ON THE DECOMPOSITION OF QUADRATIC FORMS IN SIX VARIABLES.

(DEDICATED TO PROFESSOR L. J. MORDELL ON HIS FIFTIETH BIRTHDAY.)

By

Chao Ko (Manchester).

1. Let

$$f_n(x) = \sum_{i,j=1}^{n} a_{ij}\, x_i x_j \quad (a_{ij} = a_{ji})$$

be a positive definite quadratic form with integer coefficients (i. e. a_{ij} are integers) and $A_{ij}^{(n)}$ be the cofactor of the a_{ij} in the determinant $D_n = |a_{ij}|$. A well known result by Hermite [1]) states that f_n is equivalent to a reduced form for which

$$(1) \qquad a_{11} \leqslant \gamma_n \sqrt[n]{D_n},$$

where γ_n is a number, e. g. $(4/3)^{\frac{1}{2}(n-1)}$ depending only on n. It is also known that

$$(2) \qquad \gamma_2 = \sqrt{4/3},\ \gamma_3 = \sqrt[3]{2},\ \gamma_4 = \sqrt[4]{4},\ \gamma_5 = \sqrt[5]{8}\ ^{2)},\ \gamma_6 = \sqrt[6]{64/3}\ ^{3)}.$$

Prof. Mordell proved recently the following theorems [4]):

THEOREM 1. *A decomposition*

$$f_n = X^2 + g(x),$$

[1]) Bachmann, Die Arithmetik der quadratischen Formen II (1923), 250-255.

[2]) Bachmann, II, 327-328.

[3]) Hofreiter, Monatshefte für Mathematik und Physik, 40 (1933), 129-152. Blichfeldt, Mathematische Zeitschrift, 39 (1934), 1-15.

[4]) Mordell, Annals of Mathematics, 38 (1937), 751-757.

where X ist a linear form and $g(x)$ is a positive definite or semi-definite quadratic form in $x_1, \ldots, x_n$ both with integer coefficients, is possible if either $D_n > [\gamma_n \sqrt[n]{D_n^{n-1}}]$, (the square bracket denoting the integer part), or if its adjoint form represents an integer less than or equal to D_n.

THEOREM 1a. *The form*

$$(3) \qquad \sum_{i=1}^{6} x_i^2 + \Big(\sum_{i=1}^{6} x_i\Big)^2 - 2x_1x_2 - 2x_2x_6$$

cannot be decomposed into a sum of two non-negative quadratic forms with integer coefficients.

For the decompositions of forms in six variables, I proved the [5])

THEOREM 2. *If $D_6 \not\equiv 0,4,7 \pmod 8$, $(A_{11}^{(6)}, D_6) = 1$, and if $A_{11}^{(6)}$ is odd when $(A_{11}^{(6)}/D_6) = 1$, (the symbol being that of quadratic residuacity), and $D_6 \equiv 3 \pmod 8$, then f_6 can be decomposed into a sum of eight linear squares with integer coefficients.*

These results naturally suggest the problem whether there exist non-decomposable forms in six variables other than (3). As an answer to this question, I prove in the present note the

THEOREM 3. *If f_6 is not equivalent to (3), then it can be decomposed into a sum of a linear square and a non-negative quadratic form both with integer coefficients.*

A consequence of this theorem is that any form in six variables can be decomposed into either a sum of squares or a sum of squares and a Mordell's form (3). I conjecture that the numbers of squares required are at most nine and three respectively.

2. The proof of theorem 3 requires the known lemmas:

LEMMA 1 [5]). *If $D_6 \equiv 0 \pmod 4$, the transformation*

$$x_i = y_i \quad (i = 1, \ldots, 5), \quad 2x_6 = y_6$$

carries a form $f_6' \sim f_6$ into a form with integer coefficients and determinant $D_6/4$.

LEMMA 2 [6]). *All the forms in six variables with determinant $\leqslant 3$ and not representing unity are equivalent to the form* (3)

[5]) Ko, Quarterly Journal of Mathematics (Oxford), 8 (1937) 81-98.

[6]) Ko, Journal of the London Mathematical Society, 13 (1938), 102-110.

LEMMA 3[6]). *All the forms in seven variables with determinant $\leqslant 3$ and not representing unity are equivalent to the form of determinant 2*

$$\sum_{i=1}^{7} x_i^2 + \Big(\sum_{i=1}^{7} x_i\Big)^2 - 2x_1x_2 - 2x_2x_7.$$

This form represens only even integers.

From (2), $\gamma_6 = \sqrt[6]{64/3}$ and so $D_6 < [\gamma_6 \sqrt[6]{D_6^5}]$ requires $D_6 \leqslant 14$. By theorems 1, la and lemma 2, we need deal with only those forms with determinant greater than 3 and less than 15 and

(4) $$A_{11}^{(6)} > D_6.$$

The form of determinants 4 and 8 are evidently ruled out by lemmas 1 and 2.

Let the adjoint form of f_6 be

$$F_6 = \sum_{i=1}^{6} A_{ij}^{(6)} x_i x_j.$$

Then its determinant is D_6^5. Suppose f_6 is replaced by an equivalent form for which F_6 is reduced, so that corresponding to (1)

(5) $$A_{11}^{(6)} \leqslant \sqrt[6]{(64/3)\, D_6^5}.$$

It is easy to see that from (4) and (5), we need only treat the forms with determinants

(6) $$D_6 = 5,\ 6,\ 7,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14$$

and their corresponding minors

(6′) $$A_{11}^{(6)} = 6,\ 7,\ 8,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15.$$

Examining the values of D_6, $A_{11}^{(6)}$ in (6) and (6′), the only forms which possibly cannot be expressed as a sum of eight squares by theorem 2 are those having the determinants

$$D_6 = 7,\ 11,\ 12 \text{ with } A_{11}^{(6)} = 8,\ 12,\ 13; \text{ respectively.}$$

Now we shall deal with these forms separately in the following lemmas:

LEMMA 4. *The forms having $D_6 = 7$, 12 and $A_{11}^{(6)} = 8$, 13, respectively, can be decomposed into a sum of a linear square and a non-negative quadratic form with integer coefficients.*

I. Suppose first for f_6 that $D_6 = 7$ and $A_{11}^{(6)} = 8$. We show now that there exist definite forms f_7 of determinant 3 in seven variables

$$f_7 = a_{77}\, a x_7^2 + 2a_{71} x_7 x_1 + f_6(x_1, \ldots, x_6).$$

It suffices if

$$a_{77} D_6 - a_{71}^2 A_{11}^{(6)} = 3, \quad \text{i. e.} \quad 7a_{77} - 8a_{71}^2 = 3$$

and so we may take $a_{71} = 2$, $a_{77} = 5$. By lemma 3, this form represents unity. Hence we have

$$f_7 = y_7^2 + \varphi_6(y_1, \ldots, y_6),$$

where the y's are linear forms in $x_1, \ldots, x_7$ with integer coefficients and the determinant of φ_6 is 3. It is clear that y_7 cannot contain x_7 only, as 3 is not a divisor of 7. Now by putting $x_7 = 0$, we get the required decomposition for f_6.

II. Similarly when in f_6, $D_6 = 12$ and $A_{11}^{(6)} = 13$, since $12 . 10 - 13 . 3^2 = 3$, we have correspondingly

(7) $$f_7 = 10x_7^2 + 6x_1x_7 + f_6 = y_7^2 + \psi_6(y_1, \ldots, y_6),$$

where, by lemma 2, either

(8) $$\psi_6 = y_6^2 + \psi_5(y_1, \ldots, y_5),$$

when ψ_6 represents unity, or

(9) $$\psi_6 = \sum_{i=1}^{6} y_i^2 + \Big(\sum_{i=1}^{6} y_i\Big)^2 - 2y_1y_2^2 - 2y_2y_6.$$

Suppose first (8) holds. Then y_7 and y_6 cannot be both zero, when we put $x_7 = 0$ in (7), since y_6, y_7 are not linearly dependent. Hence the required decomposition is evidently obtained in this case.

Suppose next (9) holds. Put $x_7 = 0$ in (7). The lemma is proved if $y_7 \neq 0$. Suppose then $y_7 = 0$ in (7) and let $y_i = \sum_{j=1}^{6} b_{ij} x_j$ $(i = 1, \ldots, 6)$ and $|b_{ij}|$ be the determinant of the y's. Then $|b_{ij}| = \pm 2$, since $3\,|b_{ij}|^2 = 12$.

Hence there exists an unimodular linear transformation T in the x's, for which [7])

$$y_k \to 2x_k + \sum_{j=k+1}^{6} \rho_j x_j, \quad y_i \to x_i \quad (0 < i < 7, \quad i \neq k; \quad \rho_j = 0 \text{ or } 1).$$

Since the permutations of either y_1, y_6 or y_3, y_4, y_5, leaves ψ_6 unchanged; we need consider only the cases $k = 1, 2, 3$.

Prof. Mordell has shown [4]) that the adjoint form of ψ_6 in (9) is

$$\Psi_6 = 3\left(\tfrac{1}{2} \sum_{i=1}^{2} Y_i^2 + \sum_{i=3}^{5} (Y_i - \tfrac{1}{2}Y_2 - \tfrac{1}{2}Y_1)^2 + \tfrac{4}{3}(Y_6 - \tfrac{1}{2}\sum_{i=3}^{5} Y_i + \tfrac{3}{4}Y_2 + \tfrac{1}{4}Y_1)^2\right).$$

Suppose first $k = 1$. Then the adjoint form of ψ_6 after applying the transformation T (i. e. the adjoint form of an form which is equivalent to f_6) is

$$F_6 = 12\left(\tfrac{1}{8}X_1^2 + \tfrac{1}{2}(X_2 - \tfrac{1}{2}\rho_2 X_1)^2 + \sum_{i=3}^{5} (X_i - \tfrac{1}{2}X_2 - (\tfrac{1}{4} + \tfrac{1}{2}\rho_i - \tfrac{1}{4}\rho_2) X_1)^2\right.$$

$$\left. + \tfrac{4}{3}(X_6 - \tfrac{1}{2}\sum_{i=3}^{5} X_i + \tfrac{3}{4}X_2 + (\tfrac{1}{8} - \tfrac{1}{2}\rho_6 + \tfrac{1}{4}\sum_{i=3}^{5} \rho_i - \tfrac{3}{8}\rho_2) X_1)^2\right) = 12G,$$

say. This is easily obtained by transforming the form $2^2 \Psi_6$ with the inverse transposed transformation of T:

$$Y_1 = \tfrac{1}{2}X_1, \quad Y_i = X_i - \tfrac{1}{2}\rho_i X_1 \quad (i = 2, \ldots, 6).$$

Then by theorem 1 our lemma is proved if we can show that $G \leqq 1$ for every set of the ϱ's.

a) Suppose first $\varrho_2 = 0$. Then we have evidently

$$G \leqslant 1/8 + 1/16 + 1/16 + 1/16 + (4/3)\ (1/4) = 31/48 < 1,$$

by putting $X_1 = 1$, $X_2 = 0$, $X_i = \rho_i$ $(i = 3, 4, 5)$; and choosing X_6 such that to make the last term of F_6 within the square $\leqslant \frac{1}{2}$.

b) Suppose then $\varrho_2 = 1$. Since $\rho_i = 0$ or 1, two of the ρ_3, ρ_4, ρ_5, must be equal and so we need only consider the following two cases:

b1) Suppose two of the ϱ_3, ρ_4, ρ_5, are zero. Since they are symmetrical in F_6, without loss of generality, we can assume $\varrho_3 = \rho_4 = 0$. Then

$$G \leqslant 1/8 + 1/8 + 1/4 + (4/3)\,(1/4) = 5/6 < 1,$$

by putting $X_1 = 1$, $X_i = 0$ $(i = 2, \ldots, 5)$ and choosing X_6 as above.

b2) Suppose now two ot the ρ_3, ρ_4, ρ_5 are one, say $\rho_3 = \rho_4 = 1$, then we have also

$$G \leqslant 1/8 + 1/8 + 1/4 + (4/3)\ (1/4) = 5/6 < 1,$$

by putting $X_i = 1$ $(i = 1, \ldots, 4)$, $X_5 = \rho_5$, and choosing X_6 as above.

Suppose next $k = 2$. Then the inverse transposed transformation of T is

$$Y_1 = X_1, \quad Y_2 = \tfrac{1}{2}X_2, \quad Y_i = X_i - \tfrac{1}{2}\rho_i X_2 \ (i = 3, \ldots, 6),$$

and so the adjoint form of ψ_6 after applying the substitution T becomes

$$F_6' = 12\left(\tfrac{1}{2}X_1 + \tfrac{1}{8}X_2^2 + \sum_{i=3}^{5} (X_i - (\tfrac{1}{4} + \tfrac{1}{2}\rho_i) X_2 - \tfrac{1}{2}X_1)^2\right.$$

$$\left. + \tfrac{4}{3}(X_6 - \tfrac{1}{2}\sum_{i=3}^{5} X_i + (\tfrac{3}{8} - \tfrac{1}{2}\rho_6 + \tfrac{1}{4}\sum_{i=3}^{5} \rho_i) X_2 + \tfrac{1}{4}X_1)^2\right) = 12G',$$

say. It is easy to see that $G' < 1$ for $X_1 = 0$, $X_2 = 1$, $X_i = \rho_i$ $(i = 3,4,5)$ and X_6 being chosen in such a way to make the last square of $F_6' \leqslant 1/4$.

Suppose finally $k = 3$. Then the inverse transposed transformation of T is

$$Y_1 = X_i \ (i = 1,2), \quad Y_3 = \tfrac{1}{2}X_3, \quad Y_i = X_i - \tfrac{1}{2}\rho_i X_3 \ (i = 4.\ 5,\ 6),$$

and the adjoint form

$$F_6'' = 12\left(\tfrac{1}{2}X_1^2 + \tfrac{1}{2}X_2^2 + \tfrac{1}{4}(X_3 - X_2 - X_1)^2 + \sum_{i=4}^{5} (X_i - \tfrac{1}{2}\rho_i X_3 - \tfrac{1}{2}X_2 - \tfrac{1}{2}X_1)^2\right.$$

$$\left. + \tfrac{4}{3}(X_6 - \tfrac{1}{2}\sum_{i=4}^{5} X_i - (\tfrac{1}{4} + \tfrac{1}{2}\rho_6 - \tfrac{1}{4}\sum_{i=4}^{5} \rho_i) X_3 + \tfrac{3}{4}X_2 + \tfrac{1}{4}X_1)^2\right) = 12G'',$$

say. Putting $X_1 = X_2 = 0$, $X_3 = 1$, $X_i = \rho_i$ $(i = 4,5)$, and $X_6 = 1$; clearly we get $G'' \leqslant 1$.

Hence by theorem 1, the lemma is completely proved.

[7]) Bachman, Die Arithmetik der quadratischen Formen, I, 308—310.

LEMMA 5. *The form f_6 with $D_6 = 11$ and $A_{11}^{(6)} = 12$ can be decomposed into a sum of a square and a positive definite quadratic form both with integer coefficients, if f_6 represents an odd integer.*

We show first that there exist definite forms f_7 of determinant 2 in seven variables

$$f_7 = a_{77} x_7^2 + 2a_{71} x_7 x_1 + f_6(x_1, \ldots, x_6).$$

It suffices if

$$a_{77} D_6 - a_{71}^2 A_{11}^{(6)} = 2, \quad \text{i. e.} \quad 11 a_{77} - 12 a_{71}^2 = 2$$

and so we may take $a_{77} = 10$, $a_{71} = 3$. If f_6 represents an odd integer, by lemma 3, we have

$$f_7 = y_7^2 + \varphi_6(y_1, \ldots, y_6),$$

where the y's are linear forms in $x_1, \ldots, x_7$ with integer coefficients and φ_6 has the determinant 2.

Since $2 \nmid 11$, y_7 connot be zero when we put $x_7 = 0$ in f_7, and so a required decomposition is obtained.

Now there remains for the proof of theorem 3 to discuss all the forms f_6 with $D_6 = 11$, $A_{11}^{(6)} = 12$ and representing only even integers. To do this, we requires lemma 6 due to Charve and lemma 7 of which I give a proof.

LEMMA 6[8]. *The reduced forms in four variables with $D_4 \leq 12$ and not representing odd integers are equivalent to one of the seven forms:*

$$f_{4,1} = 2 \sum_{i=1}^{3} (x_i - \tfrac{1}{2} x_4)^2 + \tfrac{1}{2} x_4^2,$$

$$f_{4,2} = 2(x_1 - \tfrac{1}{2} x_4)^2 + 2(x_2 - \tfrac{1}{2} x_3)^2 + \tfrac{3}{2}(x_3 - \tfrac{2}{3} x_4)^2 + \tfrac{5}{6} x_4^2,$$

$$f_{4,3} = 2x_1^2 + 2 \sum_{i=2}^{3} (x_i - \tfrac{1}{2} x_4)^2 + x_4^2,$$

$$f_{4,4} = 2(x_1 - \tfrac{1}{2} x_2)^2 + \tfrac{3}{2} x_2^2 + 2(x_3 - \tfrac{1}{2} x_4)^2 + \tfrac{3}{2} x_4^2,$$

$$f_{4,5} = 2x_1 + 2x_2^2 + 2(x_3 - \tfrac{1}{2} x_4)^2 + \tfrac{3}{2} x_4^2,$$

[8]) Charve, Comptes Rendus Paris, 96 (1883), 773.

$$f_{4,6} = 2 \sum_{i=1}^{2} (x_i - \tfrac{1}{2} x_4)^2 + 4(x_3 - \tfrac{1}{4} x_4)^2 + \tfrac{3}{4} x_4^2,$$

$$f_{4,7} = 2(x_1 - \tfrac{1}{2} x_3)^2 + 4(x_2 - \tfrac{1}{4} x_3 - \tfrac{1}{2} x_4)^2 + \tfrac{5}{4}(x_3 - \tfrac{6}{5} x_4)^2 + \tfrac{6}{5} x_4^2.$$

LEMMA 7. *The positive forms in five variables with $D_5 = 12$ not representing odd integers are equivalent to one of the three forms:*

$$f_{5,1} = 2 \sum_{i=1}^{2} (x_i - \tfrac{1}{2} x_4 - \tfrac{1}{2} x_5)^2 + 2(x_3 - \tfrac{1}{2} x_4)^2 + \tfrac{1}{2} x_4^2 + 3x_5^2,$$

$$f_{5,2} = 2(x_1 - \tfrac{1}{2} x_5)^2 + 2 \sum_{i=2}^{3} (x_i - \tfrac{1}{2} x_4)^2 + x_4^2 + \tfrac{3}{2} x_5^2,$$

$$f_{5,3} = 2(x_1 - \tfrac{1}{2} x_3 - \tfrac{1}{2} x_5)^2 + 4(x_2 - \tfrac{1}{4} x_3 - \tfrac{1}{2} x_4)^2 + \tfrac{5}{4}(x_3 - \tfrac{6}{5} x_4 - \tfrac{2}{5} x_5)^2 + \tfrac{6}{5}(x_4 - \tfrac{1}{2} x_5)^2 + x_5^2.$$

Suppose f_5 with $D_5 = 12$ does not represent odd integers. Let the adjoint form of f_5 be

$$F_5 = \sum_{i=1}^{5} A_{ij}^{(5)} x_i x_j.$$

Its determinant is 12^4, so we can find a form $f'_5 \sim f_5$ such that in F'_5

$$A'^{(5)}_{55} \leq [\gamma_5 \sqrt[5]{12^4}] = 12.$$

Hence

$$f_5 \sim a_{55} x_5^2 + 2 \sum_{i=2}^{4} a_{5i} x_5 x_i + f_4(x_1, \ldots, x_4),$$

where f_4 is a form in four variables with determinant $D_4 = A'^{(5)}_{55} \leq 12$. From lemma 6, f_4 is equivalent to one of the forms $f_{4,i}$ $(i = 1, \ldots, 7)$.

I. $f_4 \sim f_{4,1}$. We can write

$$f_5 \sim 2 \sum_{i=1}^{3} (x_i - \tfrac{1}{2} x_4 + a_i x_5)^2 + \tfrac{1}{2}(x_4 + a_4 x_5)^2 + 3x_5^2, \tag{10}$$

where the a's are rational numbers and $0 \geq a_i > -1$ on replacing if need be x_i by $x_i + \alpha_i x_5$ $(i = 1, 2, 3, 4)$, and the coefficients of x_5^2 is 3 since f_5 has determinant 12. Hence from the coefficients of $x_i x_5$, we get the congruences:

(11) $$2a_1 \equiv 2a_2 \equiv 2a_3 \equiv -a_1 - a_2 - a_3 + \tfrac{1}{2}a_4 \equiv 0 \pmod 1,$$

and $$Q_1 = 2a_1^2 + 2a_2^2 + 2a_3^2 + \tfrac{1}{2}a_4^2 \equiv 0 \pmod 1.$$

Multiplying Q_1 by 2 and using (11), we get $a_4^2 \equiv 0 \pmod 1$ and so $a_4 = 0$. From (11), we have

(12) $$2a_1 \equiv 2a_2 \equiv 2a_3 \equiv a_1 + a_2 + a_3 \equiv 0 \pmod 1.$$

On observing the symmetry of the expressions involving x_1, x_2, x_3 in (10), we need only consider the two following sets of solutions from (12),

$$a_1 = a_2 = a_3 = 0 \quad \text{and} \quad a_1 = a_2 = -\tfrac{1}{2},\ a_3 = 0.$$

The first set gives (10) a form representing an odd integer 3. The second set gives $f_{5,1}$.

II. $f_4 \sim f_{4,2}$. We can write.

(13) $$f_5 \sim 2(x_1 - \tfrac{1}{2}x_4 + a_1x_5)^2 + 2(x_2 - \tfrac{1}{2}x_3 + a_2x_5)^2 + \tfrac{3}{2}(x_3 - \tfrac{2}{3}x_4 + a_3x_5)^2 + \tfrac{5}{6}(x_4 + a_4x_5)^2 + \tfrac{12}{5}x_5^2.$$

Hence as above we have

(14) $$2a_1 \equiv 2a_2 \equiv \tfrac{3}{2}a_3 - a_2 \equiv 0 \pmod 1,$$

and $$Q_2 = 2a_1^2 + 2a_2^2 + \tfrac{3}{2}a_3^2 + \tfrac{5}{6}a_4^2 \equiv \tfrac{3}{5} \pmod 1.$$

Multiplying Q_2 by 6 and using (14), we get $5a_4^2 \equiv \tfrac{3}{5} \pmod 1$, which gives no solution for rational a_4. Hence (13) is impossible.

III. $f_5 \sim f_{4,3}$. We can write

(15) $$f_5 \sim 2(x_1 + a_1x_5)^2 + 2\sum_{i=2}^{3}(x_i - \tfrac{1}{2}x_4 + a_ix_5)^2 + (x_4 + a_4x_5)^2 + \tfrac{3}{2}x_5^2.$$

Hence

(16) $$2a_1 \equiv 2a_2 \equiv 2a_3 \equiv -a_2 - a_3 + a_4 \equiv 0 \pmod 1,$$

and $$Q_3 = 2a_1^2 + 2a_2^2 + 2a_3^2 + a_4^2 \equiv \tfrac{1}{2} \pmod 1.$$

Multiplying Q_3 by 2 and using (16), we get $2a_4^2 \equiv 0 \pmod 1$ and so $a_4 = 0$. From (16), a_i $(i = 1, 2, 3)$ can only be 0 or $-\tfrac{1}{2}$. Since $Q_3 \equiv \tfrac{1}{2} \pmod 1$, at least one of the a_1, a_2, a_3 must be $-\tfrac{1}{2}$. From (16), we have either $a_1 = a_2 = a_3 = -\tfrac{1}{2}$ or $a_1 = -\tfrac{1}{2}$, $a_2 = a_3 = 0$. It is easy to see that the first set gives a form representing an odd integer 3 and the second set gives $f_{5,2}$.

IV. $f_4 \sim f_{4,4}$. We can write

(17) $$f_5 \sim 2(x_1 - \tfrac{1}{2}x_2 + a_1x_5)^2 + \tfrac{3}{2}(x_2 + a_2x_5)^2 + 2(x_3 - \tfrac{1}{2}x_4 + a_3x_5)^2 + \tfrac{3}{2}(x_4 + a_4x_5)^2 + \tfrac{4}{3}x_5^2.$$

Hence

(18) $$2a_1 \equiv \tfrac{3}{2}a_2 - a_1 \equiv 2a_3 \equiv \tfrac{3}{2}a_4 - a_3 \equiv 0 \pmod 1,$$

and $$Q_4 = 2a_1^2 + \tfrac{3}{2}a_2^2 + 2a_3^2 + \tfrac{3}{2}a_4^2 \equiv \tfrac{2}{3} \pmod 1.$$

Multiplying Q_4 by 2 and using (17), we get $3(a_2^2 + a_4^2) \equiv \tfrac{1}{3} \pmod 1$. On observing the symmetry of the expressions involving x_2, x_4, we need only condider the set of solutions $a_2 = 0$, $a_4 = -{}^1/_3$ or $-{}^2/_3$. We can take $a_4 = -{}^1/_3$, for if $a_4 = -{}^2/_3$ we need only replace x_3 by $-x_3$ and x_4 by $-x_4 + x_5$. From (18), we get $a_1 = 0$, $a_3 = -\tfrac{1}{2}$ and (17) becomes

$$f_5 \sim 2(x_1 - \tfrac{1}{2}x_2)^2 + \tfrac{3}{2}x_2^2 + 2(x_3 - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5)^2 + \tfrac{3}{2}(x_4 - \tfrac{1}{3}x_5)^2 + \tfrac{4}{3}x_5^2,$$

which is easily verified to be equivalent to $f_{5,2}$ by the unimodular transformation

$$x_1 \to x_1', \quad x_2 \to x_5', \quad x_3 \to x_4', \quad x_4 \to x_3', \quad x_5 \to x_2'.$$

V. $f_4 \sim f_{4,5}$. We can write

(19) $$f_5 \sim 2\sum_{i=1}^{2}(x_i + a_ix_5)^2 + 2(x_3 - \tfrac{1}{2}x_4 + a_3x_5)^2 + \tfrac{3}{2}(x_4 + a_4x_5)^2 + x_5^2.$$

Hence

(20) $$2a_1 \equiv 2a_2 \equiv 2a_3 \equiv \tfrac{3}{2}a_4 - a_3 \equiv 0 \pmod 1$$

and $$Q_5 = 2a_1^2 + 2a_2^2 + 2a_3^2 + \tfrac{3}{2}a_4^2 \equiv 0 \pmod 1.$$

Multiplying Q_5 by 2 and using (20), we get $3a_4^2 \equiv 0 \pmod 1$ and so $a_4 = 0$. From the fourth congruence of (20), we get $a_3 = 0$. Then (20) and $Q_5 \equiv 0 \pmod 1$ give

$$2a_1 \equiv 2a_2 \equiv 2(a_1^2 + a_2^2) \equiv 0 \pmod 1.$$

The solutions of this are evidently $a_1 = a_2 = 0$ and $a_1 = a_2 = -\tfrac{1}{2}$. The first set of solutions gives (19) a form representing 1. The second set gives (19)

$$f_5 \sim 2 \sum_{i=1}^{2} (x_i - \tfrac{1}{2}x_5)^2 + 2(x_3 - \tfrac{1}{2}x_4)^2 + \tfrac{3}{2}x_4^2 + x_5^2,$$

which is evidently equivalent to $f_{5,2}$.

VI. $f_4 \sim f_{4,6}$. We can write

$$(21) \qquad f_5 \sim 2\sum_{i=1}^{2} (x_i - \tfrac{1}{2}x_4 + a_i x_5)^2 + 4(x_3 - \tfrac{1}{4}x_4 + a_3 x_5)^2 + \tfrac{3}{4}(x_4 + a_4 x_5)^2 + x_5^2.$$

Hence

$$(22) \qquad 2a_1 \equiv 2a_2 \equiv 4a_3 \equiv \tfrac{3}{4}a_4 - a_3 - a_2 - a_1 \equiv 0 \pmod 1,$$

and

$$Q_6 = 2a_1^2 + 2a_2^2 + 4a_3^2 + \tfrac{3}{4}a_4^2 \equiv 0 \pmod 1.$$

Multiplying Q_6 by 4 and using (22), we get $3a_4^2 \equiv 0 \pmod 1$ and so $a_4 = 0$. From the last congruence of (22), we get $2a_3 \equiv -2a_1 - 2a_2 \equiv 0 \pmod 1$. Substituting this value into $Q_6 \equiv 0 \pmod 1$, we have $2(a_1^2 + a_2^2) \equiv 0 \pmod 1$. Then we have

$$2a_1 \equiv 2a_2 \equiv 2a_3 \equiv a_1 + a_2 + a_3 \equiv 2a_1^2 + 2a_2^2 \equiv 0 \pmod 1.$$

This evidently give only two sets of solutions, i. e. $a_1 = a_2 = a_3 = 0$ and $a_1 = a_2 = -\tfrac{1}{2}$. $a_3 = 0$. The first set gives the form (21) representing 1. The second set gives

$$f_5 \sim 2\sum_{i=1}^{2} (x_i - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5)^2 + 4(x_3 - \tfrac{1}{4}x_4)^2 + \tfrac{3}{4}x_4^2 + x_5^2,$$

which is easily verified to be equivalent to $f_{5,1}$ by the unimodular transformation

$$x_1 \to -x_1', \quad x_2 \to -x_2', \quad x_3 \to -x_5', \quad x_4 \to x_3' - x_4' - x_5', \quad x_5 \to -x_3'.$$

VII. $f_4 \sim f_{4,7}$. We can write

$$(23) \qquad f_5 \sim 2(x_1 - \tfrac{1}{2}x_3 + a_1 x_5)^2 + 4(x_2 - \tfrac{1}{4}x_3 - \tfrac{1}{2}x_4 + a_2 x_5)^2 + \tfrac{5}{4}(x_3 - \tfrac{6}{5}x_4 + a_3 x_5)^2 + \tfrac{6}{5}(x_4 + a_4 x_5)^2 + x_5^2.$$

Hence

$$(24) \qquad 2a_1 \equiv 4a_2 \equiv \tfrac{5}{4}a_3 - a_2 - a_1 \equiv \tfrac{6}{5}a_4 - \tfrac{3}{2}a_3 - 2a_2 \equiv 0 \pmod 1,$$

and

$$Q_7 = 2a_1^2 + 4a_2^2 + \tfrac{5}{4}a_3^2 + \tfrac{6}{5}a_4^2 \equiv 0 \pmod 1.$$

Multiplying Q_7 by 20 and using (24), we get $24a_4^2 \equiv 0 \pmod 1$ and so $a_4 = 0$ or $-\tfrac{1}{2}$.

Suppose first $a_4 = 0$. Multiplying Q_7 by 4, we get $5a_3^2 \equiv 0 \pmod 1$ and so $a_3 = 0$. Then $2a_2 \equiv \tfrac{6}{5}a_4 - \tfrac{3}{2}a_3 \equiv 0 \pmod 1$ and $Q_7 \equiv 0 \pmod 1$ gives $2a_1^2 \equiv -(2a_2)^2 \equiv 0 \pmod 1$. Hence $a_1 = 0$. From $a_2 \equiv \tfrac{5}{4}a_3 - a_1 \equiv 0 \pmod 1$, we have $a_2 = 0$. This set of solutions $a_1 = a_2 = a_3 = a_4 = 0$ gives (23) a form representing 1.

Suppose next $a_4 = -\tfrac{1}{2}$. Multiplying Q_7 by 4, we get $5a_3^2 + \tfrac{1}{5} \equiv 0 \pmod 1$, and so $a_3 = -\tfrac{3}{5}$ or $-\tfrac{2}{5}$. The first solution gives the last congruence of (24)

$$4a_2 \equiv 12a_4/5 - 3a_3 \equiv -1/5 + 4/5 \pmod 1,$$

which contradicts $4a_2 \equiv 0 \pmod 1$. The second solution gives similarly

$$2a_2 \equiv 6a_4/5 - 3a_3/2 \equiv -3/5 + 3/5 \equiv 0 \pmod 1,$$

and so from $Q_7 \equiv 0 \pmod 1$, we get $2a_1^2 \equiv -4a_2^2 - \tfrac{5}{4}a_3^2 - \tfrac{6}{5}a_4^2 \equiv -1/5 - 3/10 \equiv -\tfrac{1}{2} \pmod 1$. Hence $a_1 = -\tfrac{1}{2}$. From $5a_3/4 - a_2 - a_1 \equiv 0 \pmod 1$, we get $a_2 = 0$. This set of solutions $a_1 = -\tfrac{1}{2}$, $a_2 = 0$, $a_3 = -\tfrac{2}{5}$, $a_4 = -\tfrac{1}{2}$ gives the form $f_{5,3}$.

Hence lemma 7 is completely proved.

LEMMA 8. *All the forms f_6 with $D_6 = 11$, $A_{11}^{(6)} = 12$ and not representing odd integers are equivalent to one of the three forms*

$$f_{6,1} = 2(x_1 - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5 - \tfrac{1}{2}x_6)^2 + 2(x_2 - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5)^2 + 2(x_3 - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_6)^2 + \tfrac{1}{2}x_4^2 + 3(x_5 - \tfrac{1}{6}x_6)^2 + \tfrac{11}{12}x_6^2,$$

$$f_{6,2} = 2(x_1 - \tfrac{1}{2}x_3 - \tfrac{1}{2}x_5)^2 + 4(x_2 - \tfrac{1}{4}x_3 - \tfrac{1}{2}x_4 - \tfrac{3}{4}x_6)^2 + \tfrac{5}{4}(x_3 - \tfrac{6}{5}x_4 - \tfrac{2}{5}x_5 - \tfrac{3}{5}x_6)^2 + \tfrac{6}{5}(x_4 - \tfrac{1}{2}x_5 - \tfrac{1}{3}x_6)^2 + (x_5 - \tfrac{1}{2}x_6)^2 + \tfrac{11}{12}x_6^2,$$

$$f_{6,3} = 2(x_1 - \tfrac{1}{2}x_3 - \tfrac{1}{2}x_5)^2 + 4(x_2 - \tfrac{1}{4}x_3 - \tfrac{1}{2}x_4)^2 + \tfrac{5}{4}(x_3 - \tfrac{6}{5}x_4 - \tfrac{2}{5}x_5 - \tfrac{4}{5}x_6)^2 + \tfrac{6}{5}(x_4 - \tfrac{1}{2}x_5 - \tfrac{1}{6}x_6)^2 + (x_5 - \tfrac{1}{2}x_6)^2 + \tfrac{11}{12}x_6^2,$$

and all of them can be decomposed into a sum of a square and a positive definite form both with integer coefficients.

By the same argument as in the proof of lemma 7 (with some interchanges of the variables, we can choose $A_{66}^{(6)} = 12$), we have

$$f_6 = a_{66}x_6^2 + 2\sum_{i=1}^{5} a_{6i}x_6x_i + f_5(x_1, \dots, x_5),$$

where f_5 is equivalent to one of the forms $f_{5,1}$, $f_{5,2}$, or $f_{5,3}$.

I. $f_5 \sim f_{5,1}$. We can write

$$(25)\quad f_6 \sim 2\sum_{i=1}^{2} (x_i - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5 + a_ix_6)^2 + 2(x_3 - \tfrac{1}{2}x_4 + a_3x_6)^2 + \tfrac{1}{2}(x_4 + a_4x_6)^2 + 3(x_5 + a_5x_6)^2 + \tfrac{11}{12}x_6^2.$$

Hence

$$(26)\quad 2a_1 \equiv 2a_2 \equiv 2a_3 \equiv \tfrac{1}{2}a_4 - a_3 - a_2 - a_1 \equiv 3a_5 - a_2 - a_1 \equiv 0 \pmod 1$$

and $$R_1 = 2a_1^2 + 2a_2^2 + 2a_3^2 + \tfrac{1}{2}a_4^2 + 3a_5^2 \equiv 1/12 \pmod 1.$$

Multiplying R_1 by 2 and using (26), we get $6a_5^2 \equiv \frac{1}{6} \pmod 1$ and so $a_5 = -1/6$ or $-\frac{5}{6}$. We can take $a_5 = -\frac{1}{6}$, for if $a_5 = -\frac{5}{6}$, we need only to replace x_i by $-x_i$ $(i = 1, 2, 3, 4)$, x_5 by $-x_5 + x_6$. Substituting the value of a_5 in R_1 and multiplying it by 2, we get $a_4^2 \equiv 0 \pmod 1$ and so $a_4 = 0$. From

$$\tfrac{1}{2}a_4 - a_3 - a_2 - a_1 \equiv 3a_5 - a_2 - a_1 \equiv 0 \pmod 1,$$

we get $a_3 = -\frac{1}{2}$. Since the expressions involving x_1, x_2 are symmetrical in (25), we can take $a_1 = -\frac{1}{2}$, $a_2 = 0$ for the solutions of the congruences

$$2a_1 \equiv 2a_2 \equiv \tfrac{1}{2} - a_1 - a_2 \equiv 0 \pmod 1.$$

Substituting these values into (25), we get $f_{6,1}$, which can be decomposed into a sum of x_5^2 and a positive definite form, as $A_{55}^{(6)}$ in $f_{6,1}$ is 4.

II. $f_6 \sim f_{5,2}$. We can write

$$(27)\quad f_6 \sim 2(x_1 - \tfrac{1}{2}x_5 + a_1x_6)^2 + 2\sum_{i=2}^{3} (x_i - \tfrac{1}{2}x_4 + a_ix_6)^2 + (x_4 + a_4x_6)^2 + \tfrac{3}{2}(x_5 + a_5x_6)^2 + \tfrac{11}{12}x_6^2.$$

Hence

$$(28)\quad 2a_1 \equiv 2a_2 \equiv 2a_3 \equiv a_4 - a_3 - a_2 \equiv \tfrac{3}{2}a_5 - a_1 \equiv 0 \pmod 1,$$

and $$R_2 = 2a_1^2 + 2a_2^2 + 2a_3^2 + a_4^2 + \tfrac{3}{2}a_5^2 \equiv \tfrac{1}{12} \pmod 1.$$

Multiplying R_2 by 4 and using (28), we get $6a_5^2 \equiv 1/3 \pmod 1$, which gives no solution for rational a_5. Hence (27) is impossible.

III. $f_5 \sim f_{5,3}$. We can write

$$(29)\quad f_6 \sim 2(x_1 - \tfrac{1}{2}x_3 - \tfrac{1}{2}x_5 + a_1x_6)^2 + 4(x_2 - \tfrac{1}{4}x_3 - \tfrac{1}{2}x_4 + a_2x_6)^2 + \tfrac{5}{4}(x_3 - \tfrac{6}{5}x_4 - \tfrac{2}{5}x_5 + a_3x_6)^2 + \tfrac{6}{5}(x_4 - \tfrac{1}{2}x_5 + a_4x_6)^2 + (x_5 + a_5x_6)^2 + \tfrac{11}{12}x_6^2.$$

Hence

$$(30)\quad 2a_1 \equiv 4a_2 \equiv \tfrac{5}{4}a_3 - a_2 - a_1 \equiv \tfrac{6}{5}a_2 - \tfrac{3}{2}a_3 - 2a_2 \equiv a_5 - \tfrac{3}{5}a_4 - \tfrac{1}{2}a_3 - a_1 \equiv 0 \pmod 1$$

and $$R_3 = 2a_1^2 + 4a_2^2 + \tfrac{5}{4}a_3^2 + \tfrac{6}{5}a_4^2 + a_5^2 \equiv 1/12 \pmod 1.$$

Since $120\,R_3 \equiv 120a_5^2 \equiv 0 \pmod 1$, we have $a_5 = 0$ or $-\frac{1}{2}$.

(A) Suppose first $a_5 = 0$. Since then $20R_3 \equiv 24a_4^2 \equiv 2/3 \pmod 1$, we have $a_4 = -1/6, -1/3, -2/3$ or $-5/6$. We can take $a_4 = -1/3$ or $-1/6$, for if $a_4 = -2/3$ or $-5/6$, we need only to replace x_i by $-x_i + 2x_5$ $(i=1,3)$, x_2 by $-x_2 + x_5$, x_4 by $-x_4 + x_5 + x_6$.

(A1) Let $a_4 = -\frac{1}{3}$. Substituting the values of a_4, a_5 into R_3 and multiplying it by 4, we get $5a_3^2 \equiv 4/5 \pmod 1$ and so $a_3 = -2/5$ or $-3/5$. Since $a_3 = -2/5$ gives $2a_1 \equiv -6a_4/5 - a_3 \equiv 2/5 + 2/5 \pmod 1$, which contradicts to $2a_1 \equiv 0 \pmod 1$. Hence we have $a_3 = -3/5$. From (30), we have

$$a_1 \equiv -3a_4/5 - \tfrac{1}{2}a_3 \equiv -\tfrac{1}{2}, \quad a_2 \equiv 5a_3/4 - a_1 \equiv -1/4 \pmod 1.$$

Hence $a_1 = -1/2$, $a_2 = -1/4$ $a_3 = -3/5$, $a_4 = -1/3$, $a_5 = 0$. But these values give $R_3 \equiv 1/3 \not\equiv 1/12 \pmod 1$.

(A2) Let then $a_4 = -1/6$. Substituting it into R_3 and multiplying by 4, we get $5a_3^2 \equiv 1/5 \pmod 1$ and so $a_3 = -1/5$ or $-4/5$. Since $a_3 = -1/5$, gives $2a_1 \equiv -6a_4/5 - a_3 \equiv 1/5 + 1/5 \pmod 1$, which contradicts to $2a_1 \equiv 0 \pmod 1$. Hence we have $a_3 = -4/5$. From (30), we get

$$a_1 \equiv -3a_4/5 - \tfrac{1}{2}a_3 \equiv 1/10 + 2/5 \equiv -\tfrac{1}{2}, \quad a_2 \equiv 5a_3/4 - a_1 \equiv -\tfrac{1}{2} \pmod 1.$$

Hence $a_1 = -\frac{1}{2}$, $a_2 = -\frac{1}{2}$, $a_3 = -4/5$, $a_4 = -1/6$, $a_5 = 0$. But these values give $R_3 \equiv 1/3 \not\equiv 1/12 \pmod 1$.

(B) Suppose next $a_5 = -\frac{1}{2}$. Then $20R_3 \equiv 24a_4^2 \equiv 2/3 \pmod 1$. As above we can take $a_4 = -1/3$ or $-1/6$.

(B1) Let first $a_4 = 1/3$. Substituting it into R_3 and multiplying by 4, we get $5a_3^2 \equiv 4/5 \pmod 1$, and so $a_3 = -2/5$ or $-3/5$. Since $a_3 = -2/5$ gives $2a_1 \equiv 2a_5 - 6a_4/5 - a_3 \equiv 4/5 \pmod 1$, which contradicts $2a_1 \equiv 0 \pmod 1$, we have $a_3 = -3/5$. Then from (30), we get

$$a_1 \equiv a_5 - 3a_4/5 - \tfrac{1}{2}a_3 \equiv 0, \quad a_2 \equiv -a_1 + 5a_3/4 \equiv -3/4 \pmod 1)$$

Hence we have $a_1 = 0$, $a_2 = -3/4$, $a_3 = -3/5$, $a_4 = -1/3$, $a_5 = -\frac{1}{2}$ and they give the form $f_{6,2}$.

It can be easily computed that the adjoint form of $f_{6,2}$ is

$$11\left(\tfrac{1}{2}x_1^2 + \tfrac{1}{4}x_2^2 + \tfrac{4}{5}(x_3 + \tfrac{1}{4}x_2 + \tfrac{1}{2}x_1)^2 + \tfrac{5}{6}(x_4 + \tfrac{6}{5}x_3 + \tfrac{4}{5}x_2 + \tfrac{3}{5}x_1)^2 + (x_5 + \tfrac{1}{2}x_4 + x_3 + \tfrac{1}{2}x_2 + x_1)^2 + \tfrac{12}{11}(x_6 + \tfrac{1}{2}x_5 + \tfrac{7}{12}x_4 + \tfrac{3}{2}x_3 + \tfrac{17}{12}x_2 + x_1)^2\right)$$

and it represents the value 4 for $x_1 = x_3 = x_5 = 0$, $x_2 = 1$, $x_4 = x_6 = -1$. Since $4 < 11$, the form $f_{6,2}$ can be decomposed into a sum of a square and a positive definite quadratic form both with integer coefficients.

(B2) Let then $a_4 = -\frac{1}{6}$. Substituting it into R_3 and multiplying by 4, we get $5a_3^2 \equiv 1/5 \pmod 1$ and so $a_3 = -1/5$ or $-4/5$. Since $a_3 = -1/5$ gives $2a_1 \equiv -6a_4/5 - a_3 \equiv 1/5 + 1/5 \pmod 1$, which contradicts to $2a_1 \equiv 0 \pmod 1$. Hence we have $a_3 = -4/5$. From (30), we get

$$a_1 \equiv a_5 - 3a_4/5 - \tfrac{1}{2}a_3 \equiv 0, \quad a_2 \equiv 5a_3/4 - a_1 \equiv 0 \pmod 1.$$

Hence $a_1 = a_2 = 0$, $a_3 = -4/5$, $a_4 = -1/6$, $a_5 = -\frac{1}{2}$. These values give the form $f_{6,3}$.

It can be easily computed that the adjoint form of $f_{6,3}$ is

$$11\left(\tfrac{1}{2}x_1^2 + \tfrac{1}{4}x_2^2 + \tfrac{4}{5}(x_3 + \tfrac{1}{4}x_2 + \tfrac{1}{2}x_1)^2 + \tfrac{5}{6}(x_4 + \tfrac{6}{5}x_3 + \tfrac{4}{5}x_2 + \tfrac{3}{5}x_1)^2 + (x_5 + \tfrac{1}{2}x_4 + x_3 + \tfrac{1}{2}x_2 + x_1)^2 + \tfrac{12}{11}(x_6 + \tfrac{1}{2}x_5 + \tfrac{5}{12}x_4 + \tfrac{3}{2}x_3 + \tfrac{7}{12}x_2 + x_1)^2\right),$$

which represents the value 4 for $x_1 = x_3 = x_5 = x_6 = 0$, $x_2 = 1$, $x_4 = -1$. Hence $f_{6,3}$ can be decomposed as 4 is less than the determinant value of $f_{6,3}$.

Here the proof of lemma 8 and so the theorem 3 is completed.

In closing, I should like to thank Prof. Mordell for his kind help with my manuscript.

(Received 22 November, 1937.)

On the positive definite quadratic forms with determinant unity.

By

Chao Ko (Manchester).

It will make the results of this paper more intelligible to the reader if we commence by giving a brief resumé of a little of the theory of quadratic forms.

Let

$$f_n = \sum_{i,j=1}^{n} a_{ij}x_i x_j \qquad (a_{ij} = a_{ji})$$

be a positive definite quadratic form with determinant D_n and integer coefficients a_{ij}. Denote the minor determinant of the matrix (a_{ij}) $(i,j = 1, \ldots, n)$ of f_n formed by the elements at the intersections of rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_k$ by $A^{(k)}_{i_1, \ldots, i_k; j_1, \ldots, j_k}$; the greatest common divisor of all the minors of order $k = 1, \ldots, n$ by d_k, so that $d_k \mid d_{k+1}$. Write $d_0 = 1$, $d_{n+1} = 0$. Denote the greatest common divisor of all the integers

$$A^{(k)}_{i_1, \ldots, i_k; i_1, \ldots, i_k}/d_k, \qquad 2A^{(k)}_{i_1, \ldots, i_k; j_1, \ldots, j_k}/d_k,$$

by $\sigma_k = 1$ or 2 $(k = 1, \ldots, n)$, and write $\sigma_0 = 1$. Define the numbers, really integers

$$o_k = d_{k+1}d_{k-1}/d_k^2 \quad (k = 1, \ldots, n),$$

so that