A remark on certain Hecke $L$-series which are non-negative on the real axis

by

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1. We consider certain special series of the type

$$
\sum \frac{\chi(Na)}{(Na)^s}
$$

where $a$ runs over all integral ideals of an algebraic number field $K$, $Na$ denotes the norm of the ideal $a$; and further $\chi$ is an ordinary Dirichlet character.

**Theorem.** Let $p$ denote a prime $= 1 \pmod{4}$ and let the class-number of $Q(\sqrt{p})$ be 1. Then the series

$$
L(s, \mathcal{P}) = \sum \frac{\mathcal{P}(a)}{(Na)^s}
$$

where $\mathcal{P}(a)$ is defined as $\chi(Na)$ and $\chi$ is a character of exact order 4 is a Hecke $L$-series of $Q(\sqrt{p})$ with conductor $f = (\mathcal{P})$. Moreover the series is convergent for $s > \frac{1}{4}$, and non-negative in this interval. By the functional equation $L(s, \mathcal{P})$ is non-negative on the real axis.

These are the first examples known to us of Hecke $L$-series which are non-negative on the real axis. This fact is unknown for ordinary Dirichlet $L$-series.

**Proof.** Let $g$ denote a primitive root $\pmod{p}$. Call $C_0$, $C_1$, $C_2$, $C_3$ the classes of numbers $\pmod{p}$

- $C_0$: $g^{4m}$ ($0 \leq m < \frac{p-1}{4}$),
- $C_1$: $g^{4m+1}$ ($0 \leq m < \frac{p-1}{4}$),
- $C_2$: $g^{4m+2}$ ($0 \leq m < \frac{p-1}{4}$),
- $C_3$: $g^{4m+3}$ ($0 \leq m < \frac{p-1}{4}$).

*Supported in part by NSF grant GP36418X1.
Then $\chi(n) = +1, -1, i, -i$ for $(n, p) = 1$, according as $n \equiv C_0, C_2, C_1, C_3 \pmod{p}$. $\chi(n) = 0$ if $p|n$.

Now

$$L(s, \chi) = L(s, \chi)L(s, \chi i)$$

where $\chi(n) = \left(\frac{n}{p}\right)$, the Legendre symbol. This follows from the fact that

$$\zeta_{\text{cycl}}(s) = \zeta(s)L(s, \chi).$$

Since $Na$ is a square $(\mod p)$, it follows that the number $Na$ never falls into the classes $C_2$ and $C_3$. Thus $L(s, \chi)$ is real. Consequently (or directly) the factors (for real $s$) $L(s, \chi)$ and $L(s, \chi i)$ on the right-side of (1) are conjugates. Hence

$$L(s, \chi) = P^2(s) + Q^2(s)$$

where $P(s)$ and $Q(s)$ are resp. the real and imaginary parts of $L(s, \chi)$.

Ex. $p = 5, K = Q(\sqrt{5})$

$$L(s, \chi) = \left(\frac{1}{1^5} - \frac{1}{4^5} + \frac{1}{6^5} - \frac{1}{9^5} + \ldots\right)^2 + \left(\frac{1}{2^5} - \frac{1}{3^5} + \frac{1}{7^5} - \frac{1}{8^5} + \ldots\right)^2.$$

Here we observe that $L(s, \chi) > 0$ for $\frac{1}{2} \leq s < 1$.

2. It is known (see p. 120 of Landau’s massive memoir “Über Ideale und Primideale im Idealklassen, Math. Zeitschr. 2 (1913)) that

$$\sum_{N \leq X} \Psi(a) = O(X^{1+}).$$

It follows by partial summation that the Dirichlet series

$$\sum \frac{\Psi(a)}{Na^s}$$

is convergent for $\text{Re} s > \frac{1}{2}$.

3. To see that our series is a Hecke $L$-series we simply note that $0, 1, \ldots, p - 1$ form a complete set of representatives for the ideal $I = (\sqrt{p})$. But $\chi(Na)$ is a character on the multiplicative group of non-zero representatives, and it is periodic, $\mod f$.

Hence it is a Hecke character in the narrow sense (see Landau, ibid., pp. 63–75) with conductor $f$.

4. In conclusion we remark that for each fixed prime $p \equiv 1 \pmod{4}$ and with the class-number of $Q(\sqrt{p})$ equal to 1, we have an infinite class of Hecke $L$-series which are non-negative on the real-axis, namely the

$$L(s, \chi) = \sum \frac{\Psi(a)}{(Na)^s}$$

where $\left(\frac{a}{q}\right)$ is a Legendre symbol. This series is also a product of two conjugate Dirichlet $L$-series. Hence in the functional equation the factor, often referred to as $\Psi(1)$, of absolute value 1, is indeed $-1$. This remark extends the non-negativity of $L(s, \chi)$ onto the whole real axis.

5. We conjecture that Hecke $L$-series of the type discussed above are the only ones expressible as a sum of two squares of Dirichlet series with integral coefficients.

It would be interesting to know when Hecke $L$-series are expressible as sums of squares of Dirichlet series that can be analytically continued over the whole $s$-plane.

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Received on 25. 2. 1974 (542)