecke L-series of $Q(\sqrt{p})$ with conductor $f = (\sqrt{p})$. Moreover the series is convergent for $s > \frac{1}{3}$, and non-negative in this interval. By the functional equation $L(s, \Psi)$ is non-negative on the real axis.

These are the first examples known to us of Hecke L-series which are non-negative on the real axis. This fact is unknown for ordinary Dirichlet L-series.

Proof. Let g denote a primitive root $(\bmod p)$. Call C_0, C_1, C_2, C_3 the classes of numbers $(\bmod p)$

$$C_0\colon\ g^{4m} \quad \left(0 \leqslant m < \frac{p-1}{4}\right),$$

$$C_1\colon\ g^{4m+1} \quad \left(0 \leqslant m < \frac{p-1}{4}\right),$$

$$C_2\colon\ g^{4m+2} \quad \left(0 \leqslant m < \frac{p-1}{4}\right),$$

$$C_3\colon\ g^{4m+3} \quad \left(0 \leqslant m < \frac{p-1}{4}\right).$$

* Supported in part by NSF grant GP36418X1.

Then $\chi(n) = +1, -1, i, -i$ for $(n, p) = 1$, according as $n \in C_0, C_2, C_1, C_3$ resp. $(\bmod p)$. $\chi(n) = 0$ if $p|n$.

Now

$$L(s, \Psi) = L(s, \chi) L(s, \chi\chi_1) \tag{1}$$

where $\chi_1(n) = \left(\frac{n}{p}\right)$, the Legendre symbol. This follows from the fact that

$$\zeta_{Q(\sqrt{p})}(s) = \zeta(s) L(s, \chi_1).$$

Since $N\mathfrak{a} \equiv$ a square $(\bmod p)$, it follows that the number $N\mathfrak{a}$ never falls into the classes C_2 and C_3. Thus $L(s, \Psi)$ is real. Consequently (or directly) the factors (for real s) $L(s, \chi)$ and $L(s, \chi\chi_1)$ on the right-side of (1) are conjugates. Hence

$$L(s, \Psi) = P^2(s) + Q^2(s)$$

where $P(s)$ and $Q(s)$ are resp. the real and imaginary parts of $L(s, \chi)$.

Ex. $p = 5$, $K = Q(\sqrt{5})$

$$L(s, \Psi) = \left(\frac{1}{1^s} - \frac{1}{4^s} + \frac{1}{6^s} - \frac{1}{9^s} + - \ldots\right)^2 + \left(\frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{7^s} - \frac{1}{8^s} + - \ldots\right)^2.$$

Here we observe that $L(s, \Psi) > 0$ for $\frac{1}{2} \leqslant s < 1$.

2. It is known (see p. 120 of Landau's massive memoir *Über Ideale und Primideale im Idealklassen*, Math. Zeitschr. 2 (1918)) that

$$\sum_{N\mathfrak{a} \leqslant X} \Psi(\mathfrak{a}) = O(X^{1/3}).$$

It follows by partial summation that the Dirichlet series

$$\sum \frac{\Psi(\mathfrak{a})}{N\mathfrak{a}^s}$$

is convergent for $\operatorname{Re} s > \frac{1}{3}$.

3. To see that our series is a Hecke L-series we simply note that $0, 1, \ldots, p-1$ form a complete set of representatives for the ideal $f = (\sqrt{p})$. But $\chi(N\mathfrak{a})$ is a character on the multiplicative group of non-zero representatives, and it is periodic, $\bmod f$.

Hence it is a Hecke character in the narrow sense (see Landau, ibid., pp. 63–75) with conductor f.

4. In conclusion we remark that for each fixed prime $p \equiv 1 (\bmod 4)$ and with the class-number of $Q(\sqrt{p})$ equal to 1, we have an infinite class of Hecke L-series which are non-negative on the real-axis, namely the

series

$$L(s, \Psi_1) = \sum_{\mathfrak{a}} \frac{\Psi(\mathfrak{a})\left(\frac{N\mathfrak{a}}{q}\right)}{(N\mathfrak{a})^s}$$

where $\left(\frac{n}{q}\right)$ is a Legendre symbol. This series is also a product of two conjugate Dirichlet L-series. Hence in the functional equation the factor, often referred to as $W(\Psi_1)$, of absolute value 1, is indeed $+1$. This remark extends the non-negativity of $L(s, \Psi_1)$ onto the whole real axis.

5. We conjecture that Hecke L-series of the type discussed above are the only ones expressible as a sum of two squares of Dirichlet series with integral coefficients.

It would be interesting to know when Hecke L-series are expressible as sums of squares of Dirichlet series that can be analytically continued over the whole s-plane.

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Received on 25. 2. 1974 (542)