Also:

$$M_3 - M_3' = n(3\gamma^2 + 18\gamma + 36) \equiv 4 \pmod{8},$$

weil γ gerade ist.
Von den beiden ungeraden Zahlen M_3 und M_3' ist also wenigstens eine $\not\equiv 7 \pmod{8}$ und erfüllt daher die in 1. genannte Bedingung.

7. Ist endlich $M_1 \equiv 0, 8, 16 \pmod{24}$, also $M_1 = 8k$, so bestimme ich wieder γ_1 gemäss 5. und < 6 und $\gamma_2 = \gamma_1 + 6$.
Von diesen beiden Zahlen ist eine $\equiv 2 \pmod{4}$. Diese nehme ich und bezeichne sie mit $\gamma = 2\bar{\gamma}$ wo $\gamma < 12$ und $\bar{\gamma} \equiv 1 \pmod{2}$ sein wird.
Dazu betrachte ich die Zahl $\gamma' = \gamma + 12$, die demnach auch noch < 24, also, weil sie gerade ist, $\leqq 22$ ist.
Ich erhalte dann:

$$M_1 - n\gamma^3 = 6M_3 \text{ und } M_1 - n\gamma'^3 = 6M_3',$$

wo jetzt M_3 und M_3' durch 4 teilbar, $= 4\bar{M}_3$ bezw. $= 4\bar{M}_3'$ sein werden, da γ gerade und $M_1 = 8k$ ist.
Wie oben wird dann

$$6(M_3 - M_3') = n\gamma'^3 - n\gamma^3 = n(36\gamma^2 + 3.12^2\gamma + 12^3).$$

Also

$$M_3 - M_3' = n(6\gamma^2 + 72\gamma + 2 . 12^2)$$

oder:

$$\bar{M}_3 - \bar{M}_3' = n(6\bar{\gamma}^2 + 36\bar{\gamma} + 72) \equiv 2 \pmod{4},$$

weil $\bar{\gamma}^2 \equiv 1 \pmod{4}$ und $(n, 4) = 1$ ist.
Von den beiden Zahlen $\bar{M}_3$ und $\bar{M}_3'$ ist also wenigstens eine nicht von der Form $4^\eta(8\xi + 7)$, da nicht beide $\equiv 0 \pmod{4}$ oder $\equiv 7 \pmod{8}$ sein können.
Das zugehörige M_3 oder M_3' erfüllt also die aufgestellte Bedingung, sodass nun alle Fälle erledigt sind und damit der Satz bewiesen ist.

Basel, den 4. März 1938.

(Eingegangen am 12. März 1938.)

On the fractional parts of the powers of a rational number.

By

Kurt Mahler (Manchester).

Let u and v be two coprime integers with $u > v > 1$, such that $\frac{u}{v} > 1$, suppose that

$$\rho_n = \left(\frac{u}{v}\right)^n - \left[\left(\frac{u}{v}\right)^n\right].$$

Then the following results, as special cases of more general theorems, are proved in this paper:

a: $$\lim_{n\to\infty} v^n \rho_n = \infty.$$

b: When ε is a positive constant and

$$\rho_n \leqslant u^{-\varepsilon n}$$

for an infinite sequence of positive integers $n = n_1, n_2, n_3, \ldots$ with $n_{\nu+1} > n_\nu$, then

$$\limsup_{n\to\infty} \frac{n_{\nu+1}}{n_\nu} = \infty.$$

The proofs of a) and b) depend on generalizations of the Thue-Siegel theorem, due to Schneider or myself, and are very simple.

I.

1) Some years ago, I proved the following theorem [1]):

[1]) Math. Annalen 107 (1932), 691—730, in particular Satz 2, p. 722.

LEMMA 1: *Let $F(x,y)$ be an irreducible binary form of degree $n \geqq 3$ with integer coefficients, x and y two coprime integers, $P_1, P_2, \ldots, P_t$ $(t \geqq 1)$ a finite number of different prime numbers, and $Q(x,y) = P_1^{h_1} P_2^{h_2} \ldots P_t^{h_t}$ the greatest product of powers of these primes, which divides $F(x,y)$. Then*

$$Q(x,y) \leqslant c_0 \max\ (|x|, |y|)^{2\sqrt{n}},$$

where $c_0 > 0$ is a constant, which does not depend on x and y.

From this lemma, the following one is a trivial consequence:

LEMMA 2: *Let a, b, x be three non-vanishing integers, $n \geqslant 5$ a prime number, v an integer $\geqslant 2$, and $q(x) = v^{\nu}$ the highest power of v, which divides $ax^n - b$. Then*

$$q(x) \leqslant c_1 |x|^{2\sqrt{n}+1},$$

where $c_1 > 0$ is a constant, which does not depend on x.

Proof: Since n is an odd prime, the binary form $F(x,y) = ax^n - by^n$ either is irreducible, or of the form

$$F(x,y) = (\alpha x - \beta y)\ G(x,y),$$

where α, β are integers, and $G(x,y)$ is an irreducible binary form of degree $n-1$. Suppose that $P_1, P_2, \ldots, P_t$ are the different prime factors of v. Then apply Lemma 1 with $y = 1$ to $F(x,y)$ in the first case, and to $G(x,y)$ in the second case. Then we get

$$q(x) = O(|x|^{2\sqrt{n}})$$

in the first case, and

$$q(x) = O(|x| \cdot |x|^{2\sqrt{n-1}})$$

in the second case, since $\alpha x - \beta = O(|x|)$.

THEOREM 1: *Let a, b, u, v be four non-vanishing integers with $u > v > 1$. Then the equation*

$$(1): \qquad a u^x - v^x y = b$$

has at most a finite number of solutions in integers $x \geqslant 0$ and y.

Proof: Let λ be the number

$$\lambda = \frac{\log v}{\log u};$$

thus $0 < \lambda < 1$. Take for n a prime number $\geqslant 5$, such that

$$1 + 2\sqrt{n} < \lambda n;$$

this condition is satisfied, for instance, when

$$n \geqslant \left(\frac{3}{\lambda}\right)^2.$$

Obviously, to every solution x,y of (1), there are two integers ξ and ν with

$$x = n\xi + \nu, \quad \xi \geqslant 0, \quad 0 \leqslant \nu \leqslant n-1, \quad au^{\nu}(u^{\xi})^n - b = v^{\nu} y\, (v^{\xi})^n.$$

Hence

$$a u^{\nu} X^n - b, \qquad \text{where} \qquad X = u^{\xi},$$

is divisible by a power of v, which, at least, is equal to

$$(v^{\xi})^n = X^{\lambda n}.$$

But by Lemma 2, applied to each of the n polynomials

$$au^{\nu} X^n - b \qquad (\nu = 0, 1, \ldots, n-1),$$

this power of v must be

$$O(X^{2\sqrt{n}+1}),$$

and therefore X and x connot be arbitrarily large, i. e., (1) has at most a finite number of solutions, q. e. d.

THEOREM 2: *Under the conditions of theorem 1, the congruence*

$$a u^x \equiv d \pmod{v^x}$$

can hold only for a finite number of integers $x \geqslant 0$.

THEOREM 3: *Suppose that a, u, v are integers with $a \neq 0$, $u > v > 1$, $v \nmid u$.* Then

$$\lim_{n \to \infty} v^n \left\{ a \left(\frac{u}{v}\right)^n - \left[a \left(\frac{u}{v}\right)^n \right] \right\} = \infty.$$

These two theorems are trivial consequences of Theorem 1. In the case of Theorem 3, the additional condition $v \nmid u$ makes it impossible, that $au^x - v^x y = 0$ has an infinity of solutions.

II.

2) The following theorem can be proved:

LEMMA 3: *Let $\vartheta \neq 0$ be an algebraic number and p_1/q_1, p_2/q_2, p_3/q_3,*

... an infinite sequence of simplified fractions with the following properties:

a: $$1 \leqslant q_1 < q_2 < q_3 < \cdots.$$

b: For every n, p_n *and* q_n *can be written as*

$$p_n = P_1^{h_1} \ldots P_s^{h_s} p_n^*, \qquad q_n = Q_1^{k_1} \ldots Q_t^{k_t} q_n^*,$$

where $P_1, \ldots, P_s, Q_1, \ldots, Q_t$ *is a given finite system of different prime numbers,* $h_1, \ldots, h_s, k_1, \ldots, k_t$ *are integers* $\geqslant 0$ *and* p_n^*, q_n^* *are integers, such that as* $n \to \infty$

$$p_n^* = O(p_n^{\alpha}), \qquad q_n^* = O(q_n^{\beta}),$$

where α, β *are given constants with* $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant 1$.

c: For every n

$$\left| \vartheta - \frac{p_n}{q_n} \right| \leqslant q_n^{-\gamma},$$

where γ *is a constant with* $\gamma > \alpha + \beta$.

Then

$$\limsup_{n \to \infty} \frac{\log q_{n+1}}{\log q_n} = \infty.$$

For $\alpha = \beta = 1$, $s = t = 0$, this theorem was proved by Th. Schneider [2], and by using his method, I proved it [3]) for $\alpha = 0$, $\beta = 1$, $t = 0$, or for $\alpha = 1$, $\beta = 0$, $s = 0$, or for $\alpha = \beta = 0$. The same method, however, leads also to the general result of Lemma 3, as a study of the proof shows. (It is sufficient for this purpose, to use approximation polynomials of the form

$$R(z_1, z_2, \ldots, z_n) = \sum R_{l_1 l_2 \ldots l_k} z_1^{l_1} z_2^{l_2} \ldots z_k^{l_k},$$

where the summation sign refers to all integers $l_1, l_2, \ldots, l_k$ with

$$0 \leqslant l_1 \leqslant r_1,\ 0 \leqslant l_2 \leqslant r_2, \ldots, 0 \leqslant l_k \leqslant r_k,\quad \frac{k}{2}(1-\varepsilon) \leqslant \sum_{\varkappa=1}^{k} \frac{l_\varkappa}{r_\varkappa} \leqslant \frac{k}{2}(1+\varepsilon).$$

Compare Kapitel 1 of my paper, in particular § 6 and § 8).

[2]) Journal reine u. angew. Math. 175 (1937), „Über die Approximation algebraischer Zahlen".

[3]) Proceedings Royal Academy Amsterdam, 39 (1937), 633—640, 729—737.

THEOREM 4: *Suppose that* $\vartheta \neq 0$ *is an algebraic number and that u and v are integers with* $u > v > 1$, $v \nmid u$, *that* ε *is a positive constant, and that* $n = n_1, n_2, n_3, \ldots$ *is an infinite increasing sequence of positive integers, for which*

(2): $$\vartheta\left(\frac{u}{v}\right)^n - \left[\vartheta\left(\frac{u}{v}\right)^n\right] \leqslant u^{-\varepsilon n}.$$

Then

$$\limsup_{\nu \to \infty} \frac{n_{\nu+1}}{n_\nu} = \infty.$$

Proof: If again

$$\lambda = \frac{\log v}{\log u},$$

then (2) obviously is equivalent to

$$0 \leqslant \vartheta - \frac{v^n\left[\vartheta\left(\frac{u}{v}\right)^n\right]}{u^n} \leqslant \left(\frac{v}{u}\right)^n u^{-\varepsilon n} = u^{-(1-\lambda+\varepsilon)n}.$$

Hence, Lemma 3 can be applied with

$$p = v^n\left[\vartheta\left(\frac{u}{v}\right)^n\right],\quad p^* = \left[\vartheta\left(\frac{u}{v}\right)^n\right],\quad q = u^n,\quad q^* = 1,$$

so that

$$\alpha = 1 - \lambda,\ \beta = 0,\ \alpha + \beta < \gamma = 1 - \lambda + \varepsilon,$$

and the assertion follows at once.

Probably, (2) has only a finite number of solutions for n.

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