(B1) Let first $a_4 = 1/3$. Substituting it into R_3 and multiplying by 4, we get $5a_3^2 \equiv 4/5 \pmod 1$, and so $a_3 = -2/5$ or $-3/5$. Since $a_3 = -2/5$ gives $2a_1 \equiv 2a_5 - 6a_4/5 - a_3 \equiv 4/5 \pmod 1$, which contradicts $2a_1 \equiv 0 \pmod 1$, we have $a_3 = -3/5$. Then from (30), we get

$$a_1 \equiv a_5 - 3a_4/5 - \tfrac{1}{2}a_3 \equiv 0,\ a_2 \equiv -a_1 + 5a_3/4 \equiv -3/4 \pmod 1)$$

Hence we have $a_1 = 0$, $a_2 = -3/4$, $a_3 = -3/5$, $a_4 = -1/3$, $a_5 = -\frac{1}{2}$ and they give the form $f_{6,2}$.

It can be easily computed that the adjoint form of $f_{6,2}$ is

$$11\left(\tfrac{1}{2}x_1^2 + \tfrac{1}{4}x_2^2 + \tfrac{4}{5}(x_3 + \tfrac{1}{4}x_2 + \tfrac{1}{2}x_1)^2 + \tfrac{5}{6}(x_4 + \tfrac{6}{5}x_3 + \tfrac{4}{5}x_2 + \tfrac{3}{5}x_1)^2 + (x_5 + \tfrac{1}{2}x_4 + x_3 + \tfrac{1}{2}x_2 + x_1)^2 + \tfrac{12}{11}(x_6 + \tfrac{1}{2}x_5 + \tfrac{7}{12}x_4 + \tfrac{3}{2}x_3 + \tfrac{17}{12}x_2 + x_1)^2\right)$$

and it represents the value 4 for $x_1 = x_3 = x_5 = 0$, $x_2 = 1$, $x_4 = x_6 = -1$. Since $4 < 11$, the form $f_{6,2}$ can be decomposed into a sum of a square and a positive definite quadratic form both with integer coefficients.

(B2) Let then $a_4 = -\frac{1}{6}$. Substituting it into R_3 and multiplying by 4, we get $5a_3^2 \equiv 1/5 \pmod 1$ and so $a_3 = -1/5$ or $-4/5$. Since $a_3 = -1/5$ gives $2a_1 \equiv -6a_4/5 - a_3 \equiv 1/5 + 1/5 \pmod 1$, which contradicts to $2a_1 \equiv 0 \pmod 1$. Hence we have $a_3 = -4/5$. From (30), we get

$$a_1 \equiv a_5 - 3a_4/5 - \tfrac{1}{2}a_3 \equiv 0,\quad a_2 \equiv 5a_3/4 - a_1 \equiv 0 \pmod 1.$$

Hence $a_1 = a_2 = 0$, $a_3 = -4/5$, $a_4 = -1/6$, $a_5 = -\frac{1}{2}$. These values give the form $f_{6,3}$.

It can be easily computed that the adjoint form of $f_{6,3}$ is

$$11\left(\tfrac{1}{2}x_1^2 + \tfrac{1}{4}x_2^2 + \tfrac{4}{5}(x_3 + \tfrac{1}{4}x_2 + \tfrac{1}{2}x_1)^2 + \tfrac{5}{6}(x_4 + \tfrac{6}{5}x_3 + \tfrac{4}{5}x_2 + \tfrac{3}{5}x_1)^2 + (x_5 + \tfrac{1}{2}x_4 + x_3 + \tfrac{1}{2}x_2 + x_1)^2 + \tfrac{12}{11}(x_6 + \tfrac{1}{2}x_5 + \tfrac{5}{12}x_4 + \tfrac{3}{2}x_3 + \tfrac{7}{12}x_2 + x_1)^2\right),$$

which represents the value 4 for $x_1 = x_3 = x_5 = x_6 = 0$, $x_2 = 1$, $x_4 = -1$. Hence $f_{6,3}$ can be decomposed as 4 is less than the determinant value of $f_{6,3}$.

Here the proof of lemma 8 and so the theorem 3 is completed.

In closing, I should like to thank Prof. Mordell for his kind help with my manuscript.

(Received 22 November, 1937.)

On the positive definite quadratic forms with determinant unity.

By

Chao Ko (Manchester).

It will make the results of this paper more intelligible to the reader if we commence by giving a brief resumé of a little of the theory of quadratic forms.

Let

$$f_n = \sum_{i,j=1}^{n} a_{ij}x_ix_j \qquad (a_{ij} = a_{ji})$$

be a positive definite quadratic form with determinant D_n and integer coefficients a_{ij}. Denote the minor determinant of the matrix (a_{ij}) $(i,j = 1, \ldots, n)$ of f_n formed by the elements at the intersections of rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_k$ by $A^{(k)}_{i_1, \ldots, i_k; j_1, \ldots, j_k}$; the greatest common divisor of all the minors of order $k = 1, \ldots, n$ by d_k, so that $d_k \mid d_{k+1}$. Write $d_0 = 1$, $d_{n+1} = 0$. Denote the greatest common divisor of all the integers

$$A^{(k)}_{i_1, \ldots, i_k; i_1, \ldots, i_k}/d_k, \qquad 2A^{(k)}_{i_1, \ldots, i_k; j_1, \ldots, j_k}/d_k,$$

by $\sigma_k = 1$ or 2 $(k = 1, \ldots, n)$, and write $\sigma_0 = 1$. Define the numbers, really integers

$$o_k = d_{k+1}d_{k-1}/d_k^2 \quad (k = 1, \ldots, n),$$

so that

(1) $$D_n = d_n = d_1\, o_{n-1}\, o_{n-2}^2 \dots o_2^{n-2} o_1^{n-1}.$$

Then the o's and σ's are arithmetical invariants of f_n. All the forms having the same system of o's and σ's belong to the same *order*.

Write

(2) $$A^{(k)}{}_{1,2,\dots,k;\,1,2,\dots,k}/d_k\sigma_k = A_k \qquad (k = 1, \dots, n);$$

then by a linear transformation, we can make all the A's relatively prime to $2o_1o_2\dots o_{n-1}$[1]. For sufficiently large t, we also can transform f_n by a unimodular linear substitution into f_n', a form in n variables ξ, ξ', such that

(3) $$f_n' \equiv \sum_{s=1}^{\lambda} 2^{v_s}\varphi_s \pmod{2^t} \qquad v_\lambda > v_{\lambda-1} > \dots > v_1 \geqslant 0,$$

where the v's are integers and φ_s is either of the type

(3.1) $$\varphi_s = \sum_{i=1}^{\varkappa_s} a_{is}\xi_{is}\xi_{is} \qquad (\varkappa_s \text{ an integer}),$$

or

(3.2) $$\varphi_s = \sum_{i=1}^{\frac{1}{2}\varkappa_s} (2\alpha_{is}\xi_{is}\xi_{is} + 2\alpha'_{is}\xi_{is}\xi'_{is} + 2\alpha''_{is}\xi'_{is}\xi'_{is}) \qquad (\varkappa_s \text{ even}),$$

the α_{is}, α'_{is} being odd and $\sum_{s=1}^{\lambda}\varkappa_s = n$[2].

Write

(4) $$\omega_i = v_i - v_{i-1} \qquad (i = 1, \dots, n-1),$$

and

(5) $$2^{\mu_i} = \sigma_{i-1} 2^{\omega_i} \sigma_{i+1}, \qquad o_i = 2^{\omega_i} e_i \qquad (i = 1, \dots, n-1).$$

Then the following characters are also arithmetical invariants of f_n[3])

[1]) Bachman, Zahlentheorie, vol. 4, part 1, 452.
[2]) Ibid. 444.
[3]) Ibid. 473.

(6.1) $\left(\frac{A_i}{p_i}\right)$, the Legendre symbols of quadratic residuacity, where p_i runs through the different odd prime divisors of o_i;

(6.2) when $\mu_i \geqslant 2$, the units $(-1)^{\frac{1}{2}(A_i-1)}$;

(6.3) when $\mu_i \geqslant 3$, the units $(-1)^{\frac{1}{2}(A_i-1)}$ and $(-1)^{(A_i-1)/8}$,

(6.4) when $\mu_{i-1} \geqslant 2$, $\mu_i = 0$, $\mu_{i+1} \geqslant 2$, $\sigma_i = 1$ and $e_i A_{i-1} A_{i+1} \equiv 1 \pmod 4$, the units $(-1)^{\frac{1}{2}(A_i-1)}$.

All the forms having the same system of characters belong to the same *genus;* and all the forms equivalent to each other belong to the same *class.* In one order, there may be several genera; and in one genus, there may be several classes of forms.

For given n and D_n, we shall denote the number of orders, genera, and classes by O_{n,D_n}, G_{n,D_n} and C_{n,D_n}.

An *automorph* is defined as an integral linear transformation with determinant ± 1, which transforms f_n into itself. The reciprocal of the number of automorphs of f_n is the *mass of the class* represented by f_n; and the *mass of a genus* of forms is the sum of the masses of all the different classes of the genus.

Hermite[4]) proved that for $n = 1, \dots, 7$; $C_{n,1} = 1$ with a numerical error corrected by Stouff when $n = 7$, and so $G_{n,1} = O_{n,1} = 1$. For $n = 8$, I[5]) proved that there is only one class of forms which represents odd integers and Mordell[6]) proved that $C_{8,1} = 2$, from which we easily deduce that $G_{8,1} = O_{8,1} = 2$. For $n = 9$, I[7]) proved that $C_{9,1} = 2$ and one can deduce that $G_{9,1} = O_{9,1} = 1$. By using my method, Ketley[8]) proved that $C_{10,1} = 2$. and so $G_{10,1} = O_{10,1} = 1$. A long time ago, Minkowski[9]) proved that $C_{n,1} \geqslant [n/8] + 1$. Magnus[10]) proved that the mass of the principal genus of forms in n variables with determinant D_n is greater than $n^{n^2(1-\varepsilon)/4}$ for $n > n_0$, where $\varepsilon = \varepsilon(n_0)$ is a small fixed positive quantity, and so, as Dr. Mahler pointed out, $C_{n,D_n} > n^{n^2(1-\varepsilon)/4}$ for $n > n_0$. By other considerations, Erdös and I[11]) proved that $C_{n,1} \geqslant 2^{\sqrt{n}}$ for large n.

[4]) Hermite, Oeuvres, vol. 1, 129.
[5]) Ko, Quart. J. Math. (Oxford) 8 (1937), 81—98.
[6]) Mordell, J. de Mathématique, 17 (1938), 41—46.
[7]) Ko, J. London Math. Soc., 13 (1938), 102—110.
[8]) M. Sc. dissertation of Manchester University, (1938).
[9]) Minkowski, Gesammelte Abhandlungen, 1 (1909), 77.
[10]) Magnus, Math. Annalen, 114 (1937), 465—475.
[11]) Erdös and Ko, "On the decomposition of a definite quadratic form into a sum of two definite or semi-definite quadratic forms," may appear in Acta Arithmetica.

In the present note, I shall prove

THEOREM 1. *$G_{n,1}=O_{n,1}=1$ for all $n \not\equiv 0 \pmod 8$ and $G_{n,1}=O_{n,1}=2$ for all $n\equiv 0 \pmod 8$.*

THEOREM 2. *$C_{10,1}=C_{11,1}=2$ and $C_{12,1}=C_{13,1}\geqslant 3$.*

To prove these theorems, we require the following known lemmas.

LEMMA 1 [12]. *There exist forms f_n with $D_n=1$ not representing odd integers, if and only if $n\equiv 0 \pmod 8$.*

LEMMA 2 [13]. *Let $\upsilon=[\frac{1}{2}(n-1)]$ and B_i, E_i be the absolute values of the Bernoulli and Euler's numbers, respectively. Then when $D_n=1$, for the mass of the principal genus*

$$M_n=(2/\beta_n)\prod_{i=1}^{\upsilon}(B_{2i}/2i)(1-2^{-2i});\qquad (n\ \text{odd}),$$

$$(7)\qquad M_n=(2B_{\frac{1}{2}n}/n\beta_n)(2^{\frac{1}{2}n}-1)\prod_{i=1}^{\upsilon}(B_{2i}/2i)(1-2^{-2i});\qquad (n\equiv 0 \pmod 4)),$$

$$M_n=(2/\beta_n)E_{\upsilon}2^{-\frac{1}{2}n-1}\prod_{i=1}^{\upsilon}(B_{2i}/2i)(1-2^{-2i});\qquad (n\equiv 2 \pmod 4)),$$

where

$$\beta_n=\tfrac{1}{2}\prod_{i=1}^{n}\delta_i,$$

and for

$i \equiv 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8 \pmod 8;$

$\delta_i=1+2^{-j}-2^{2-i},\ 1+2^{-j},\ 1+2^{-j},\ 1,\ 1-2^{-j}-2^{2-i},\ 1-2^{-j},\ 1-2^{-j},\ 1;$

where $j=[\frac{1}{2}(i-1)]$.

Proof of theorem 1. Since $D_n=1$, from (1), all the o's are 1; and from (3), $\lambda=1$, $\nu_1=0$ for otherwise; $D_n\equiv 0 \pmod 2$.

Suppose first $n\not\equiv 0 \pmod 8$. By lemma 1, all the forms represent odd integers, and so the corresponding φ_1 are of the type (3.1). Hence all the σ's are 1. This proves that $O_{n,1}=1$. Since the o's are 1, all the characters of (6.1) are 1. Then from (4) and (5), we have $\mu_i=0$ and so the characters of (6.2), (6.3) and (6.4) do not occur. This proves $G_{n,1}=1$.

When $n\equiv 0 \pmod 8$, by lemma 1, there exist forms which represent only even integers; and so the corresponding φ_1 are of the type (3.2). Hence from the definition of σ on noting that α_{is}, α'_{is} in (3.2) are odd, we obtain

$$\sigma_1=2,\ \sigma_2=1,\ \sigma_3=2,\ \sigma_4=1,\dots,\ \sigma_{n-1}=2,\ \sigma_n=1.$$

This proves $O_{n,1}=2$ for $n\equiv 0 \pmod 8$.

From (4) and (5),

$$\mu_1=0,\ \mu_2=2,\ \mu_3=0,\ \mu_4=2,\dots,\mu_{n-1}=0.$$

From (6), the only characters, which occur with respect to 2; are

$$(-1)^{\frac{1}{2}(A_{2i}-1)}.$$

But from (3.2), it is easy to see that

$$A_{4i+2}\equiv -1 \pmod 4,\qquad A_{4i}\equiv 1 \pmod 4,$$

hence all these characters are fixed for all the even forms and define only one genus for all these even forms. This proves $G_{n,1}=2$ for $n\equiv 0 \pmod 8$.

Proof of theorem 2. From lemma 2, we have [14]

$$M_9=17/2^9.9!.3.5,\ M_{10}=5/2^{10}.10!.3,\ M_{11}=31/2^{11}.11!.3^2,$$
$$M_{12}=31/2^{12}.12!.3,\ M_{13}=691/2^{13}.13!.5.$$

Now $C_{9,1}=2$ and the representatives of the two classes are

$$f_9=\sum_{i=1}^{9}x_i^2,\qquad f'_9=f_8+x_9^2,$$

where

$$f_8=\sum_{i=1}^{8}x_i^2+(\sum_{i=1}^{8}x_i)^2-2x_1x_2-2x_2x_3$$

representing only even integers. Since the number of automorphs of f_9 is $2^9.9!$, the number of automorphs of f'_9 is

[12]) Ibid.

[13]) Magnus (10), 475. The value given by Magnus $\delta_i=1+2^{-j}+2^{2-i}$ for $i\equiv 1 \pmod 8$ is incorrect.

[14]) In Magnus (10), the values given by Magnus $M_9=3^2.17/2^9.9!.137$, $M_{10}=3^2.5/2^{10}.10!.137$, $M_{11}=3.5.31/2^{11}.11!.137$ are incorrect. The number 137 standing in the denominator should read 135.

(8) $$1/(M_9 - 1/2^9 \cdot 9!) = 2^8 \cdot 9!\,.\,3\,.\,5.$$

Since f_8 does not represent unity, it is clear from (8) that the number of automorphs of f_8 is

$$2^7\,.\,9!\,.\,3\,.\,5.$$

Thus the number of automorphs of the form

$$f_8 + \sum_{i=9}^{n} x_i^2$$

is

$$A_n = 2^{n-8}(n-8)!\,.\,2^7 \cdot 9!\,.\,3\,.\,5.$$

From the values of M_{10}, M_{11},

$$M_{10} = 1/2^{10}\,.\,10!\,. + 1/A_{10}, \quad M_{11} = 1/2^{11}\,.\,11! + 1/A_{11}.$$

Since the number of automorphs of $\sum_{i=1}^{n} x_i^2$ for $n = 10$, 11 are respectively 2^{10}. 10! and 2^{11} . 11!, the principal genus contains only two classes of forms in each case. From theorem 1, it follows that $C_{10,1} = C_{11,1} = 2$.

From the values of M_{12}, M_{13},

$$M_{12} = 1/2^{12}\,.\,12! + 1/A_{12} + 1/2^{11}\,.\,12!,$$

(9)

$$M_{13} = 1/2^{13}\,.\,13! + 1/A_{13} + 1/2^{12}\,.\,12!,$$

hence $C_{12,1} \geqslant 3$, $C_{13,2} \geqslant 3$. Suppose that the number of automorphs of the classes of the representative forms $f_{12}^{(1)}, \ldots, f_{12}^{(a)}$, other than the two known ones in 12 variables are $A_{12,1}, \ldots, A_{21,a}$. It is clear that these forms do not represent unity and so the auomorphs of the classes of forms

$$f_{12}^{(1)} + x_{13}^2, \ldots, f_{12}^{(a)} + x_{13}^2$$

are in number, respectively, $2A_{12,1}, \ldots, 2A_{12,a}$. From (9), we have

$$1/2^{11}\,.\,12! = 1/A_{12,1} + \ldots + 1/2A_{12,a}.$$

But

$$1/2^{12}\,.\,12! = 1/2A_{12,1} + \ldots + 1/2A_{12,a},$$

it follows that $C_{12,1} = C_{13,1}$. Hence theorem 2 is proved.

Remark. For $n = 12$, the form [15])

$$f_{12} = 23x_1^2 + 2\sum_{i=2}^{12} x_i^2 + 10x_1x_2 + 2\sum_{i=2}^{11} x_i x_{i+1},$$

does not represent unity and so is not equivalent to the other two forms

$$\sum_{i=1}^{12} x_i^2 \quad \text{and} \quad f_8 + \sum_{i=9}^{12} x_i^2.$$

Probably the number of automorphs of f_{12} is 2^{10}. 12! and so $C_{12,1} = C_{13,1} = 3$, but this number is very tedious to calculate.

For $n = 14$, 15, we have from (7),

$$M_{14} = 61.691/2^{14}\,.\,14!\,.\,3^3\,.\,5 = 1/2^{14}\,.\,14! + 1/2^6\,.\,6!\,.\,2^7\,.\,9!\,.\,3\,.\,5$$
$$+ 1/2^2\,.\,2!\,.\,2^{11}\,.\,12! + 1/2^{13}\,.\,10!\,.\,3^4\,.\,7,$$
$$M_{15} = 43.691/2^{15}\,.\,15!\,.\,3^2 = 1/2^{15}\,.\,15! + 1/2^7\,.\,7!\,.\,2^7\,.\,9!\,.\,3\,.\,5$$
$$+ 1/2^3\,.\,3!\,.\,2^{11}\,.\,12! + 1/2\,.\,2^{13}\,.\,10!\,.\,3^4\,.\,7 + 1/2^5\,.\,15!.$$

Hence $C_{14,1} \geqslant 4$ and $C_{15,1} \geqslant 5$ and four and five forms can be written down, since it is known that for $n = 14$, 15, the forms[16])

$$8x_1^2 + 6x_1x_2 + 2\sum_{i=2}^{13} x_i^2 + x_8^2 + 2\sum_{i=2}^{12} x_i x_{i+1} + 6x_{13}x_{14} + 8x_{14}^2$$

and

$$15x_1^2 + 8x_1x_2 + 2\sum_{i=2}^{15} x_i^2 + 2\sum_{i=2}^{14} x_i x_{i+1}$$

do not represent unity. It is very probably that $C_{14,1} = 4$ and $C_{15,1} = 5$.

In closing, I should like to thank Prof. Mordell for his kind help with my manuscript.

(Received 12 March, 1933.)

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[15]), [16]) See [11]).