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Remark on a problem of Schinzel

by

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The following problem was proposed by A. Schinzel at the Institute of Number Theory at Stony Brook, 1969:

Let K be an algebraic number field. Does there exist a sequence $\{a_i\}$ of integers in K such that, for every ideal $\boldsymbol{q}$ of K, integers $a_1, a_2, \ldots, a_{N(\boldsymbol{q})}$ represent all residue classes modulo $\boldsymbol{q}$?

The problem has been treated in special cases by J. Latham, and the writer (see [3], [4]) and in general case by D. Barsky in [1]. However, his argument has proved to be erroneous [1a].

In [4] the problem was reformulated in the following way: A sequence $\{a_i\}^m$ of integers in K is an F-sequence of length m iff for every initial segment $\{a_i\}^m$ and for every ideal $\boldsymbol{q}$ such that $N(\boldsymbol{q}) \geqslant m$, $a_1, a_2, \ldots, a_m$ represent m different residue classes modulo $\boldsymbol{q}$. Clearly all the terms in an F-sequence are distinct and the problem of Schinzel is equivalent to the problem of existence of an infinite F-sequence. In the same paper the following theorem on F-sequences was proved.

THEOREM 1. *A sequence of distinct terms $\{a_i\}^m$ is an F-sequence of length m in K if and only if*

$$|N(a_i - a_j)| < \max(i, j) \quad \textit{for all } i, j \leqslant m.$$

It is not difficult to see that $\{i\}_{i=1}^{\infty}$ is an F-sequence of infinite length in Q and that $\langle 0, 1\rangle$ is an F-sequence of length 2 in any algebraic number field. Following [3] and [4] we denote the maximal length of F-sequences in K (possibly ∞) by $m(K)$. At present no field besides Q with $m(K) = \infty$ is known. In the cases where it has been possible to determine $m(K)$ the values have been very small and often just equal 2.

The aim of this paper is to prove the following

THEOREM 2. *For any algebraic number field K, if $m(K) > C_K\sqrt{|d|}$, then $h(K) = 1$, where C_K is the Minkowski constant of K* (see [2], p. 120) *and d is its discriminant.*

Hence infinite F-sequences can only exist in simple fields. However, the converse does not hold, there are simple fields that do not contain infinite F-sequences (see [3], [4]).

Proof of Theorem 2. It is well-known that the class number of K equals the number of equivalence-classes among the ideals of norm not exceeding $C_K\sqrt{|d|}$. Hence it is enough to show that every ideal $\boldsymbol{m}$, with norm m not exceeding $[C_K\sqrt{|d|}]$ is principal if $m(K) > C_K\sqrt{|d|}$. Suppose that $m(K) > [C_K\sqrt{|d|}]$, i.e. since $[C_K\sqrt{|d|}] \geqslant m$ and every initial segment of an F-sequence is an F-sequence there is an F-sequence, say $\langle a_1, a_2, \ldots, a_m, a_{m+1}\rangle$, of length $m+1$. By the definition of F-sequences there exists one $a_i \neq a_{m+1}$ such that $a_i \equiv a_{m+1} (\operatorname{mod} \boldsymbol{m})$. Hence $(a_i - a_{m+1}) = \boldsymbol{m}\cdot\boldsymbol{h}$ and

$$|N(a_i - a_{m+1})| = N(\boldsymbol{m})\cdot N(\boldsymbol{h}) = m\cdot N(\boldsymbol{h}) < \max(i, m+1) = m+1$$

by Theorem 1 and so $\boldsymbol{h} = (1)$ and $(a_i - a_{m+1}) = \boldsymbol{m}\cdot(1) = \boldsymbol{m}$.

This completes the proof.

References

[1] D. Barsky, *Sur les systèmes complets de restes modulo les idéaux d'un corps de nombres*, Acta Arith. 22 (1972), pp. 49–56.

[1a] — *Erratum*, ibid. 26 (1974), pp. 115–116.

[2] S. Lang, *Algebraic Number Theory*, London 1968.

[3] J. Latham, *On sequences of algebraic integers*, Journ. London Math. Soc. 6 (1973), pp. 555–560.

[4] R. Wasén, *On sequences of algebraic integers in pure extensions of prime degree*, Coll. Math. 30 (1974), pp. 89–104.

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Corrections to the paper "Quasiperfect numbers"

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by

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1. Replace lines 21–24 on page 443 namely "Thus if $3^4 5^6 17^2 q^2 r^2 \ldots$ and N would not be QP" by the following:

Thus if $N = 3^4 5^6 17^2 q^2 r^2$ is QP, both q and r must be greater than 120. For, if $q < 120$, then

$$\frac{\sigma(Mq^2)}{Mq^2} > \frac{\sigma(M)}{M}\left(1 + \frac{1}{q}\right) > 2$$

and so by Proposition 0, no non-trivial multiple of Mq^2 can be a QP; so that N cannot be a QP.

2. Replace the penultimate sentence on page 443 "But $\sigma(239^2) \equiv 0 \pmod{29} \ldots$ disallowed by Proposition 0" by the following:

If $N = 3^4 5^6 17^2 239^2 r^2$, then $2 < \frac{\sigma(M)}{M}\cdot\frac{239}{238}\cdot\frac{r}{r-1}$; which gives

$$1 - \frac{1}{r} < \frac{173398815623}{174104437500},$$

from which it follows that $r < 247$. Since r is a prime, $r = 241$. This cannot hold, since the exponent on 241 must be at least 4, by Lemma 4A(f).

3. Replace p^2 in line 1 on page 444 by 17^2.

4. Replace "$\sigma(N) \equiv 2 \pmod 3$", in line 16 on page 444 by "$\sigma(N) \equiv 0$ or $2 \pmod 3$, which cannot hold, since $\sigma(N) = 2N+1 \equiv 1 \pmod 3$".

5. Replace "$\sigma(N) \equiv 2 \pmod 3$" in line 9 on page 445 by "$\sigma(N) \equiv 0$ or $2 \pmod 3$, which cannot hold, since $\sigma(N) = 2N+1 \equiv 1 \pmod 3$".